Exercise 9.1: (10 points)
Consider the load balancing problem with identical machines. The following pseudocode describes a greedy algorithm for a given sequence of jobs with weights \( w_1, \ldots, w_n \).

1: initialize \( \ell_j = 0 \) for all machines \( j \in [m] \).
2: for \( i = 1 \) to \( n \) do
3: assign job \( i \) to a machine \( j \) with minimal load \( \ell_j \)
4: update \( \ell_j \leftarrow \ell_j + w_i \)
5: end for

Let \( A \) denote the assignment computed by the greedy algorithm and let \( \text{cost}(A) \) denote the respective cost, i.e., the maximum load of a machine for this assignment.

1. Prove that \( \text{cost}(A) \leq \left(2 - \frac{1}{m}\right) \cdot \text{OPT} \).
   Hint: Start by proving an upper bound of 2.
2. Prove that this bound is tight by showing a way to construct an input to the greedy algorithm such that \( \text{cost}(A) = \left(2 - \frac{1}{m}\right) \cdot \text{OPT} \).
3. The input constructed in part 2 cannot be a Nash equilibrium since this would violate the upper bound of \( \left(2 - \frac{2}{m+1}\right) \cdot \text{OPT} \) on the cost of Nash equilibria. Show that in the assignment computed by the greedy algorithm for the input you constructed in part 2, there is always an “unhappy” job that can move to a machine such that after the move the new machine has less load than the old one had before the move.
4. Find an algorithm that computes a Nash equilibrium in polynomial time.

Exercise 9.2: (5 points)
Now consider machines with speeds.

1. When using the construction used in the proof of the lower bound on the Price of Anarchy (Theorem 4, Section 2.2.2), what is the smallest number of machines that yields a Price of Anarchy greater than 2?
2. Try to find a Nash equilibrium with Price of Anarchy greater than 2 with as few machines as possible.
3. Does your algorithm from exercise 9.1.4 also work for machines with speeds? If not, find an algorithm that computes a Nash equilibrium in polynomial time for machines with speed.

Deadline: Monday, June 26, 11:00, letterbox in front of i1.