Algorithmic Graph Theory (SS2016)

Chapter 1
Planar Graphs

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Definition: Graph

**Definition (Undirected Graph)**

- Let \( V(G) = \{v_1, ..., v_n\} \) be a non-empty set of nodes and
- \( E(G) \) be a set or multiset of pairs from \( V(G) \) (set of edges).
- The sets \( V(G) \) and \( E(G) \) define the graph \( G = (V(G), E(G)) \).
- If \( G \) is uniquely determined, then we just write: \( V \) and \( E \).
- Or in other words \( G = (V, E) \).
- We always use as default writing: \( n = |V| \) and \( m = |E| \).
Way of Speaking for Graphs

Definition (Way of Speaking)

- Let $G = (V(G), E(G))$ and $e = (v, w) \in E(G)$.
- The nodes $v, w$ are called connected (adjacent) by an edge $e$.
- An edge $e$ is called loop, if $v = w$ holds.
- Two edges are called parallel, if they are the same.
- A graph without parallel edges is called simple.

- As long as we do not state differently we will use in the following simple graph without loops.
Degree of a Node

Definition (Degree of a Node)

- Let $v \in V(G)$.
- With

$$\text{deg}(v) = |\{e \in E(G) \mid e = (v, v'), v' \in V(G) \setminus \{v\}\}|$$

we denote the degree of a Node (degree) of $v$.

- $\text{deg}(v_0) = 4$.
- $\text{deg}(v_1) = 3$.
- $\text{deg}(v_4) = 6$.
- $\text{deg}(v_5) = 6$. 

![Graph with vertices and edges](image)
Regular and Complete

**Definition (Regular)**

A graph $G$ is called $k$-regular, iff for all $v \in V(G)$ we have: $d(v) = k$.

**Definition (Complete)**

A graph $G$ is called complete, iff all pairs of nodes $a, b$ from $V$ holds: $(a, b) \in E$.

Notation: $K_n$. 
Special Graphs

**Definition (Bipartite)**

A Graph $G$ is called bipartite, iff $V$ may be split in to disjoint set $V', V''$, such that each edge connects only nodes from both partitions.

- Notation: $G = (V', V'', E)$

**Definition (Complete bipartite)**

A Graph $G$ is called complete bipartite, iff $V$ may be split in to disjoint set $V', V''$, and $E = \{(a, b) \mid a \in V', b \in V''\}$.

- Notation: $K_{p,q}$ with $p = |V'|$ and $q = |V''|$.
- Star, iff $S_n = K_{1,n-1}$. 
Examples
**Definition (Subgraph)**

- A Graph $H = (V(H), E(H))$ is called a subgraph of $G = (V(G), E(G))$, iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
Subgraphs

Definition (node-induced subgraph)

- A graph $H = (V(H), E(H))$ is a node-induced subgraph of $G = (V(G), E(G))$,
- iff $V(H) \subseteq V(G)$ and $E(H) = \{(a, b) \in E(G) \mid a, b \in V(H)\}$. 

![Graph Diagram](https://via.placeholder.com/150)
A graph $G = (V, E)$ is called connected, iff between any two different nodes $a, b$ exists a path from $a$ to $b$. 
Node-Separator

**Definition**

Let $G = (V, E)$, $V' \subset V$ is called a node-separator (vertex cut), iff $G - V'$ is not connected.

**Notation:** $G - V' := (V \setminus V', \{(a, b) \in E \mid a, b \in V \setminus V'\})$

**Definition**

If $\{v\}$ is a node-separator, then $v$ is called articulation point.

**Theorem**

*Only cliques $K_n$ do not have any node-separator.*
**Definition**

Let $G = (V, E)$. $E' \subset E$ is called edge-separator (edge cut), iff $G - E'$ is not connected.

**Notation:** $G - E' := (V, E \setminus E')$

**Definition**

If $\{v, w\}$ is an edge-separator, then $\{v, w\}$ is called a bridge.

**Theorem**

An minimal edge-separator $E'$ of $G = (V, E)$ induces a 2-partite graph. Or in other words: $G = (V, E')$ is a 2-partite graph.
Example
Connectivity

**Definition**

A Graph $G = (V, E)$ is called $k$-connected, iff $\forall V' \subseteq V : |V'| = k - 1$ we have $G - V'$ is connected.

A $k$-connected Graph is also $k - 1$-connected.

**Notation:** $\kappa(G) = k$

**Definition**

Let $G = (V, E)$ and $k$ minimal with: $\exists E' \subseteq E : |E'| = k$ and $G - E'$ is not connected or trivial. Then we call $G$ $k$-edge-connected.

A $k$-edge-connected Graph is also $k - 1$-edge-connected.

**Notation:** $\lambda(G) = k$
Statements on Connectivity

Theorem

For any graph $G = (V, E)$ we have:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Notation: $\delta(G) := \min\{\deg(v) \mid v \in V\}$

Theorem

For all integer numbers $0 < a \leq b \leq c$ there are graphs $G$ with:

$$\kappa(G) = a, \ \lambda(G) = b, \ \delta(G) = c$$

Theorem

Let $G = (V, E)$ be a graph with: $|V| = n$ and $\delta(G) \geq n/2$. Then we have:

$$\lambda(G) = \delta(G)$$
Statements on Node-Connectivity

Theorem

Let $G = (V, E)$ with: $|V| = n$ and $|E| = m$. Then is the maximal connectivity (maximal $k$ with $G$ is $k$-connected) of $G$:

$$\begin{align*}
0 & \text{ falls if } m < n - 1 \\
2 \cdot \frac{m}{n} & \text{ if } m \geq n - 1
\end{align*}$$

Theorem

Let $G = (V, E)$ connected. The following statements are equivalent:

1. $v \in V$ is a node-separator.
2. $\exists a, b \in V: a, b \neq v$: each path from $a$ to $b$ traverses via $v$.
3. $\exists A, B: A \cup B = V \setminus \{v\}$ and each path from $a \in A$ to $b \in B$ traverses via $v$. 
Statements on Edge-Connectivity

**Theorem**

Let $G = (V, E)$ be connected. The following statements are equivalent:

1. $e \in E$ is a edge-separator.
2. $e$ is not in any simple cycle of $G$.
3. $\exists a, b \in E$: each path from $a$ to $b$ traverses via $e$.
4. $\exists A, B$: $A \cup B = V$ and each path from $a \in A$ to $b \in B$ traverses via $e$. 
Definition

Let $G = (V, E)$ and $(a, b) = e \in E$. The subdivision of an edge $e$ results in graph $G = (V \cup \{v\}, E \cup \{(a, v), (v, b)\} \setminus \{e\})$.

Definition

A set of paths of $G = (V, E)$ is called intern-node-disjoint, iff no two paths share an internal-node. The internal nodes are all except the start and the end node.
Theorem

Let $G = (V, E)$ with $|V| \geq 3$. The following statements are equivalent:

1. $G$ is 2-connected.
2. Each node pair is connected by two intern-node-disjoint paths.
3. Each node pair is on a common simple cycle.
4. There exits an edge and each node together with this edge is on a common simple cycle.
5. There exit two edges and each pair of edges is on a common simple cycle.
6. For each pair of nodes $a, b$ and an edge $e$ exists a simple path from $a$ to $b$ traversing $e$.
7. For three nodes $a, b, c$ exists a path from $a$ to $b$ traversing $c$.
8. For three nodes $a, b, c$ exists a path from $a$ to $b$ avoiding $c$. 
Theorem

Let $G = (V, E)$ be $k$-connected. Then any $k$ nodes are on a common simple cycle.

Notation: Let $(G = V, E)$ and $(H = W, F)$ graphs $G + W = (V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\})$

Theorem

A graph $G$ is 3-connected, iff $G$ may be constructed from the wheel $W_i = K_1 + C_i$ ($i \geq 4$) by the following operations:

1. Adding a new edge.
2. Splitting a node of degree $\geq 4$ into two connected nodes of degree $\geq 3$. 
Statements on k-Connectivity

Theorem (Menger’s Theorem)

\( G \) is \( k \)-connected, iff any two node are connected by \( k \) intern-node-disjoint paths.

Theorem (Menger’s Theorem)

\( G \) is \( k \)-edge-connected, iff any two node are connected by \( k \) edge-disjoint paths.
Computing the Connectivity

**Theorem**

The 1-connectivity of a graph may be computed by DFS/BFS.

**Theorem**

The 1-edge-connectivity of a graph may be computed by DFS/BFS.

**Theorem**

The 2-connectivity of a graph may be computed by DFS/BFS.

**Theorem**

The k-connectivity of a graph may be computed by flow algorithms.

**Theorem**

The k-edge-connectivity of a graph may be computed by flow algorithms.
Definitions

Definition
A graph \( G = (V, E) \) is called planar, iff it could be drawn in the plane without crossing edges.
A connected area of such an embedding is called window.
The unlimited window is called outer window.

Definition
A graph \( G = (V, E) \) is called maximal planar, iff the adding of an edge makes \( G \) non-planar.
Example: planar Graph
Results I

Theorem

If \( G = (V, E) \) is planar and 2-connected, then each window is a simple cycle and each edge separates two different windows.

Theorem (Euler)

Let \( G = (V, E) \) be a planar graph with \( |V| = n, |E| = m \). Let \( f \) be the number of windows and \( k \) be the number of connected components. Then the following holds:

\[
    n - m + f = 1 + k.
\]

Proof by simple induction.
Proof

- \( n - m + f = 1 + k \) holds for a single node.
- new node:
  \[(n + 1) - m + f = 1 + (k + 1)\]
- new edge connects components:
  \( n - (m + 1) + f = 1 + (k - 1) \) or
- new edge separates window:
  \( n - (m + 1) + (f + 1) = 1 + k \).
Results II

Theorem

Let $G = (V, E)$ be a planar graph with $|V| = n$, $|E| = m$ and each window is a simple cycle of length $k$. Then the following holds:

$$m = k \cdot \frac{n - 2}{k - 2}$$

Note: $k \cdot f = 2 \cdot m$ und $n - m + f = 2$

Theorem

Let $G = (V, E)$ be a planar graph with $|V| = n$, $|E| = m$ and each window is a 3-clique. Then the following holds: $m = 3 \cdot n - 6$.

If each window is a simple cycle of length 4, then we get: $m = 2 \cdot n - 4$.

Theorem

Let $G = (V, E)$ be a planar graph with $|V| = n \geq 3$, $|E| = m$. Then we get: $m \leq 3 \cdot n - 6$. If $G$ contains no triangles, then we have: $m \leq 2 \cdot n - 4$.
## Results III

### Theorem

**Theorem**

\[ n - m + f = 1 + k \]

\[ e \leq 3 \cdot n - 6 \]

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**Theorem**

\( K_5 \) and \( K_{3,3} \) are non-planar graphs.

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**Theorem**

Let \( G = (V, E) \) be a planar graph with \(|V| \geq 4\). Then \( G \) contains at least four nodes with degree \( \leq 5 \).

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**Theorem**

Let \( G = (V, E) \) be a planar graph. Then each window could become the outer window.
Results IV

Theorem

Let $G = (V, E)$ be a maximal planar graph with $|V| \geq 4$. Then $G$ is 3-connected.

Theorem

Each 3-connected planar graph is embeddable in a unique way on the sphere.

Theorem

Any planar graph could be drawn with straight lines on the plane.
Recognition-Problem

Definition

The following problem is the recognition-problem on graphs:

- Given a graph $G = (V, E)$
- and a graph-class $\mathcal{G}$.
- Question; does $G \in \mathcal{G}$ hold.

Theorem

The recognition-problem for planar graphs is solvable in linear time.
Definition

A planar graph $G$ is called outer-planar, iff it could be drawn without crossing in the plane, such that all nodes are on one (the outer) window.

Definition

A graph $G = (V, E)$ is called maximal outer-planar, iff the addition of any edge makes $G$ non-outer-planar.
Example: outer-planar Graph
Definition

A planar graph $G$ is called outer-planar, iff it could be drawn without crossing in the plane, such that all nodes are on one (the outer) window.

Definition

A graph $G = (V, E)$ is called maximal outer-planar, iff the addition of any edge makes $G$ non-outer-planar.

Definition

A planar graph $G = (V, E)$ is called $k$-outer-planar, iff it could be drawn in the plane, such that

- no two edges cross and
- after deletion $k - 1$ times the nodes of the outer window,
- the remaining is a embedded outer-planar graph.
Results 1

Theorem

Let $G = (V, E)$ be a maximal outer-planar graph with $|V| = n \geq 3$. Then $G$ will have $n - 2$ inner windows.

Theorem

Let $G = (V, E)$ be a maximal outer-planar graph with $|V| = n$ and $|E| = m$. Then the following holds:

1. $2 \cdot n - 3 = m$
2. At least three nodes have a degree of $\leq 3$.
3. At least two nodes have a degree of two.
4. $G$ is exactly two-connected.

Theorem

$K_4$ and $K_{2,3}$ are not outer-planar graphs.
Definition

A SP-graph is constructed by a sequence of series and parallel operations from the graphs $(\{a, b\}, \{(a, b)\})$ and $(\{a, b\}, \emptyset)$.

The parallel operation merges the corresponding connector nodes.

The series operation merges two connector nodes. This new may not be used as a connector node in any future operation.

Theorem

$K_4$ is not a SP-graph, but the $K_{2,3}$ is a SP-graph.
Definition

Two graphs $G$ and $H$ are called homeomorph, iff they could be constructed from the same graph by a sequence of subdivisions.
Theorem

\( G = (V, E) \) is outer-planar, iff no subgraph is homeomorphic to the \( K_4 \) or the \( K_{2,3} \) with the exception of the \( K_4 - e \).

Theorem

\( G = (V, E) \) is a SP-graph, iff no subgraph is homeomorphic to the \( K_4 \) with the exception of the \( K_4 - e \).

Theorem (Kuratowski)

\( G = (V, E) \) is planar, iff no subgraph is homeomorphic to the \( K_5 \) or \( K_{3,3} \).

Theorem

A outer-planar graph is a SP-graph.
A SP-Graph is a planar graph.
Results I

Theorem

Any planar graph is 5-colourable.

Theorem

Any planar graph is 4-colourable.

Theorem

Any planar graph with at most two triangles is 3-colourable.
A Proof

**Theorem**

*Any planar graph is 5-colourable.*

**Idea of Proof:**

- Choose a node $v$ of degree less than 6.
- Colour recursively $G - \{v\}$.
- If $\deg(v) < 5$ holds, $v$ can be coloured.
- If all neighbours of $v$ use just four colours, $v$ can be coloured.
- If $\deg(v) = 5$ holds and all neighbours of $v$ are coloured with different colours, note:
  - Within $G - \{v\}$ there is a component, which uses just two colours and can be recoloured.
  - A short case discussion shows:
    - There exists two colours and a component using these colours,
    - such that just one neighbour of $v$ receives a new colour.
Recolouring one Component
Theorem

A planar graph is 4-colourable, iff each hamilton planar graph is 4-colourable.

Theorem

A planar graph is 4-colourable, iff each cubic planar graph without bridges is 3-colourable.

Theorem

The 3-colouring-problem on planar graphs if degree \( \leq 4 \) is NP-complete.
Idea and Structure of Proof

Theorem

The 3-colouring-problem on planar graphs if degree $\leq 4$ is NP-complete.

- Problem $L_1$ is easier than $L_2$: $L_1 \leq_P L_2$.
- If $L_2$ is in $\mathcal{P}$, then is also $L_1$ in $\mathcal{P}$.
- If $L_1$ is hard, i.e. $L_1 \in \text{NP}$, then is also $L_2 \in \text{NP}$.
- Structure of proof:
  - Let $L_1 \in \text{NP}$ and we assume $L_2 \in \mathcal{P}$.
  - We transform input of $L_1$ with function $f$ into input for $L_2$ such that:
    - $x \in L_1 \iff f(x) \in L_2$.
    - If $f \in \mathcal{P}$ holds, then we get $L_1 \in \mathcal{P}$, which is a contradiction.
- Here we have: $L_1$ is the 3-colouring-problem and $L_2$ 3-colouring-problem on planar graphs of degree $\leq 4$. 
Theorem

The 3-colouring-problem on planar graphs if degree \( \leq 4 \) is NP-complete.

- Let \( G = (V, E) \) be the input of the 3-colouring-problem
- Construct planar \( f(G) \) as input of the 3-colouring-problem
  - Draw \( G \) in the plane. We get some crossings.
  - Replace each crossing with a 3-colorable planar graph, such that
  - \( G \) is 3-colorable, \( f(G) \) is 3-colorable.

Lemma

There exists a planar graph \( H \) with nodes \( a, c, b, d \):

- The nodes \( a, c, b, d \) are on the outer face in that order.
- The nodes \( a, b \) take in any 3-coloring of \( H \) the same color.
- The nodes \( c, d \) take in any 3-coloring of \( H \) the same color.
Proof (planar)

- The central nodes are coloured w.l.o.g. as follows.
- Case 1: Colour $a$ blue.
Proof (planar 2.case)

- The central nodes are coloured w.l.o.g. as follows.
- Case 2: Colour a red.
Proof (planar)

Each crossing is replaced by such a component.
Proof (planar, degree 4)

There exists a component $H$ with three nodes $a, h, d$ of degree 2 which are coloured the same in each 3-colouring of $H$. 
Proof (planar, degree 4)

There exists a component $H_x$ with $x$ nodes of degree 2 which are coloured the same in each 3-colouring of $H_x$. 
Summary (Proof)

- Replace edge-crossings by the above construction, such that
  - each crossing is replaced by one component.
  - I.e. an edge with $x$ crossings will be replaced by $x$ components
  - and one edge.
- Replace a node of degree $g > 4$ by $\lceil (g - 6)/2 \rceil + 1$ components of the second construction.
  - Note: $x$ tree-wise connected components have $x + 2$ nodes of degree 2 coloured by the same colour.
  - $2 \cdot (\lceil (g - 6)/2 \rceil + 1 + 2) \geq 2 \cdot ((g - 6)/2 + 3) = g$
Introduction

- Basis for all divide and conquer algorithms.
- We would like to have small separators.
- Split the graph at the separator.
- Solve the problem recursively on the disconnected components.
- Construct the solution by using the sub-solutions.
- Here: separators for planar graphs.
Definition

Let $G = (V, E)$ be a graph and $n = |V|$.

Let $0 \leq \alpha \leq 1$ be a constant.

Let $f(n)$ be a function.

We call $C \subseteq V$ a $(f(n), \alpha)$-separator, iff
- $|C| \leq f(n)$ and
- each component of $G[V \setminus C]$ contains at most $\alpha \cdot n$ nodes.
**Example 1**

**Lemma**

A tree $T$ has a $(1, 1/2)$-separator.

**Proof:**

- Choose an arbitrary node $c$ as a candidate.
- Let $T_i$ be the trees in $T - c$.
- If one component $T_i$ contains more than $n/2$ nodes,
  then choose a new candidate $c := \Gamma(c) \cap V(T_i)$.
- After such a step the size of the largest component decreases by at least one.
- Repeat till a separator is found.
Example Outer-planar Graph
Lemma

A outer-planar graph \( G \) has a \((3, 1/2)\)-separator.

Proof:

- Maximise the outer-planar graph \( G \).
- Use the above technique.
- Thus use the tree of inner windows.
- Choose as separator the node of the selected window.
Let $G = (V, E)$ be a graph and $n = |V|$.

Let $f(n)$ be a function.

Then $C \subseteq V$ is called a $f(n)$-separator, iff

- $V$ may be split in $C, T_1, T_2$.
- $|C| \leq f(n)$.
- $T_1, T_2$ are not connected.
- $T_i$ has at most $2/3 \cdot n$ nodes.

$\leq 2/3 \cdot n$

$\leq f(n)$

$\leq 2/3 \cdot n$
Comparing above Definitions

Lemma

G has a \((f(n), 2/3)\)-separator, iff \(G\) has a \(f(n)\)-separator.

Show \(\Leftarrow\)

- Each component \(K\) contains at most in one \(T_i\).
- Thus \(|V(T_i)| \leq 2/3 \cdot n\) holds.

Show \(\Rightarrow\)

- If a component \(K\) contains at least \(1/3 \cdot n\) nodes.
- then choose \(T_1 = K\).
- If all components contain less then \(1/3 \cdot n\) nodes,
- then enlarge \(T_1\) step by step till \(T_1\) contains more than \(1/3 \cdot n\) nodes.
- Then \(T_2\) contains at most \(2/3 \cdot n\) nodes.
Introduction

- Planar graphs are important with many applications.
- How large could be a minimal separator in a planar Graph?
- First example:
Introduction

- Planar graphs are important with many applications.
- There is no separator of constant size.
- Aim: $O(\sqrt{n})$-separator.
- Consider maximal planar graphs.
- Consider cycles as separators.
Overview

Theorem (Lipton, Tarjan 1979)

Each planar graph with \( n \) nodes has a \((2 \cdot \sqrt{2n}, \frac{2}{3})\)-separator.

Theorem (Lipton, Tarjan 1979)

A \((2 \cdot \sqrt{2n}, \frac{2}{3})\)-separator can be constructed on planar graphs in time \( O(n) \).

Theorem (Lipton, Tarjan 1979)

Let \( G = (V, E) \) be a planar graph and \( \varepsilon \leq 1 \) with \( \varepsilon \cdot n \geq 1 \). Then contains \( G \) a \(((2 + \frac{\sqrt{2}}{\varepsilon \cdot n}) \cdot \sqrt{6n/\varepsilon}, \varepsilon)\)-separator, which could be constructed in time \( O(n \log 1/\varepsilon) \).
Basic Idea

- We could hope for a good separator.
- But in general we may need $O(\sqrt{n})$.
- In the worse case the planar graph is maximal.
**Definition (Diameter and Radius)**

- The diameter of $G = (V, E)$ is:
  \[ \text{diam}(G) = \max\{\text{dist}(v, w) \mid v, w \in V\}. \]

- The radius of a node $v \in V$ is:
  \[ \text{rad}(v, G) = \max\{\text{dist}(v, x) \mid x \in V\} \]

- The radius of $G$ is:
  \[ \text{rad}(G) = \min\{\text{rad}(v, G) \mid v \in V\}. \]
Lemma

Let $G = (V, E)$ be a planar graph and $B = (V, T)$ be a spanning-tree of $G$ with radius $s$. Then $G$ contains a $(2 \cdot s + 1, 2/3)$-separator.

Proof:

- Let $G$ be triangulated and embedded in the plane as a planar Graph.
- Let $e \in E \setminus T$.
- $e$ assembles with some edges from $T$ a unique cycle $C_e$.
- By $int(C_e)$ we denote the number of nodes which are inside $C_e$.
- $ext(C_e)$ we denote the number of nodes which are outside $C_e$.
- Aim: Search $e$ with $int(C_e) \leq 2/3 \cdot n$ and $ext(C_e) \leq 2/3 \cdot n$.
- Then is $C_e$ a $(2 \cdot s + 1, 2/3)$-separator.
Example
Proof (continued)

Search step by step for an edge $e$ with

- $\text{int}(C_e) \leq 2/3 \cdot n$ and $\text{ext}(C_e) \leq 2/3 \cdot n$.
- Choose any $e$.
- If $\text{int}(C_e) \leq 2/3 \cdot n$ and $\text{ext}(C_e) \leq 2/3 \cdot n$ holds, terminate.
- Let w.l.o.g.: $\text{int}(C_e) > 2/3 \cdot n$.

Let $e = \{x, y\}$ and $z$ be the missing node of the window attached at $e$ and in the inside of $C_e$.

- If $e' = \{x, z\}$ on the cycle $C_e$, continue with considering $C_{e''}$.
- If $e'' = \{y, z\}$ on the cycle $C_e$, continue with considering $C_{e'}$.
- Otherwise let w.l.o.g. $\text{int}(C_{e'}) \leq \text{int}(C_{e''})$
  and consider now $C_{e''}$.

In the last step $\text{int}(C_e) \leq 2/3 \cdot n$ und $\text{int}(C_e) \geq 1/3 \cdot n$ holds.

It follows that $\text{int}(C_e) \leq 2/3 \cdot n$ und $\text{ext}(C_e) \leq 2/3 \cdot n$ holds.
Proof (continued)

- Last step in detail:
- The inside of \( e = \{x, y\} \) is too large:

\[
2/3 \cdot n < \text{int}(C_{\{x,y\}}) = \text{int}(C_{\{x,z\}}) + \text{int}(C_{\{y,z\}}) + |C_{\{x,z\}} \cap C_{\{y,z\}}| - 1
\]

- The inside of \( e'' = \{x, z\} \) is the larger part of \( (\text{int}(C_e') \leq \text{int}(C_{e''})) \):

\[
2/3 \cdot n < \text{int}(C_{\{x,z\}}) + \text{int}(C_{\{y,z\}}) + |C_{\{x,z\}} \cap C_{\{y,z\}}| - 1 \\
\leq 2 \cdot \text{int}(C_{\{x,z\}}) + |C_{\{x,z\}}| - 1
\]

- This way we get:

\[
\text{ext}(C_{\{x,z\}}) = n - |C_{\{x,z\}}| - \text{int}(C_{\{x,z\}}) \\
< n - 1/3 \cdot n \\
= 2/3 \cdot n
\]
Example

Start BFS from some node $r$. If the radius is smaller than $\sqrt{2n}$ we apply the lemma.
Example

Consider the case that the radius is large then $\sqrt{2n}$.
Each intermediate level disconnects the graph.
We could only hope for a small separator.
Example

None of the levels is a separator.
Check set of levels of distance $s = \lceil \sqrt{n/2} \rceil$.
One set is smaller then $\lfloor n/s \rfloor$. 
Example

If this set is no separator, consider the largest component.
And apply the lemma.
Planarer Separator (Teil 1)

Theorem (Lipton, Tarjan 1979)

*Any planar graph with $n$ nodes has a $(2 \cdot \sqrt{2n}, 2/3)$-separator.*

Proof:

- Choose node $w$ as the root.
- Determine $S_i$ ($1 \leq i \leq l$) the set of nodes at distance $i$ from $w$.
- If $2 \cdot l + 1 \leq 2\sqrt{2n}$ holds, the proof follows from the above lemma.
- Otherwise let $s = \lceil \sqrt{n/2} \rceil$.
- Define $L_j = \bigcup_{i \equiv j \mod s} S_i$ for $0 \leq j < s$.
- For a $k$ hold: $|L_k| \leq \lfloor n/s \rfloor$.
- Consider $H = G[V \setminus L_k]$.
- Assume now, that one component of $H$ has more than $2/3 \cdot n$ nodes.
Proof (continued)

- $H$ contains at most $s - 1$ continuous levels $S_i$.
- Let $S_l, S_{l+1}, \ldots, S_{l+s-2}$ be those levels.
- Show that $H$ could be embedded as a planar graph $H'$ with radius $s - 1$.
  - If $l = 0$ holds, is $w$ part of $H$ and we have an embedding.
  - Otherwise $l > 0$ holds and we connect all nodes from $S_l$ with a node $w'$.
- We have by the above lemma for $H'$ a $(2 \cdot s - 1, 2/3)$-separator $C'$.
- The separator for $G$ is $C = C' \cup L_k$.
- $|C| \leq \lfloor n/s \rfloor + 2 \cdot s - 1$.
- Note: $s = \lceil \sqrt{n/2} \rceil$.
- Thus we have: $|C| \leq \sqrt{2n} + \sqrt{2n}$.
Theorem 2

Theorem (Lipton, Tarjan 1979)

A \((2 \cdot \sqrt{2n}, 2/3)\)-separator for a planar graph may be computed in time \(O(n)\).

- Computing the levels: breath-first-search.
- Counting the nodes: run through a tree.
- Planar embedding: depth-first-search.
- Triangulation: local search.
- Construction of the tree of windows: depth-first-search.
- Counting the nodes: run through the tree of windows
- Used also: dynamic programming.
Theorem

Any graph with genus $g$ and $n$ nodes has a $(6 \cdot \sqrt{gn} + 2 \cdot \sqrt{2n} + 1, 2/3)$-separator, which could be computed in time $O(n + g)$.

Theorem

Any graph without $H$-minor and $n$ nodes has a $(|H|^{3/2} \sqrt{n}, 2/3)$-separator.
Theorem

The independent set problem on planar Graphs of degree three is NP-complete.

Proof:

- Construct component for the crossing of two edges.
- This component will increase the size of the independent set by six.
- We could replace a polynomial number of crossing with this component.
- Replace a node of degree $\geq 4$ by a special binary tree.
- The leaves will take the role of the original node.
- There could be two cases:
  - All leaves are in the independent set and the total number within the tree is $x$.
  - No leave is in the independent set and the total number within the tree is $x - 1$. 
Crossings (1)
Crossings (2a)
Crossings (2b)
Crossings (3a)
Crossings (3b)
Crossings (4a)
Crossings (4b)
Gradkomponente

\[ g_0 \]

\[ f_0 \quad f_1 \]

\[ e_0 \quad e_1 \]

\[ d_0 \quad d_1 \quad d_2 \quad d_3 \]

\[ c_0 \quad c_1 \quad c_2 \quad c_3 \]

\[ b_0 \quad b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5 \quad b_6 \quad b_7 \]

\[ a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \]
Theorem

The independent set problem on planar graphs is solvable in time $2^{O(\sqrt{n})}$.

- **Algorithm:**
  - Compute a $C$.
  - For each independent set $I$ on $C$:
    - Remove all nodes $\Gamma(I)$ from the components of $G[V \setminus C]$.
    - Solve the independent set problem recursively on each component.

- Running time: $t(n) \leq O(n) + 2^{\sqrt{8n}} + O(n) \cdot t(2/3 \cdot n)$.

- Let $k_1, k_2, n_0$ be the constants of the $O$ terms,

- I.e. $(k_1 + k_2) \cdot n \leq 2^{\sqrt{8n}}$ for all $n > n_0$.

- Let $L \geq t(n)$ for all $n \leq n_0$.

- Show by induction: $t(n) \leq L \cdot 2^{\frac{2\sqrt{8n}}{1 - \sqrt{2/3}}}$. 
Show by induction: $t(n) \leq L \cdot 2^{\frac{2\sqrt{8n}}{1-\sqrt{2/3}}}$.

Holds for $n \leq n_0$.

Let $n > n_0$:

Reminder: $(k_1 + k_2) \cdot n \leq 2\sqrt{8n}$.

\[
t(n) \leq k_1 \cdot n + 2\sqrt{8n} + k_2 \cdot n \cdot t(2/3 \cdot n)
\leq 2^{2\cdot\sqrt{8n}} \cdot t(2/3 \cdot n)
\leq 2^{2\cdot\sqrt{8n}} \cdot L \cdot 2^{\frac{2\sqrt{2/3 \cdot 8n}}{1-\sqrt{2/3}}}
= L \cdot 2^{\frac{2\sqrt{8n}}{1-\sqrt{2/3}}}
\]
Legend

■ : Not of relevance
■ : implicitly used basics
■ : idea of proof or algorithm
■ : structure of proof or algorithm
■ : Full knowledge