1 Static Analysis of the Price of Anarchy

1.1 Some Examples

1.1.1 Pigou’s Example

\[ \ell(x) = 1 \]

\[ \ell(x) = x \]

- Optimal solution: split flow \( \frac{1}{2} / \frac{1}{2} \) via both edges. Then,

\[ OPT = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \]

- But then: Agents on top edge are unhappy and start to migrate to bottom edge.

- Equilibrium (all agents happy): Total flow on bottom edge. Then \( cost = 1 \)
1.1.2 Braess’ Paradox

- First ignore dashed edge.
- Optimal solution: split flow $\frac{1}{2}$ via both paths. Then,

$$OPT = \frac{1}{2} \cdot \left( \frac{1}{2} + 1 \right) + \frac{1}{2} \cdot \left( 1 + \frac{1}{2} \right) = \frac{3}{2}$$

- Now, insert dashed edge:
- Optimal solution stays the same
- Equilibrium: All agents use zigzag path. Then, $\text{cost} = 1 \cdot (1 + 1) = 2$.

1.2 The Wardrop Model

We are given a
- directed graph $G = (V, E)$
- $k$ commodities with source-sink pairs $s_i, t_i$ and flow demands $r_i$, $i \in [k]$, normalise $\sum_{i \in [k]} r_i = 1$.
- denote set of paths between $s_i$ and $t_i$ by $\mathcal{P}_i$, let $\mathcal{P} = \bigcup_{i \in [k]} \mathcal{P}_i$ (for simplicity, let the $\mathcal{P}_i$ be disjoint)
- latency functions on the edges $\ell_e : [0, 1] \mapsto \mathbb{R}_0^+$ (non-negative, non-decreasing, differentiable)
- The triple $(G, r, \ell)$ is an instance of the routing problem.

Agents induce flow and latency:
- flow vector $(f_P)_{P \in \mathcal{P}}, f_e = \sum_{P \ni e} f_P$.
- a flow is feasible if it satisfies the flow demands and flow conservation
• edge latency: $\ell_e(f) = \ell_e(f_e)$

• path latency: $\ell_P(f) = \sum_{e \in P} \ell_e(f)$

• edge cost: $c_e(f) = f_e \cdot \ell_e(f)$.

• total social cost (average latency):

$$C(f) = \sum_{e \in E} c_e(f) = \sum_{e \in E} f_e \ell_e(f) = \sum_{P \in P} f_P \ell_P(f)$$

There are at least two different interpretations for this model:

1. We consider an infinite set of agents each of which controls an infinitesimal amount of flow. Each agent picks a path of its own.

2. There is one agent per commodity $i$ that controls a flow of size $r_i$, but this flow can be split over several paths.

Agents strive to minimise their own latency. A flow is at equilibrium if no agent has an incentive to shift their flow unilaterally:

**Definition 1** (Nash equilibrium). A feasible flow $f$ is at a Nash equilibrium iff for every commodity $i \in [k]$, every two paths $P_1, P_2 \in P_i$ with $f_{P_1} > 0$, and every $\delta \in [0, f_{P_1}]$ the following holds. For the modified flow $\tilde{f}$ where a flow amount of $\delta$ is shifted from $P_1$ to $P_2$, i.e.,

$$\tilde{f}_P = \begin{cases} 
  f_P - \delta & \text{if } P = P_1 \\
  f_P + \delta & \text{if } P = P_2 \\
  f_P & \text{otherwise}, 
\end{cases}$$

we have $\ell_{P_1}(f) \leq \ell_{P_2}(\tilde{f})$.

Using continuity and monotonicity of the latency functions and letting $\delta$ tend to 0 we obtain from this definition a very useful characterisation due to Wardrop, commonly called Wardrop’s principle or Wardrop equilibrium.

**Lemma 1** (Wardrop’s principle). A feasible flow vector $f$ is at a Nash equilibrium iff for every commodity $i \in [k]$, every two paths $P_1, P_2 \in P_i$ with $f_{P_1} > 0$, we have $\ell_{P_1}(f) \leq \ell_{P_2}(f)$.

Furthermore, if $f$ is a Nash equilibrium we see that all used paths of the same commodity have minimal latency whereas unused paths may have
larger latency. Let $L_i(f)$ denote this minimal latency for commodity $i$ and a Nash equilibrium $f$. Then,

$$C(f) = \sum_{i \in [k]} r_i \cdot L_i(f).$$

Remember, the price of anarchy is

$$\rho(G, r, \ell) = \frac{\max_{f \text{ Nash}} C(f)}{\min_{f \text{ feasible}} C(f)}.$$

### 1.3 Characterising Optimal Flows

- Optimal flow: minimise the value of $C(f)$.
- among all feasible flows $f$.
- Formulation as a convex program:

$$\min \sum_{e \in E} c_e(f)$$

subject to

$$\sum_{P \in \mathcal{P}} f_P = r_i \quad \forall i \in [k]$$

$$f_e = \sum_{P \ni e} f_P \quad \forall e \in E$$

$$f_P \geq 0 \quad \forall P \in \mathcal{P}$$

- The linear constraints of this program define a convex polyhedron.
- We assume that the terms $c_e(f) = f_e \cdot \ell_e(f)$ are convex.
- However, the number of variables is exponential in the network size $(f_P)$. This can be reduced.

#### 1.3.1 Marginal Cost

- What does shifting flow from path $P_1$ to $P_2$ imply?
- Cost of the edges in $P_1$ reduces, cost of $P_2$ increases.
- If cost of $P_1$ reduces more than cost of $P_2$ increases, total cost decreases, i.e., we are not minimal.
• Define marginal cost
\[ \ell'_e(x) = c'_e(x) = c_e(x) \frac{d}{dx}, \quad c'_p(f) = \sum_{e \in P} c'_e(f_e). \]

• Observation: A flow is optimal iff marginal cost of used paths is equal for all paths of the same commodity.

**Lemma 2** (Characterisation of optima via marginal cost). A feasible flow \( f \) is optimal iff for all commodities \( i \in [k] \), paths \( P_1, P_2 \in [k] \), \( f_{P_1} > 0 \) we have \( c'_{P_1}(f) \leq c'_{P_2}(f) \).

Now, observe the similarity of this characterisation of optimal flows and Nash flows.

**Theorem 1** (Equivalence of Nash equilibria and optima w.r.t. marginal cost). Assume \( x \cdot \ell_e(x) \) is convex for all \( e \in E \) and let \( \ell'_e(x) = (x \cdot \ell_e(x)) \frac{d}{dx} \) for all \( e \in E \). Then a feasible flow \( f \) is optimal for \((G, r, \ell)\) if and only if \( f \) is at a Nash equilibrium for \((G, r, \ell^*)\).

**Proof.** The characterisations of optima for \((G, r, \ell)\) by Lemma 2 and Nash equilibria for \((G, r, \ell^*)\) by Lemma 1 are identical. \(\square\)

Observe that \( \ell'_e(x) = (\ell_e(x)x)' = \ell_e(x) + \ell'_e(x)x \) consists of two terms. The first is the per-unit latency incurred by additional flow whereas the second accounts for the increased cost of the flow that is already using the edge.

### 1.4 Existence of Nash Equilibria

**Theorem 2** (Existence and essential uniqueness of Nash equilibria). An instance \((G, r, \ell)\) with nondecreasing latency functions admits a flow at Nash equilibrium. Moreover, if \( f \) and \( \tilde{f} \) are Nash equilibria, \( C(f) = C(\tilde{f}) \).

**Proof.** Remember Rosenthal’s potential for congestion games
\[ \Phi = \sum_{e \in E} \sum_{i=1}^{x_e} d_e(i) \]

In the continuous case we use the integral:
\[ H(f) = \sum_{e \in E} h_e(f_e) \quad \text{with} \quad h_e(x) = \int_0^x \ell_e(u) \, du. \]
Consider the convex program:

\[
\min \sum_{i \in [k]} h_e(f)
\]

subject to

\[
\begin{array}{l}
\sum_{P \in \mathcal{P}} f_P = r_i \quad \forall i \in [k] \\
f_e = \sum_{P : e \in P} f_P \quad \forall e \in E \\
f_P \geq 0 \quad \forall P \in \mathcal{P}
\end{array}
\]

Proof of existence:

- Note that \( h'_e(x) = \ell_e(x) \)
- Hence, the optimality condition from Lemma 2 matches the characterisation for Nash equilibria from Lemma 1.
- Objective function \( H \) is convex (by monotonicity of the \( \ell_e \)), and the solution space is nonempty and convex \( \Rightarrow \) an optimum exists.

Proof of essential uniqueness:

- Consider Nash equilibria \( f \) and \( \tilde{f} \)
- Since both are feasible, the convex combination \( f_\alpha = \alpha \cdot f + (1 - \alpha) \cdot \tilde{f} \) for \( \alpha \in [0, 1] \) is also feasible by convexity of the solution space.
- \( h_e \) must be linear between \( f_e \) and \( \tilde{f}_e \) since otherwise for \( \alpha \in (0, 1) \), we would have \( H(f_\alpha) < H(f) = H(\tilde{f}) \) which would contradict optimality of \( f \) and \( \tilde{f} \).
- Hence, \( \ell_e = h'_e \) must be constant between \( f_e \) and \( \tilde{f}_e \) for all \( e \in E \).
- Also \( L_i(f) = L_i(\tilde{f}) \) for all \( i \) and finally \( C(f) = C(\tilde{f}) \).

1.5 Upper Bounding the Price of Anarchy

1.5.1 A Bound for Latency Functions of Limited Steepness

As we have seen, the price of anarchy depends on the steepness of the latency function. This is formalised in the following theorem.

**Theorem 3.** Suppose for every \( e \in E \) and all \( x \in [0, 1] \),

\[
x \cdot \ell_e(x) \leq \alpha \cdot \int_0^x \ell_e(u) \, du.
\]

Then, the price of anarchy \( \rho(G, r, \ell) \leq \alpha \).
Proof. Let \( f \) denote a Nash flow and \( f^* \) denote a system optimal flow for \((G, r, \ell)\).

\[
C(f) = \sum_{e \in E} \ell_e(f_e) f_e \quad \text{(definition)}
\]

\[
\leq \alpha \sum_{e \in E} \int_0^{f_e} \ell_e(u) \, du \quad \text{(hypothesis)}
\]

\[
\leq \alpha \sum_{e \in E} \int_0^{f_e^*} \ell_e(u) \, du \quad \text{(since } f \text{ minimises } H)\)
\]

\[
\leq \alpha \sum_{e \in E} \ell_e(f^*) f_e^* \quad \text{(by monotonicity of } \ell_e)\)
\]

\[
= \alpha C(f^*) .
\]

This yields an upper bound on the price of anarchy for polynomial latency functions:

**Corollary 1.** Suppose latency functions have the form \( \ell_e(x) = \sum_{i=1}^{d} a_{e,i} x^i \) for a positive integer \( d \) and \( a_{e,i} \geq 0 \). Then

\[
\rho(G, r, \ell) \leq d + 1 .
\]

Proof.

\[
x \cdot \ell_e(x) = \sum_{i=1}^{d} a_{e,i} x^{i+1} \leq (d + 1) \sum_{i=1}^{d} \frac{a_{e,i}}{i+1} x^{i+1} = (d + 1) \int_0^x \ell_e(u) \, du .
\]

For linear latency functions, this yields an upper bound on the price of anarchy of two. A more involved analysis yields a better upper bound that matches our examples from the beginning:

**Theorem 4.** Suppose latency functions are linear with positive slope and offset. Then

\[
\rho(G, r, \ell) \leq \frac{4}{3} .
\]
1.5.2 A Bicriteria Bound

**Theorem 5.** If \( f \) is a flow at Nash equilibrium for \((G, r, \ell)\) and \( f^* \) is feasible for \((G, 2r, \ell)\), then \( C(f) \leq C(f^*) \).

**Proof.**

- We construct a helper instance with latency functions where the cheap part is cut away:
  \[
  \bar{\ell}_e(x) = \begin{cases} 
  \ell_e(f_e) & \text{if } x \leq f_e \\
  \ell_e(x) & \text{if } x \geq f_e
  \end{cases}
  \]

- Denote the respective cost function by
  \[
  \bar{C}(f) = \sum_{e \in E} f_e \cdot \bar{\ell}_e(f_e)
  \]

- We bound the cost we can “lose” by using these latency functions. For any edge \( e \in E \) and \( x \geq 0 \):
  \[
  x(\bar{\ell}_e(x) - \ell_e(x)) \leq \ell_e(f_e)f_e
  \]
  (If \( x \geq f_e \) the difference is zero, and if \( x \leq f_e \), the difference is maximised to \( \ell_e(f_e)f_e \) if \( \ell_e \) drops to 0 left of \( f_e \).)

- Now consider the total additional cost by evaluating \( f^* \) with respect to \( \bar{C} \) instead of \( C \):
  \[
  \bar{C}(f^*) - C(f^*) = \sum_{e \in E} f^*_e(\bar{\ell}_e(f^*_e) - \ell_e(f^*)) \leq \sum_{e \in E} \ell_e(f_e)f_e = C(f) \quad (1)
  \]

- Let 0 denote the zero flow. Then, for commodity \( i \in [k] \) and \( P \in P_i \)
  \[
  \bar{\ell}_P(f^*) \geq \bar{\ell}_P(0) \geq L_i(f)
  \]
  (Hence,
  \[
  \bar{C}(f^*) \geq \sum_{i \in [k]} \sum_{P \in P_i} L_i(f) f^*_P = \sum_{i \in [k]} 2L_i(f) r_i = 2C(f) \quad (2)
  \]

- Combining Equations (1) and (2) we obtain
  \[
  C(f^*) \geq \bar{C}(f^*) - C(f) \geq 2C(f) - C(f) = C(f)
  \]

**References**