Exercise 7.1: (5 points)

1. Prove the following inequality: For every non-negative random variable \( X \) and every \( t > 0 \):
\[
\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t} \cdot .
\]

2. Now, let \( X \) be a non-negative random variable with bounded second moment \( \mathbb{E}[X^2] \leq \alpha, \alpha \geq 0 \). Prove that for any \( t > 0 \),
\[
\Pr[X \geq t \cdot \sqrt{\alpha}] \leq \frac{1}{t^2} .
\]

Hint: Both proves are one-liners.

Exercise 7.2: (5 points)

Consider \( n \) queries that are mapped by a 2-wise independent (i.e. \((1, 2)\)-universal) hash function to \( n \) modules.

1. Prove that the contention is bounded by \( O\left(\sqrt{n}\right) \) with probability at least 1/2.

2. What is the expected contention?

Proof idea: Consider an arbitrary fixed module \( j \) and consider random indicator variables \( X_1, \ldots, X_n \) where \( X_i \) indicates whether query \( i \) is mapped to module \( j \). Then \( C_j = \sum_{i \in [n]} X_i \) is the load of this module. Compute \( \mathbb{E}[C_j^2] \) and apply the inequality proved in Exercise 7.1.

Exercise 7.3: (5 points)

How can the analysis in Exercise 7.2 be extended to \( k \)-wise independent hash functions? How does the resulting bound relate to the one presented in the lecture?

Deadline: Tuesday, June 6, 11:00, letterbox in front of i1.