Contents I

1 Introduction
   - Recall and Motivation
   - First Results

2 Simple Graphs
   - Lines
   - Trees
   - Graphs with Bridges

3 Networks
   - Cycles
   - HQ
   - Hypercube

4 Complexity
   - CCC and BF

5 Telephone-Mode
   - Even Number of Nodes
   - Odd Number of Nodes

6 Telegraph-Mode
   - Upper Bound
   - Lower Bound

7 Summary
   - Telefon-Mode
   - Telegraph-Mode
Recall

**Definition (Gossip):**

Given is $G = (V, E)$.

- Each node $w \in V$ has some information $I(w)$ and no node of $V \setminus \{w\}$ knows $I(w)$.
- Construct algorithm, where each node $v \in V$ collects information $\bigcup_{w \in V} I(w)$.

- By $comm(A)$ we denote the complexity (number of rounds) of a communication-algorithm.
- $r(G) = \min\{comm(A) \mid A \text{ is a one-way algorithm for the gossip-problem on } G\}$
- $r_2(G) = \min\{comm(A) \mid A \text{ is a two-way algorithm for the gossip-problem on } G\}$
Motivation

- Broadcast is a part of gossip.
- Many broadcasts have to “cooperate”. This makes the problem interesting.
- More important for algorithms on networks.
- Example: Distribute lower bounds for “Branch and Bound”.
- For gossip we get a difference between telegraph- and telephone-mode.
- We start with gossiping in the telephone-mode.
Lemma:

Let $G = (V,E)$ a graph with $n$ nodes. Then we have:

$$r(G) \geq r_2(G) \geq \begin{cases} \lceil \log_2 n \rceil & n \text{ even}, \\ \lceil \log_2 n \rceil + 1 & n \text{ odd}. \end{cases}$$

Proof: Only the case, where $n$ is odd, has to be proven.

- **Show:** $r_2(G) \geq \lceil \log_2 n \rceil + 1$.

- Let $A$ be a communication-algorithm for the gossip-problem.
  $A$ has communication rounds (matchings) $E_1, E_2, \cdots, E_k$.

- **Show by induction:** After $i$ rounds has each node at most $2^i$ pieces of information.
  
  - $i = 0$: Each node has $2^0 = 1$ pieces of information.
  - $i - 1 \rightarrow i$: at most $2^{i-1} + 2^{i-1} = 2^i$ pieces of information may be collected by any node.

- In round $k$ is at least one node $v$ inactive.

- $v$ has after $k$ rounds at most $2^{k-1}$ pieces of information.
Lemma:
For any graph $G = (V, E)$ with $|V| = n$ we have:

- $r(G) \leq 2n - 2$, and
- $r_2(G) \leq 2n - 3$.

Proof: Follows from the following known statements:

- $\min b(G) \leq n - 1$ for any graph $G = (V, E)$ with $|V| = n$.
- $r(G) \leq 2 \cdot \min b(G)$
- $r_2(G) \leq 2 \cdot \min b(G) - 1$
Simple Algorithm (Continuation)

Lemma:

We have:
- \( r(T_k(1)) = 2k \)
- \( r_2(T_k(1)) = 2k - 1 \)

Proof:

- Show: \( r(T_k(1)) \geq 2k \).
- \( r(T_k(1)) \) has one root and \( k \) leaves.
- The maximal matching is 1.
- In each round is only one leaf active.
- Each leaf has to send at least once.
- Each leaf has to receive at least once.
- Thus in total \( 2k \) rounds necessary.
- \( r_2(T_k(1)) \geq 2k - 1 \), is a simple exercise.
Gossip on Lines

Theorem:

We have:

- \( r_2(L(n)) = n - 1 \) for any even number \( n \geq 2 \),
- \( r_2(L(n)) = n \) for any odd number \( n \geq 3 \),
- \( r(L(n)) = n \) for any even number \( n \geq 2 \) and
- \( r(L(n)) = n + 1 \) for any odd number \( n \geq 3 \).

Proof:

- Show: \( r_2(L(n)) \geq n - 1 \).
- Note: \( r_2(L(n)) \geq b(L(n)) \geq diam(L(n)) = n - 1 \).
Gossip on Lines (Proof I)

- Show: $r_2(L(n)) \leq n - 1$ for $n$ even.

- Consider algorithm $A$, given by the following matchings:

  1. $\{\{0, 1\}, \{n - 1, n - 2\}\}$,
  2. $\{\{1, 2\}, \{n - 2, n - 3\}\}$,
  3. $\{\{2, 3\}, \{n - 3, n - 4\}\}$,
  4. $\ldots$
  5. $\{\{n/2 - 1, n/2\}\}$
  6. $\ldots$
  7. $\{\{2, 3\}, \{n - 3, n - 4\}\}$,
  8. $\{\{1, 2\}, \{n - 2, n - 3\}\}$,
  9. $\{\{0, 1\}, \{n - 1, n - 2\}\}$
Gossip on Lines (Proof II)

- Show: $r_2(L(n)) \leq n$ for $n$ odd.

- Consider algorithm $A$, given by the following matchings:

1. $\{0, 1\}$
2. $\{1, 2\}, \{n - 1, n - 2\}$
3. $\{2, 3\}, \{n - 2, n - 3\}$
4. \ldots
5. $\{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$
6. \ldots
7. $\{2, 3\}, \{n - 2, n - 3\}$
8. $\{1, 2\}, \{n - 1, n - 2\}$
9. $\{0, 1\}$

\[
\begin{align*}
r_2(L(n)) &= n - 1 \quad (n \equiv 0 \pmod{2}) \\
r_2(L(n)) &= n \quad (n \equiv 1 \pmod{2}) \\
r(L(n)) &= n \quad (n \equiv 0 \pmod{2}) \\
r(L(n)) &= n + 1 \quad (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on Lines (Proof II)

- Show: \( r_2(L(n)) \geq n \) for \( n \) odd.
- Consider the flow of messages from the left to the right node.
- These could not be forwarded without delay.
- Because we would get a time-conflict in the center.
- Thus at least one message has to be delayed.
- This provides the lower bound.

\[
\begin{align*}
r_2(L(n)) &= n - 1 \quad (n \equiv 0 \pmod{2}) \\
r_2(L(n)) &= n \quad (n \equiv 1 \pmod{2}) \\
r(L(n)) &= n \quad (n \equiv 0 \pmod{2}) \\
r(L(n)) &= n + 1 \quad (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on Lines (Proof III)

- Show: \( r(L(n)) \leq n \) for \( n \) even.

- Consider algorithm \( A \), given by the following matchings:

  1. \( \{(0, 1), (n - 1, n - 2)\} \),
  2. \( \{(1, 2), (n - 2, n - 3)\} \),
  3. \( \{(2, 3), (n - 3, n - 4)\} \),
  4. \( \ldots \)
  5. \( \{(n/2 - 1, n/2)\} \)
  6. \( \{(n/2, n/2 - 1)\} \)
  7. \( \ldots \)
  8. \( \{(3, 2), (n - 4, n - 3)\} \),
  9. \( \{(2, 1), (n - 3, n - 2)\} \),
  10. \( \{(1, 0), (n - 2, n - 1)\} \)

\[
\begin{align*}
r_2(L(n)) &= n - 1 \quad (n \equiv 0 \pmod{2}) \\
r_2(L(n)) &= n \quad (n \equiv 1 \pmod{2}) \\
r(L(n)) &= n \quad (n \equiv 0 \pmod{2}) \\
r(L(n)) &= n + 1 \quad (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on Lines (Proof IV)

- Show: \( r(L(n)) \geq n \) for \( n \) even.
- The proof is similar to the above one:
  - Consider the flow of messages from the left to the right node.
  - These could not be forwarded without delay.
  - Because we would get a time-conflict in the center.
  - Thus at least one messages has to be delayed.
  - This provides the lower bound.

\[
\begin{align*}
r_2(L(n)) &= n - 1 \quad (n \equiv 0 \pmod{2}) \\
r_2(L(n)) &= n \quad (n \equiv 1 \pmod{2}) \\
r(L(n)) &= n \quad (n \equiv 0 \pmod{2}) \\
r(L(n)) &= n + 1 \quad (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on Lines (Proof V)

- Show: \( r(L(n)) \leq n + 1 \) for \( n \) odd.

- Consider algorithm \( A \), given by the following matchings:

\[
\begin{align*}
1 & \quad \{(0, 1)\}, \\
2 & \quad \{(1, 2), (n - 1, n - 2)\}, \\
3 & \quad \{(2, 3), (n - 2, n - 3)\}, \\
4 & \quad \ldots \\
5 & \quad \{([n/2], [n/2])\} \\
6 & \quad \{([n/2], [n/2])\} \\
7 & \quad \ldots \\
8 & \quad \{(3, 2), (n - 3, n - 2)\}, \\
9 & \quad \{(2, 1), (n - 2, n - 1)\}, \\
10 & \quad \{(1, 0)\}
\end{align*}
\]
Gossip on Lines (Proof VI)

Show: \( r(L(n)) \geq n + 1 \) for \( n \) odd.

The proof is similar to the above one:

Consider the flow of messages from the left to the right node.

These could not be forwarded without delay.

Because we would get a time-conflict in the center.

Thus at least one messages (w.l.o.g. the right) has to be delayed.

Now the right message has to move, because otherwise we would have already a delay of two.

But now we still do get a further delay.

Thus we have proven the lower bound.
Gossip on arbitrary Trees

Lemma:
For any tree $T$ we have:
- $r(T) = 2 \cdot \min b(T)$
- $r_2(T) = 2 \cdot \min b(T) - 1$

Idea of the proof:
- We have already for any graph $G$: $r(G) \leq 2 \cdot \min b(G)$.
- We have to show: $r(G) \geq 2 \cdot \min b(G)$.
- Let $W = \bigcup_{v \in V} l(v)$ be the total information.
- Let $A$ be any communication algorithm on $T$.
- Let $t$ be the point in time, when some node knows $W$.
- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
Gossip on arbitrary Trees (Proof I)

1. Let \( u \neq v \) be an other node which knows \( W \) after \( t \) steps.
2. Let \( (u, y_1, y_2, \ldots, y_k, v) \) be the unique path connecting \( u \) and \( v \).
3. If \( v \) sends to \( y_k \) at time \( t \), then \( v \) did know \( W \) at time \( t - 1 \).
4. So we have to consider the case: \( y_k \) sends to \( v \) at time \( t \):
   - In this case \( y_k \) sends \( v \) some missing information.
   - \( y_k \) knows at time \( t - 1 \) the full information, which has to be send from \( y_k \) to \( v \).
   - The information, which has to be send from \( v \) to \( y_k \), is already send.
   - Then the node \( y_k \) know \( W \) at time \( t - 1 \).
5. Contradiction, the node \( u \) does not exist.
6. Thus we have: \( t \geq \minb(T) = b(v, T) \).
Gossip on arbitrary Trees (Proof II)

- Consider the situation at node \( v \) after round \( t \).
- Let w.l.o.g. \( v \) be the root of \( T \).
- Let \( v_1, v_2, \ldots, v_k \) be the successors of \( v \).
- Let \( T_1, T_2, \ldots, T_k \) be the subtrees with roots \( v_1, v_2, \ldots, v_k \).
- In each subtree \( T_i \) is some information \( w_i \) missing.
- Only the node \( v \) knows \( \bigcup_{j=1}^{k} w_j \).
- Thus there are \( b(v, T) \) steps to be done.
- We finally have \( r(T) \geq \min b(T) + b(v, T) \geq 2 \cdot \min b(T) \)
Gossip on arbitrary Trees (Proof III)

- Consider the two-way mode: by a similar way we may prove:
- At time $t$ only two neighbours nodes $u$ and $v$ know the total information. We get in the similar way the second statement.
Lemma:

For all $m \geq 1$ and $k \geq 2$ we have:

- $r(T_k(m)) = 2 \min b(T_k(m)) = 2 \cdot k \cdot m.$
- $r_2(T_k(m)) = 2 \min b(T_k(m)) - 1 = 2 \cdot k \cdot m - 1.$
Graphs with Bridges

Lemma:

Let \( G = (V, E) \) be a graph with bridge \( e \in E \), which is separated by \( e \) in components \( G_1 \) and \( G_2 \), then we have

\[
r(G) \geq \min b(G) + 1 + \min \{\min b(G_1), \min b(G_2)\}
\]

Proof: Let \( W = \bigcup_{v \in V} I(v) \) be the total information.

Let \( t \geq \min b(G) \) the time, when a node \( w \) knows \( W \).

- If \( w \in G_1 \) hold, then do no node from \( G_2 \) know \( W \).
- Then there are still \( 1 + \min b(G_2) \) steps to do.
- If \( w \in G_2 \) hold, then do no node from \( G_1 \) know \( W \).
- Then there are still \( 1 + \min b(G_1) \) steps to do.
- Thus we have: \( r(G) \geq \min b(G) + 1 + \min \{\min b(G_1), \min b(G_2)\} \).
Lemma:

Let $G = (V, E)$ be a graph with bridge $e \in E$, which is separated by $e$ in components $G_1$ and $G_2$, then we have:

$$r_2(G) \geq \min b(G) + \min\{\min b(G_1), \min b(G_2)\}$$

Proof: Let $t \geq \min b(G)$ be the time, when node $w$ knows $W$ the first time. As before we may prove:

- Let $i \in \{1, 2\}$. If $w \in G_i$ and $v_{3-i}$ does not know $W$, then no node from $G_{3-i}$ knows $W$. There are still $1 + \min b(G_{3-i})$ steps to do.

- If $v_1$ and $v_2$ know $W$ at time $t$, then no other node knows $W$. There are still $\min\{\min b(G_1), \min b(G_2)\}$ steps to do.

Thus we have: $r_2(G) \geq \min b(G) + \min\{\min b(G_1), \min b(G_2)\}$. 
Theorem:
We have:
- \( r_2(C(k)) = \frac{k}{2} \) for even \( k \).
- \( r_2(C(k)) = \lceil \frac{k}{2} \rceil + 1 \) for odd \( k \).

Idea of the proof (\( k \) even): [\( k \) odd: an easy exercise]
- Let \( k \) be even.
- \( r_2(C(k)) \geq \frac{k}{2} \) results by the diameter.
- \( r_2(C(k)) \leq \frac{k}{2} \) is true by the following algorithm:
  1. \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i+1\}, \ldots, \{n - 2, n - 1\}\}
  2. \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\}\}
  3. \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i+1\}, \ldots, \{n - 2, n - 1\}\}
  4. \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\}\}
  5. \ldots

- Note: After \( i \) rounds knows each node \( 2 \cdot i \) Informationen.
1-Way Gossip on Cycles (Idea)

- Messages should traverse in both directions.
- Activate each $f(n)$-th node on the cycle.
- This will result in an additional $\Theta(f(n))$ steps.
- During the distribution we get $\Theta\left(\frac{n}{2 \cdot f(n)}\right)$ delays.
- Thus we will choose $f(n) = \Theta(\sqrt{n})$.
- By this idea we may get a lower and upper bound.
Gossip on Cycles (Idea)
Gossip on Cycles (Idea of the algorithm)

- Split the cycle in $\Theta(\sqrt{n})$ blocks $B_i$.
- Within block $B_i$ ($i \in \{1, 2, 3, \cdots, k\}$ with $k \in \Theta(\sqrt{n})$) do the following:
  - Phase 1:
    - The nodes $v_i$ [$u_i$] start a “wave” to the left [right].
    - The messages of $v_i$ and $u_i$ are delayed $\Theta(\sqrt{n})$ times by the other messages.
    - After $n/2 + \Theta(\sqrt{n})$ round know nodes $z_i$ the total information.
  - Phase 2:
    - Each node $z_i$ distributes the total information to $\Theta(\sqrt{n})$ nodes.
- Note: If $n$ is even, we have always a delay of one and the synchronization is easy.
Gossip on Cycles (Idea)

Theorem:
We have:
- \( r(C(n)) \leq \frac{n}{2} + \sqrt{2n} - 1 \) for even \( n \).
- \( r(C(n)) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lceil 2 \cdot \sqrt{\left\lceil \frac{n}{2} \right\rceil} \right\rceil - 1 \) for odd \( n \).
- \( r(C(n)) \geq \frac{n}{2} + \sqrt{2n} - 1 \) for even \( n \).
- \( r(C(n)) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \sqrt{2n} - 1/2 \right\rceil - 1 \) for odd \( n \). 

Proof: See literature.
Gossip on the Hypercube

Theorem:
For all $m \in \mathbb{N}$ we have: $r_2(HQ(m)) = m$

Proof:

- The lower bound is the diameter.
- Upper bound by the following algorithm:
  
  for $i = 1$ to $m$ do
  
  for all $a_1, a_2, \ldots, a_{m-1} \in \{0, 1\}$ do in parallel
  
  $a_1 a_2 \cdots a_{i-1} 0 a_i a_{i+1} \cdots a_{m-1}$ sends to
  
  $a_1 a_2 \cdots a_{i-1} 1 a_i a_{i+1} \cdots a_{m-1}$

Corollary:
For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
Consider one-way mode:

- Start with the first phase of the gossip-algorithm for cycles on all cycles.
- Then each $\Theta(\sqrt{n})$-th node on each cycle knows the total information of its cycles.
- In $\Theta(\sqrt{n})$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each $\Theta(\sqrt{n})$-th node of each cycle the total information.
- The final part is the second phase of the gossip-algorithm of cycles on all cycles.
- All nodes know now the total information.
Consider two-way mode:

- Start with the gossip algorithm for cycles on all cycles.
- Each node of the cycle knows now the total information of its cycle.
- In $\Theta(n/2)$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each node the total information.
Theorem:

Let $k \geq 3$, then we have:

1. $r(\text{CCC}(k)) \leq r(\text{C}(k)) + 3k - 1 \leq \left\lceil \frac{7k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 2$.
2. $r(\text{BF}(k)) \leq r(\text{C}(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 1$.
3. $r_2(\text{CCC}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.
4. $r_2(\text{CCC}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$.
5. $r_2(\text{BF}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.
6. $r_2(\text{BF}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$. 
Definition:

The two-way gossip-problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r_2(G) \leq k$ hold.

Definition:

The one-way gossip-problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r(G) \leq k$ hold.
Theorem:
The two-way and one-way gossip-problem on trees is in $\mathcal{P}$

Proof: simple exercise.

Theorem:
The two-way and one-way gossip-problem is in $\mathcal{NPC}$

Proof: Same way as the for the broadcast-problem.
Gossip on Graphs with $2 \cdot m$ Nodes (0. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes (1. Idea)

Implication:

- For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
- For all $m \in \mathbb{N}$ we have: $r_2(K(m)) \leq \lceil \log m \rceil + 1$
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)

- Too many nodes where inactive for too long time.
- These nodes could not double their information.
- Idea: Try to double the information of any node.
- Detailed idea: In each step each node has an “interval” of information.
- To make the doubling easy split the nodes into two groups.
- Both groups should be the same size.
- In the first step pairs of node from each group share their information.
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes

**Theorem:**

For all $m \in \mathbb{N}$ we have: $r_2(K(2m)) = \lceil \log_2 m \rceil$

**Proof:** Split the nodes in groups $Q[i]$ and $R[i]$ ($0 \leq i \leq m - 1$).

- **algorithm:**
  
  for all $i \in \{0, \ldots, m - 1\}$ do in parallel
  
  Exchange the information between $Q[i]$ and $R[i]$

  for $t = 1$ to $\lceil \log_2 m \rceil$ do
    
    for all $i \in \{0, \ldots, m - 1\}$ do in parallel
      
      Exchange the information between $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$

- **Invariant:**

  - Let $\alpha[i]$ be the information of $Q[i]$ and $R[i]$ after their initial exchange.
  - After round $t$ know nodes $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$: $\bigcup_{0 \leq j \leq 2^t - 1} \alpha[(i + j) \mod m]$

  The invariant is easy to be shown.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (a try)

- We need an extra round.
- A nice proof with this idea will become complicated.
- We will try to put some structure into the proof.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (Idea)

- How could this be an idea?
- We only have the edges of the first step.
- Idea: We could now choose a small even number of Nodes, which together have the total information.
- These nodes may perform the above gossip algorithm.
- In the last step we repeat the first round.
Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be all nodes.
- For all $i \in \{0, 1, \ldots, m-1\}$ the node $v_{m+2+i}$ sends to $v_i$.
- The node $\{v_0, v_1, v_2, \ldots, v_m\}$ have now the total information.
- If $m + 1$ is even, perform a gossip on the nodes $\{v_0, v_1, v_2, \ldots, v_m\}$.
- If $m + 1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \ldots, v_{m+1}\}$.
- For all $i \in \{0, 1, \ldots, m-1\}$ the nodes $v_i$ send to $v_{m+2+i}$.
- Correctness follows direct by the construction.

Running time for $m + 1$ even:

$$r_2(K(m + 1)) + 2 = \lfloor \log_2(m + 1) \rfloor + 2 = \lfloor \log_2 \left( \frac{n+1}{2} \right) \rfloor + 2$$

$$= \lfloor \log_2(n + 1) \rfloor + 1 = \lfloor \log_2 n \rfloor + 1$$

Running time for $m + 1$ odd:

$$r_2(K(m + 2)) + 2 = \lfloor \log_2(m + 2) \rfloor + 2 = \lfloor \log_2 \left( \frac{n+3}{2} \right) \rfloor + 2$$

$$= \lfloor \log_2(n + 3) \rfloor + 1 = \lfloor \log_2 n \rfloor + 1$$
1st Idea (Let the Knowledge grow)

We need more rounds.

A nice proof with this idea will become complicated.

We will try to put some structure into the proof.
2\textsuperscript{nd} Idea (Let the Knowledge grow in a structured way)

- We need an additional two rounds.
- $v_x$ and $w_y$ alternate as sender and receiver.
- The information grows in blocks (intervals) in the nodes.
- With this idea we may do the proof.
- Only the first two rounds are special.
2\textsuperscript{nd} Idea (Let the Knowledge grow in a structured way)

- After the first two rounds some node-pairs share their information.

- Consider this situation as the start:
  - All $v_x$ and $w_x$ have one information pair.
  - $v_i$ sends to $w_j$ and the $w_x$ have 2 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 3 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 5 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 8 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 13 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 21 information pairs.
  - Thus the grow-rate and the algorithm is clearly visible.
Let $n = 2m$.

Gossip-Algorithm:

- $t := 0$;
- for all $i \in \{0, \ldots, m - 1\}$ do in parallel $R[i]$ sends to $Q[i]$;
- for all $i \in \{0, \ldots, m - 1\}$ do in parallel $Q[i]$ sends to $R[i]$;
- while $\text{fib}(2t + 1) < m$ do begin
  - $t := t + 1$;
  - for all $i \in \{0, \ldots, m - 1\}$ do in parallel
    - $R[(i + \text{fib}(2t - 1)) \mod m]$ sends to $Q[i]$;
  - if $\text{fib}(2t) < m$ then
    - for all $i \in \{0, \ldots, m - 1\}$ do in parallel
      - $Q[(i + \text{fib}(2t)) \mod m]$ sends to $R[i]$
  end;

$\text{fib}(0) = \text{fib}(1) = 1$

$\text{fib}(i) = \text{fib}(i - 1) + \text{fib}(i - 2)$
Theorem:
Let $n = 2m$ and $k = \min\{x \mid \text{fib}(x) \geq m\}$. Then we have $r(K(n)) \leq k + 1$.

Proof:

- The algorithm stops, if $\text{fib}(2t + 1) \geq m$ or $\text{fib}(2t) \geq m$ holds.
- The number of rounds within the loop is $2t$ or $2(t - 1) + 1$.
- The total number of rounds is $(k - 1) + 2$.
- Correctness may be proven by the following invariant:
- Let $a[i]$ be the information, which share $R[i]$ and $Q[i]$ after two rounds.
- After $t$ loops we have:
  - $Q[i]$ knows $\bigcup_{0 \leq j \leq \text{fib}(2t+1)-1} \alpha[(i + j) \mod m]$
  - $R[i]$ knows $\bigcup_{0 \leq j \leq \text{fib}(2t+2)-1} \alpha[(i + j) \mod m]$
- The correctness is a direct result of this.

$\text{fib}(0) = \text{fib}(1) = 1$
$\text{fib}(i) = \text{fib}(i - 1) + \text{fib}(i - 2)$
One-Way-Gossip

Theorem:
Let \( n = 2m - 1 \) and \( k = \min\{x \mid \text{fib}(x) \geq m\} \). Then we have \( r(K(n)) \leq k + 2 \).

Proof: Using the same idea as for the two-way mode.

Theorem:
Let \( n \) even. Then we have: \( r(K(n)) \geq 2 + \lceil \log_{\frac{1}{2}(1+\sqrt{5})} \frac{n}{2} \rceil \).

Proof: See literature (Idea is given the following).

<table>
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<th>6</th>
<th>8</th>
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<td>7</td>
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</tr>
<tr>
<td>Lower Bound</td>
<td>2</td>
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<td>5</td>
<td>5</td>
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<td>6</td>
<td>7</td>
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<td>7</td>
</tr>
</tbody>
</table>
Idea for the lower Bound

- Situation:
  - Algorithm with “fibonacci growth”.
  - No idea to enlarge this growth.

- Construction of a lower bound:
  - Start with an arbitrary algorithm.
  - Use only the restriction of the algorithm.
  - Abstract.

- We will now try to do the abstraction.

- Try the get the core-problem.

- The core-problem ist:
  - “Fibonacci growth” could not be improved.
1. Abstraction

Definition:

The Network Counting Problem:
- Given a directed graph $G = (V, E)$.
- Each node stores a number.
- Initial just the number 1 is stored.
- The receiver add the number from the sender to his number after one communication.
- The objective is: all nodes should store a number larger then $|V|$.
- With $nc(G)$ we denote the minimal rounds to achieve this objective.

Lemma:

For any graph $G$ we have: $r(G) \geq nc(G)$. 
2. Abstraction

- Let $G = (\{v_1, v_2, v_3, \cdots, v_n\}, E)$ be a directed Graph.
- Each node $v_i$ stores after $t$ rounds the number $z_i^t$.
- One situation of the network counting problem could be described by a vector:
  - Initial: $(1, 1, 1, \cdots, 1)^T$.
  - After $t$ rounds: $(z_1^t, z_2^t, z_3^t, z_n^t)^T$.
- One round of an algorithm for the network counting problem is given by a matrix $B$:
  - $A$ is a $n \times n$ matrix.
  - $a_{ij} = 1$ node $j$ sends to node $i$.
  - $A$ contains on the diagonal only ones.
  - $A$ has in each row at most two ones.
  - $A$ has in each column at most two ones.
  - If $a_{ij} = a_{kl} = 1$ ($i \neq j \neq k \neq l$), then we have $l \neq i \neq k$ and $l \neq j \neq k$.
  - Thus we get: $A \cdot (z_1^t, z_2^t, z_3^t, z_n^t)^T = (z_1^{t+1}, z_2^{t+1}, z_3^{t+1}, z_n^{t+1})^T$.
2. Abstraction (Continuation)

- We consider now matrices of the above form.
- These are matrices $A$, for which there is a transformation $T$ with:

$$TAT^{-1} = \begin{pmatrix}
B & 0 \\
B & B \\
& & 1 \\
& & & 1
\end{pmatrix}.$$

and $B = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}$.

- We will estimate the growth, which these matrices provide for the network counting problem.
Recollection (Norm, 3. Abstraction)

- Let \( \|..\| \) be the vector norm over \( \mathbb{R}^n \). Then we have:
  - \( \|x\| = 0 \iff x = 0^n \),
  - \( \|\alpha \cdot x\| = |\alpha| \cdot \|x\| \),
  - \( \|x + y\| \leq \|x\| + \|y\| \)
  - this holds for all \( \alpha \in \mathbb{R}, x, y \in \mathbb{R}^n \)

- The matrix norm for a vector norm \( \|..\| \) is defined by \( \|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \).
  Then we have:
  - \( \|A\| = 0 \iff A = 0 \)
  - \( \|A + B\| \leq \|A\| + \|B\| \)
  - \( \|\alpha A\| = \alpha \cdot \|A\| \)
  - \( \|A \cdot B\| \leq \|A\| \cdot \|B\| \)
  - \( \|A \cdot x\| \leq \|A\| \cdot \|x\| \)
  - this holds for all \( A, B \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \alpha \geq 0. \)

- Here we use: \( \|x\| = \sqrt{\sum_{i=1}^n |x_i|^2} \) for ein \( x = (x_1, .., x_n) \),

- Known: \( \|A\| = \text{Spectral Norm}(A) = \sqrt{|\lambda_{\text{max}}(A^T \cdot A)|} \) with: \( \lambda_{\text{max}} \) is the largest Eigenvalue.
2. Abstraction (Continuation)

- We compute the spectral norm:
  - $\|A\| = \|TAT^{-1}\| = \|B\|$.  
  - $B^T \cdot B = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \begin{pmatrix} 11 \\ 01 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix}$. 
  - $\Rightarrow (2 - \lambda)(1 - \lambda) - 1 = 0$ 
  - $\Rightarrow \lambda^2 - 3\lambda + 1 = 0$ 
  - $\Rightarrow \lambda_{\text{max}}(B^TB) = \frac{3}{2} + \sqrt{\frac{5}{4}}$ 
  - $\|A\| = \sqrt{\lambda_{\text{max}}(A^TA)} = \frac{1}{2}(1 + \sqrt{5})$
Theorem:

A algorithm, solving the network counting problem needs $2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) \frac{n}{2} \rceil$ rounds.

Proof:

- Let $A_j$, $1 \leq j \leq r$ be matrices, which solve the problem in $r$ rounds.
- $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdot \cdots \cdot A_2 \cdot A_1 \cdot (1, 1, \cdots, 1)$.
- $||\alpha|| \leq \left( \prod_{i=1}^{r-2} ||A_i|| \right) \cdot ||(1, \ldots, 1)|| \leq \left( \frac{1}{2}(1 + \sqrt{5}) \right)^{r-2} \cdot \sqrt{n}$
- Let $inf(i, t)$ be the number, which have the nodes $v_i$ after $t$ rounds.
- After round $t$ we have: $inf(i, t) \geq n$ for all $i \in \{1, 2, \cdots, n\}$.
- After round $t-1$ we have: $inf(i, t-1) \geq n$ for at least $n/2$ nodes.
- There could be some $i$ with: $inf(i, t-2) \geq n$.
- But if $\alpha_i < n$ and $inf(i, t-1) \geq n$, then there exists $j$ with: $\alpha_i + \alpha_j \geq n$. 
Continuation

\[ \alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdots A_2 \cdot A_1 \cdot (1, 1, \cdots, 1) \]

- Let
  - \( c_1 \) be the number of cases with: \( \alpha_i \geq n \),
  - \( c_2 \) be the number of cases with: \( \alpha_i < n \) and \( \alpha_j \geq n \),
  - \( c_3 \) be the number of cases with: \( \alpha_i < n, \alpha_j < n \) and \( \alpha_i + \alpha_j \geq n \).

- Then we have: \( c_1 \geq c_2 \) and \( c_1 + c_2 + c_3 \geq n/2 \).
- Thus we also get: \( 2c_1 + c_3 \geq \frac{n}{2} \)
- \( \|\alpha\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \sqrt{c_1 n^2 + c_3 \cdot 2 \cdot \frac{n^2}{4}} \geq n \cdot \sqrt{\frac{1}{2} (2c_1 + c_3)} \geq \frac{n}{2} \sqrt{n} \).
- We already have:
  \( \|\alpha\| \leq (\prod_{i=1}^{r-2} \|A_i\|) \cdot \|(1, \ldots, 1)\| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n} \).
- And we get:
  \( \frac{n}{2} \cdot \sqrt{n} \leq \|\alpha\| \leq \Phi^{r-2} \cdot \sqrt{n} \),
- From which we conclude:
  \( r \geq 2 + \left\lceil \log_{\frac{1}{2} (1 + \sqrt{5})} \frac{n}{2} \right\rceil \).
Quality of these Bounds

Lemma:

Let \( n = 2m \) and let:

- \( t_1 := 1 + k \), with \( k \) is the smallest number with \( m \leq F(k) \) and
- \( t_2 := 2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) m \rceil \).

Then we have \( t_1 = t_2 \) for infinite many \( m \) and \( t_1 \leq t_2 + 1 \) for all \( m \).

Proof:

- Let \( \Phi = \frac{1}{2}(1 + \sqrt{5}) \).
- Then we have: \( \Phi^2 = \Phi + 1 \).
- Furthermore we have \( \Phi^{i-2} \leq F(i) \leq \Phi^{i-1} \) for all \( i \geq 2 \).
- Consider \( n \in \mathbb{N} \) with: \( n = 2 \cdot F(k) \) for some \( k \).
  - Then we have: \( t_1 = k + 1 \) and
    \( t_2 = 2 + \lceil \log_\Phi F(k) \rceil = 2 + k - 1 = k + 1 \).
  - From which we get: \( t_1 = t_2 \) for these \( n \).
Lemma:

Let $n = 2m$ and let:

- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Setze $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
- Then we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Let $n = 2 \cdot m$ arbitrary.
  - Let $i$ be defined by: $\Phi^{i-1} < m \leq \Phi^i$, then we have: $t_2 = 2 + i$.
  - Let $k$ be the smallest number with $F(k) \geq m$.
  - Note: $\Phi^{k-2} \leq F(k) \leq \Phi^{k-1}$.
  - Then we have: $i = k - 1$ oder $i = k - 2$.
  - From which we conclude: $t_1 = k + 1 \leq i + 3$. 
## Summary (Telefon-Mode)

| Graph   | $|V|$ | diam | Lower Bound                  | Upper Bound                  |
|---------|------|------|------------------------------|------------------------------|
| $K_n$   | $n$  | $1$  | $\lceil \log_2 n \rceil + \text{odd}(n)$ | $\lceil \log_2 n \rceil + \text{odd}(n)$ |
| $H_k$   | $2^k$| $k$  | $n - \text{even}(n)$        | $n - \text{even}(n)$        |
| $P_n$   | $n$  | $n - 1$ | $\left\lceil \frac{n}{2} \right\rceil + \text{odd}(n)$ | $\left\lceil \frac{n}{2} \right\rceil + \text{odd}(n)$ |
| $C_n$   | $n$  | $\left\lfloor \frac{n}{2} \right\rfloor$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2, k \text{ even}$ |
| $CCC_k$ | $k \cdot 2^k$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2, k \text{ odd}$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2, k \text{ even}$ |
| $SE_k$  | $2^k$ | $2k - 1$ | $2k - 1$ | $2k + 5$ |
| $BF_k$  | $k \cdot 2^k$ | $\left\lfloor \frac{3k}{2} \right\rfloor$ | $1.9770k$ | $2.25 \cdot k + o(k)$ |
| $DB_k$  | $2^k$ | $k$ | $1.5965k$ | $2k + 5$ |
### Summary (Telegraph-Mode)

| Graph | $|V|$ | diam | Lower Bound | Upper Bound |
|-------|------|------|-------------|-------------|
| $K_n$ | $n$  | 1    | $1.44 \log_2 n$ | $1.44 \log_2 n$ |
| $H_k$ | $2^k$ | $k$  | $1.44k$ | $1.88k$ |
| $P_n$ | $n$  | $n-1$ | $n + \text{odd}(n)$ | $n + \text{odd}(n)$ |
| $C_n$ | $n$ even | $\left\lfloor \frac{n}{2} \right\rfloor$ | $\frac{n}{2} + \left\lceil \sqrt{2n} \right\rceil - 1$ | $\frac{n}{2} + \left\lceil \sqrt{2n} \right\rceil - 1$ |
|       | $n$ odd | $\left\lceil \frac{n}{2} \right\rceil$ | $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil - 1$ | $\left\lfloor \frac{n}{2} \right\rfloor + 2\sqrt{\left\lceil \frac{n}{2} \right\rceil} - 1$ |
| $CCC_k$ | $k \cdot 2^k$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lceil \frac{7k}{2} \right\rceil + 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} - 2$ |
| $SE_k$ | $2^k$ | $2k-1$ | $2k-1$ | $3k+3$ |
| $BF_k$ | $k \cdot 2^k$ | $\left\lfloor \frac{3k}{2} \right\rfloor$ | $1.9770k$ | $\left\lfloor \frac{5k}{2} \right\rfloor + 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} - 1$ |
| $DB_k$ | $2^k$ | $k$ | $1.5965k$ | $3k+3$ |
J. Hromkovič, et al.:
Dissemination of Information in Communication Networks:
Broadcasting, Gossiping, Leader Election, and Fault-Tolerance.