Contents I

1. Basic Definitions
   - Graphs
   - Special Graphs

2. Connectivity of Graphs
   - Definition
   - Statements
   - Directed Graph

3. Flows
   - Introduction

4. Matchings
   - Definitions
   - Matching on Bipartite Graphs
   - Applications
   - Probleme

5. Factors of Graphs
   - Introduction
   - Statements

6. Posets
   - Definition and Statements
Definition: Graph

Definition (Undirected Graph)

- Let $V(G) = \{v_1, ..., v_n\}$ be a non-empty set of nodes and
- $E(G)$ be a set or multiset of pairs from $V(G)$ (set of edges).
- The sets $V(G)$ and $E(G)$ define the graph $G = (V(G), E(G))$.
- If $G$ is uniquely determined, then we just write: $V$ and $E$.
- Or in other words $G = (V, E)$.
- We always use as default writing: $n = |V|$ and $m = |E|$.
Way of Speaking for Graphs

Definition (Way of Speaking)

- Let $G = (V(G), E(G))$ and $e = (v, w) \in E(G)$.
- The nodes $v, w$ are called connected (adjacent) by an edge $e$.
- An edge $e$ is called loop, if $v = w$ holds.
- Two edges are called parallel, if they are the same.
- A graph without parallel edges is called simple.

As long as we do not state differently we will use in the following simple graph without loops.
Degree of a Node

Definition (Degree of a Node)

- Let $v \in V(G)$.
- With
  \[
  \deg(v) = |\{ e \in E(G) \mid e = (v, v'), v' \in V(G) \setminus \{v\}\}|
  \]
  we denote the degree of a Node (degree) of $v$.

- $\deg(v_0) = 4$.
- $\deg(v_1) = 3$.
- $\deg(v_4) = 6$.
- $\deg(v_5) = 6$. 
Handshake Theorem

**Theorem**

\[ \sum_{v \in V(G)} \deg(v) = 2|E(G)|. \]

**Proof:** Each edge connects two nodes.

**Theorem**

*The number of nodes of odd degree is even.*

**Proof:**

\[ \sum_{v \in V(G)} \deg(v) + \sum_{v \in V(G)} \deg(v) \equiv 0 \pmod{2} \]

\[ \sum_{\deg(v) \equiv 0} \deg(v) + \sum_{\deg(v) \equiv 1} \deg(v) = 2|E(G)| \]
**Definition (Regular)**

A graph $G$ is called $k$-regular, iff for all $v \in V(G)$ we have: $d(v) = k$.

**Definition (Complete)**

A graph $G$ is called complete, iff all pairs of nodes $a, b$ from $V$ holds: $(a, b) \in E$.

- Notation: $K_n$. 

---

**Regular and Complete**
Special Graphs

Definition (Bipartite)

A Graph $G$ is called bipartite, iff $V$ may be split into disjoint set $V'$, $V''$, such that each edge connects only nodes from both partitions.

- Notation: $G = (V', V'', E)$

Definition (Complete bipartite)

A Graph $G$ is called complete bipartite, iff $V$ may be split into disjoint set $V'$, $V''$, and $E = \{(a, b) \mid a \in V', b \in V''\}$.

- Notation: $K_{p,q}$ with $p = |V'|$ and $q = |V''|$.
- Star, iff $S_n = K_{1,n-1}$. 
Examples
**Subgraphs**

**Definition (Subgraph)**

- A Graph $H = (V(H), E(H))$ is called a subgraph of $G = (V(G), E(G))$,
- iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
**Subgraphs**

**Definition (node-induced subgraph)**

- A graph \( H = (V(H), E(H)) \) is a node-induced subgraph of \( G = (V(G), E(G)) \),
- iff \( V(H) \subseteq V(G) \) and \( E(H) = \{(a, b) \in E(G) \mid a, b \in V(H)\} \).
Definition (Isomorph)

- Two graphs $G$ and $H$ are called **isomorph**, if there is a bijective mapping $f : V(G) \mapsto V(H)$, such that for all $v, w \in V(G)$ hold:
  - $(v, w) \in E(G)$, iff $(f(v), f(w)) \in E(H)$. 

![Graphs](image_url)
A graph $G = (V, E)$ is called connected, iff between any two different nodes $a, b$ exists a path from $a$ to $b$. 
Definition

Let \( G = (V, E) \), \( V' \subset V \) is called a node-separator (vertex cut), iff \( G - V' \) is not connected.

Notation: \( G - V' := (V \setminus V', \{(a, b) \in E \mid a, b \in V \setminus V'\}) \)

Definition

If \( \{v\} \) is a node-separator, then \( v \) is called articulation point.

Theorem

*Only cliques \( K_n \) do not have any node-separator.*
Example
**Edge-Separator**

**Definition**

Let $G = (V, E)$. $E' \subset E$ is called edge-separator (edge cut), iff $G - E'$ is not connected.

**Notation:** $G - E' := (V, E \setminus E')$

**Definition**

If $\{v, w\}$ is an edge-separator, then $\{v, w\}$ is called a bridge.

**Theorem**

An minimal edge-separator $E'$ of $G = (V, E)$ induces a 2-partite graph. Or in other words: $G = (V, E')$ is a 2-partite graph.
Example
Connectivity

Definition
Let $G = (V, E)$ and $k$ minimal with: $\exists V' \subset V : |V'| = k$ and $G - V'$ is not connected or trivial. Then we call $G$ $k$-connected. A $k$-connected Graph is also $k - 1$-connected.

Notation: $\kappa(G) = k$

Definition
Let $G = (V, E)$ and $k$ minimal with: $\exists E' \subset E : |E'| = k$ and $G - E'$ is not connected or trivial. Then we call $G$ $k$-edge-connected. A $k$-edge-connected Graph is also $k - 1$-edge-connected.

Notation: $\lambda(G) = k$
Statements on Connectivity

Theorem

For any graph $G = (V, E)$ we have:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Notation: $\delta(G) := \min\{\deg(v) \mid v \in V\}$

Theorem

For all integer numbers $0 < a \leq b \leq c$ there are graphs $G$ with:

$$\kappa(G) = a, \quad \lambda(G) = b, \quad \delta(G) = c$$

Theorem

Let $G = (V, E)$ be a graph with: $|V| = n$ and $\delta(G) \geq n/2$. Then we have:

$$\lambda(G) = \delta(G)$$
Statements on Node-Connectivity

**Theorem**

Let $G = (V, E)$ with: $|V| = n$ and $|E| = e$. Then is the maximal connectivity (maximal $k$ with $G$ is $k$-connected) of $G$:

- $0$ falls if $e < n - 1$
- $2 \cdot e/n$ if $e \geq n - 1$

**Theorem**

Let $G = (V, E)$ connected. The following statements are equivalent:

1. $v \in V$ is a node-separator.
2. $\exists a, b \in V: a, b \neq v$: each path from $a$ to $b$ traverses via $v$.
3. $\exists A, B: A \cup B = V \setminus \{v\}$ and each path from $a \in A$ to $b \in B$ traverses via $v$. 

Statements on Edge-Connectivity

**Theorem**

Let $G = (V, E)$ be connected. The following statements are equivalent:

1. $e \in E$ is a edge-separator.
2. $e$ is not in any simple cycle of $G$.
3. $\exists a, b \in E$: each path from $a$ to $b$ traverses via $e$.
4. $\exists A, B: A \cup B = V$ and each path from $a \in A$ to $b \in B$ traverses via $e$. 
**Definition**

Let $G = (V, E)$ and $(a, b) = e \in E$. The subdivision of an edge $e$ results in graph $G = (V \cup \{v\}, E \cup \{(a, v), (v, b)\} \setminus \{e\})$.

A set of paths of $G = (V, E)$ is called intern-node-disjoint, iff no two paths share an internal-node. The internal nodes are all except the start and the end node.
Theorem

Let $G = (V, E)$ with $|V| \geq 3$. The following statements are equivalent:

1. $G$ is 2-connected.
2. Each node pair is connected by two intern-node-disjoint paths.
3. Each node pair is on a common simple cycle.
4. There exit an edge and each node together with this edge is on a common simple cycle.
5. There exit two edges and each pair of edges is on a common simple cycle.
6. For each pair of nodes $a, b$ and an edge $e$ exists a simple path from $a$ to $b$ traversing $e$.
7. For three nodes $a, b, c$ exists a path from $a$ to $b$ traversing $c$.
8. For three nodes $a, b, c$ exists a path from $a$ to $b$ avoiding $c$. 
Theorem

Let \( G = (V, E) \) \( k \)-connected. Then any \( k \) nodes are on a common simple cycle.

Notation: Let \((G = V, E)\) and \((H = W, F)\) graphs
\[ G + W = (V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\}) \]

Theorem

A graph \( G \) is 3-connected, iff \( G \) may be constructed from the wheel \( W_i = K_1 + C_i \) \((i \geq 4)\) by the following operations:

1. Adding a new edge.
2. Splitting a node of degree \( \geq 4 \) into two connected nodes of degree \( \geq 3 \).
Statements on $k$-Connectivity

Theorem (Menger’s Theorem)

$G$ is $k$-connected, iff any two node are connected by $k$ intern-node-disjoint paths.

Theorem (Menger's Theorem)

$G$ is $k$-edge-connected, iff any two node are connected by $k$ edge-disjoint paths.
Computing the Connectivity

**Theorem**

*The 1-connectivity of a graph may be computed by DFS/BFS.*

**Theorem**

*The 1-edge-connectivity of a graph may be computed by DFS/BFS.*

**Theorem**

*The 2-connectivity of a graph may be computed by DFS/BFS.*

**Theorem**

*The k-connectivity of a graph may be computed by flow algorithms.*

**Theorem**

*The k-edge-connectivity of a graph may be computed by flow algorithms.*
Definition: Graph

Definition (Directed Graph)

- Let $V(G) = \{v_1, ..., v_n\}$ be a non-empty set of nodes and
- $E(G)$ a set or multiset of pairs from $V \times V$ (set of edges).
- The sets $V(G)$ and $E(G)$ define the graph $G = (V(G), E(G))$.
- If $G$ is uniquely determined, then we just write: $V$ and $E$.
- Or in other words $G = (V, E)$.
- We always use as default writing: $n = |V|$ and $m = |E|$.
Strong Connectivity

Definition
A directed graph \( G = (V, E) \) is called strongly connected, iff for any two different nodes \( a, b \) exists a path from \( a \) to \( b \).

Theorem
The strong connectivity of a graph may be computed by DFS/BFS.
The Flow Problem

Definition (Flow)

- Let $G = (V, E)$ a directed graph with cost-function $c : E \mapsto \mathbb{N}$. Let $s, t \in V$ be the source and drain.

- A function $f : E \mapsto \mathbb{N}$ is a flow-function, iff
  - $\forall e \in E : 0 \leq f(e) \leq c(e)$
  - $\forall v \in V \setminus \{s, t\} : \sum_{e = (v, w) \in E} f(e) = \sum_{e = (w, v) \in E} f(e)$

- The value of the flow is: $\sum_{e = (s, w) \in E} f(e) - \sum_{e = (w, s) \in E} f(e)$

Definition (Maximal Flow Problem)

Given: Graph $G = (V, E)$, $s, t \in V$ and $c : E \mapsto \mathbb{N}$
Compute: Maximal flow-function $f$.

Theorem (Maximal Flow Problem)

The problem to compute the maximal flow is in $\mathcal{P}$.
Definition (Cut)

- Let $G = (V, E)$ be a directed graph with cost-function $c : E \rightarrow \mathbb{N}$
- Let $s, t \in V$ source and drain.
- $A, B \subset V$ are called a cut, iff
- $s \in A$ and $t \in B$
- $A \cap B = \emptyset$ and $A \cup B = V$
- The capacity of the cut $A, B$ is: $\sum_{e=(v,w) \in E, v \in A, w \in B} c(e)$

Theorem (Min-Cut-Max-Flow)

*The capacity of the minimal cut is the same as the maximal flow.*
Maximal Matching Problem

Definition
Let \( G = (V, E) \) be a graph. The edges \( e, e' \in E \) are called independent, iff they share no common node.

Definition (Matching)
Let \( G = (V, E) \) be a graph. 
\( M \subseteq E \) is called a matching, iff \( \forall e, f \in M, e \neq f : e \cap f = \emptyset \).
\( M \) is a set of independent edges.

Definition
Let \( G = (V_1, V_2, E) \) be a bipartite graph, and there exists a set \( M \) of \( |V_1| \) independent edges. We call \( M \) complete matching from \( V_1 \) to \( V_2 \).
Theorem of Hall

Definition

Let \( G = (V_1, V_2, E) \) be a bipartite graph, and \( A \subseteq V_1 \). We denote:

\[
\Gamma(A) = \{ v \in V_2 \mid (v, w) \in E, w \in A \}.
\]

Theorem (Hall)

Let \( G = (V_1, V_2, E) \) be a bipartite graph. There exits a complete matching from \( V_1 \) to \( V_2 \), iff for each \( A \subseteq V_1 \) we have

\[
|\Gamma(A)| \geq |A|.
\]

Corollary

Every regular bipartite Graph \( G = (V_1, V_2, E) \) with \( |V_1| = |V_2| \) contains a complete matching.
Proof (Hall)

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exits a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$ 

$\implies$ simple:

- Let $M$ be a matching with $|M| = |V_1|$ and let $A \subseteq V_1$ arbitrary.
- $|\Gamma(A)| = |\{v \in V_2 \mid (v, w) \in E, w \in A\}|$.
- $|\Gamma(A)| \geq |\{v \in V_2 \mid (v, w) \in M, w \in A\}|$.
- $|\Gamma(A)| \geq |A|$. 
Proof (Hall)

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exits a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$  

$\Leftarrow$ by contradiction:

- Let $M$ be the largest matching with $|M| < |V_1|$.
- Let $A_1 = \{v \in V_1 \mid \exists b \in V_2 : \{v, b\} \in M\}$.
- Let $A_2 = \{v \in V_2 \mid \exists b \in V_1 : \{v, b\} \in M\}$.
- Let $a \in V_1 \setminus A_1$.
- $\Gamma(a) \subset A_2$, because $M$ is the largest matching.
- Any alternating path starting from $a$ reaches only nodes in $A_1' \cup A_2'$ with $A_i' \subset A_i$ and $|A_1'| = |A_2'|$.
- Thus we have $\Gamma(A_1' \cup \{a\}) \subset A_2'$.
- $|A_1' \cup \{a\}| > |A_2'|$. 
Corollary

Let $G = (V_1, V_2, E)$ be a bipartite graph and $|\Gamma(A)| \geq |A| - d$ for every $A \subseteq V_1$. Then contains $G$ at least $|V_1| - d$ independent edges.

$\implies$ By contradiction:

- Let $M$ be the largest matching with $m = |M| < |V_1| - d$.
- Let $A_1 = \{v \in V_1 \mid \exists b \in V_2 : \{v, b\} \in M\}$.
- Let $A_2 = \{v \in V_2 \mid \exists b \in V_1 : \{v, b\} \in M\}$.
- Let $a_0, a_1, \ldots, a_d \in V_1 \setminus A_1$.
- $N(a_i) \subseteq A_2$, because $M$ is the largest matching.
- Any alternating path starting from $a_i$ reaches only nodes in $A_1 \cup A_2$.
- Thus we get $\Gamma(A_1 \cup \{a_i\}) \subseteq A_2$.
- $m + d + 1 = |A_1 \cup \{a_i \mid 0 \leq i \leq d\}| \geq |A_2| = m$. 
Corollary

Let $G = (V_1, V_2, E)$ be a bipartite graph with $V_1 = (x_1, \ldots, x_m)$ and $V_2 = (y_1, \ldots, y_n)$. Then contains $G$ a spanning graph $H$ with $\deg_H(x_i) = d_i$ and $0 \leq \deg_H(y_i) \leq 1$, iff for each $A \subseteq V_1$ we get

$$|\Gamma(A)| \geq \sum_{x_i \in A} d_i.$$

$\implies$ simple:

- Let $S$ be a spanning graph with $|S| = |V_1|$.
- Let $A \subset V_1$ arbitrary.
- $|\Gamma(A)| = |\{v \in V_2 \mid (v, w) \in E, w \in A\}|$.
- $|\Gamma(A)| \geq |\{v \in V_2 \mid (v, w) \in S, w \in A\}|$.
- $|\Gamma(A)| \geq \sum_{x_i \in A} d_i$. 
Corollary

Let $G = (V_1, V_2, E)$ be a bipartite graph with $V_1 = (x_1, \ldots, x_m)$ and $V_2 = (y_1, \ldots, y_n)$. Then contains $G$ a spanning graph $H$ with $\deg_H(x_i) = d_i$ and $0 \leq \deg_H(y_i) \leq 1$, iff for each $A \subseteq V_1$ we get

$$|\Gamma(A)| \geq \sum_{x_i \in A} d_i.$$

$\iff$ by contradiction:

- Let $S$ be the largest spanning graph with $|S| < |V_1|$.
- Let $A_1 = \{v \in V_1 \mid \exists b \in V_2 : \{v, b\} \in S\}$.
- Let $A_2 = \{v \in V_2 \mid \exists b \in V_1 : \{v, b\} \in S\}$.
- Let $a \in V_1 \setminus A_1$.
- $N(a) \cap A_2 \neq \emptyset$, because $S$ is the largest spanning graph.
- Thus we get $|\Gamma(A_1 \cup \{a\})| < \sum_{x_i \in A_1 \cup \{a\}} d_i$. 
Definition

Let $A = (a_{ij})$ be a matrix, $i = 1, \ldots, r$, $j = 1, \ldots, n$, with $a_{ij} \in \{1, \ldots, n\}$. The matrix $A$ is called Latin rectangle, iff no two element in a row or a column are the same.

Theorem

Let $A$ be $r \times n$ Latin rectangle. Then we may enlarge $A$ to a $n \times n$ Latin square.

Proof: Exercise.
Matching-Problems

Definition (Maximal Matching Problem)
Given: Graph $G = (V, E)$
Compute: Matching $M$ with: $\forall e \in E$: $M \cup \{e\} \text{ is no matching.}$

Definition (Maximum Matching Problem)
Given: Graph $G = (V, E)$
Compute: Matching $M$ with: $\forall M': M'$ is a matching $\implies |M'| \leq |M|$. 
The Maximal Matching Problem

Theorem (Maximal Matching Problem)

The maximal matching problem is in \( P \) for bipartite graphs.

Algorithm:

- **Input:** \( G = (V, E) \) bipartite graph.
- Let \( M = \emptyset \).
- While \( E \neq \emptyset \) do
  - Choose \( e \in E \)
  - Let \( M = M \cup \{e\} \)
  - Let \( E := E \setminus \{f \in E \mid e \cap f \neq \emptyset\} \)
Alternating Paths

- Let $G = (V, E)$ be a graph and $M \subset E$ be a matching.
- A node $v \in V$ is called free, iff $v \notin \bigcup_{e \in M} e$.
- A path $v_0, \{v_0, v_1\}, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, v_{l-1}, \{v_{l-1}, v_l\}, v_l$ is called alternating, iff $\{v_{i-1}, v_i\} \in M \Leftrightarrow \{v_i, v_{i+1}\} \notin M$ ($0 < i < l$).

  $A \oplus B = (A \cup B) \setminus (A \cap B)$

- A alternating path $v_0, \{v_0, v_1\}, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, v_{l-1}, \{v_{l-1}, v_l\}, v_l$ is called enlarging, iff $v_0, v_l$ are free.

- Note: An edge between free nodes is an enlarging path.

- We get the following algorithm:
  1. Let $M = \emptyset$.
  2. While there is an enlarging path $P$, do:
     1. Enlarge $M$ by the following operation $M = M \oplus E(P)$. 
Example arbitrary Graph

Try enlarging paths on arbitrary graphs:

\[ A \oplus B = (A \cup B) \setminus (A \cap B) \]

Odd cycles could become a problem
Theorem of Berge

A matching $M'$ of a graph $G$ is a maximum Matching, iff there exists no enlarging path.

Proof:

$\implies$ simple.

$\iff$ by contradiction.

- Let $M$ be a matching with $|M| > |M'|$ and assume there is no enlarging path for $M'$.
- Consider the graph $H$ containing only edges from $M \cup M' \setminus (M \cap M')$.
- $H$ consists of disjoint paths and cycles.
- Thus there is a enlarging path $M'$.
The Maximum Matching Problem

Theorem (Maximum Matching Problem)

The Maximum Matching Problem ist in $\mathcal{P}$.

Algorithm:

- Input $G = (V, E)$ [bipartite] graph.
- Let $M = \emptyset$.
- While there is an enlarging path $(a_0, a_1, a_2, \cdots a_l)$ in $G$, with odd $l$, $\{a_{2i}, a_{2i+1}\} \not\in M$ and $\{a_{2i+1}, a_{2i}\} \in M$ do
  - Exchange the edges of $P$:
    - Add the edges of the form $\{a_{2i}, a_{2i+1}\}$ to $M$ and
    - delete the edges of the form $\{a_{2i+1}, a_{2i}\}$ from $M$.
- If $G = (V, E)$ is not bipartite graph, then resolve the odd cycles recursively.
Factors

**Definition**
Let $G$ be a graph. A $k$-regular spanning graph $H$ of $G$ is called $k$-factor.

**Theorem**
The graph $K_{2t}$ is the sum of $2t - 1$ 1-factors.

**Theorem**
The graph $K_{2t+1}$ is the sum of $t$ spanning cycles.
Example I
Theorem

The graph $K_{2t}$ is the sum of $2t - 1$ 1-factors.

- Draw $2t - 1$ nodes $a_1, a_2, \cdots, a_{2t-1}$, as a regular $(2t - 1)$-gon.
- Draw $a_{2t}$ as the top of a pyramid above the nodes $a_1, a_2, \cdots, a_{2t-1}$.
- Choose a 1-Factor:
  - Choose one edge of the $(2t - 1)$-gon.
  - Choose all parallel diagonals in the $(2t - 1)$-gon.
  - Choose one edge from the only free node of the $(2t - 1)$-gon to the top.
Example II
**Theorem**

The graph $K_{2t+1}$ is the sum of $t$ spanning cycles.

- Draw $2t$ nodes $a_1, a_2, \ldots, a_{2t}$, as a regular $(2t)$-gon.
- Draw $a_{2t+1}$ as the top of a pyramid above the nodes $a_1, a_2, \ldots, a_{2t}$.
- Choose one 2-factor:
  - connect two opposing nodes as follows:
  - Move in a zig-zag way over all nodes of the $(2t)$-gon.
  - Move first to the direct right neighbour,
  - and then to the direct left neighbour (i.e. two nodes back).
  - Continue in the same fashion.
  - Connect the two opposing end-nodes through $a_{2t+1}$
- We may identify for each edge a unique 2-factor.
**Definition**

Let $G$ be a graph. A spanning graph $H$ of $G$ is called \([k, k']\)-factor, iff for all nodes $v$ of $H$ we have: $k \leq \deg(v) \leq k'$. The $k, k'$-factor is called perfect, iff each connectivity component is regular.

**Theorem (Tutte 1953)**

A graph $G = (V, E)$ contains a perfect [1,2]-factor, iff for each $S \subset V$ hold: $|S| \leq |\Gamma(S)|$.

**Proof ($\implies$)**

- Let $S$ be a perfect [1,2]-factor.
- $S_1 = \{x \in S \mid \deg_S(x) = 1\}$ and $S_2 = \{x \in S \mid \deg_S(x) = 2\}$.
- Thus we get $|S_1| = |\Gamma_H(S_1)|$ and $|S_2| \leq |\Gamma_H(S_2)|$.
- Because $\Gamma_H(S_2)$ and $\Gamma_H(S_1)$ are disjoint, we get:
- $|S| = |S_1| + |S_2| \leq |\Gamma_H(S_1)| + |\Gamma_H(S_2)| = |\Gamma_H(S)| \leq |\Gamma_G(S)|$. 
**Proof (Part 2)**

Theorem (Tutte 1953)

A graph $G = (V, E)$ contains a perfect $[1,2]$-factor, iff for each $S \subset V$ hold: $|S| \leq |\Gamma(S)|$.

Proof ($\iff$):

- Let $V = \{x_1, x_2, \cdots, x_n\}$, and define: $V_1 = \{x'_1, x'_2, \cdots, x'_n\}$ and $V_2 = \{x''_1, x''_2, \cdots, x''_n\}$.
- $G' = (V_1, V_2, \{(x'_i, x''_j) \mid (x_i, x_j) \in E\})$ is a bipartite graph.
- Let $S' = \{x'_i \mid x_i \in S\}$.
- Then we get: $\Gamma(S') = \{x''_i \mid x_i \in \Gamma(S)\}$
- And: $|S'| = |S| \leq |\Gamma(S)| = |\Gamma(S')|$.
- Thus $G'$ contains a 1-factor $M$ (matching).
- Let $H = \{(x_i, x_j) \mid (x'_i, x''_j) \in M\}$.
- Then is the graph $H$ a $[1,2]$-factor.
**Proof (Part 3)**

Proof ($\iff$):

- Let $V = \{x_1, x_2, \cdots, x_n\}$, and define: $V_1 = \{x'_1, x'_2, \cdots, x'_n\}$ and $V_2 = \{x''_1, x''_2, \cdots, x''_n\}$.
- $G' = (V_1, V_2, \{(x'_i, x''_j) \mid (x_i, x_j) \in E\})$ is a bipartite graph.
- Let $S' = \{x'_i \mid x_i \in S\}$.
- Then we get: $\Gamma(S') = \{x''_i \mid x_i \in \Gamma(S)\}$
- And: $|S'| = |S| \leq |\Gamma(S)| = |\Gamma(S')|$
- Thus $G'$ contains a 1-factor $M$ (matching).
- Let $H = \{(x_i, x_j) \mid (x'_i, x''_j) \in M\}$.
- Then is the graph $H$ a $[1,2]$-factor.
- Show now: If $\deg_H(x_i) = 1$ and $\{x_i, x_j\} \in H$, then does $\deg_H(x_j) = 1$ hold:
  - There exist $k, l$: $(x'_i, x''_k), (x'_l, x''_i) \in M$.
  - Then we get $k = l$ and $\deg_H(x_j) = 1$. 
Definition

A connectivity component of a graph $G$ is called odd (reps. even), if it contains an odd (resp. even) number of nodes. Let $q(G)$ be the number of odd connectivity components of $G$.

Theorem (Tutte 1947)

A graph $G = (V, E)$ contains a 1-factor, iff for each $S \subset V$ we have:

$q(G - S) \leq |S|$.

Theorem (Petersen 1891)

Let $G$ be a 3-regular 2-edge connected graph. Then is $G$ the sum of a 1-factor and a 2-factor.

Theorem (Petersen 1891)

A Graph $G = (V, E)$ is the sum of $k$ 2-Factors, iff $G$ is $2k$-regular.
Proof I (Part 1)

Theorem (Tutte 1947)

A graph \( G = (V, E) \) contains a 1-factor, iff for each \( S \subseteq V \) we have:
\[
q(G - S) \leq |S|.
\]

Proof (\( \implies \))

- Let \( S \subseteq V \) and \( G \) has a 1-factor.
- Let \( U_1, U_2, \cdots U_p \) be the odd components of \( G - S \).
- From each \( U_i \) must be an edge of the factor, which goes to \( S \).
- Let \( \{u_i, s_i\} \) be that edge.
- Then we get: \( q(G - S) = p = |\{s_1, s_2, \cdots, s_p\}| \leq |S| \).
Proof I (Part 2)

Theorem (Tutte 1947)

A graph \( G = (V, E) \) contains a 1-factor, iff for each \( S \subset V \) we have:
\[
q(G - S) \leq |S|.
\]

Proof (\( \iff \)) by induction over \( n = |V| \):

- Note: For all odd \( n \) holds the statement.
- Note also for this: \( S = \emptyset \).
- Start of induction \( n = 2 \):
- Because of \( S = \emptyset \) is there an edge.
- Thus we have the start of the induction.
Proof I (Part 3)

Theorem (Tutte 1947)

A graph $G = (V, E)$ contains a 1-factor, iff for each $S \subset V$ we have:
$q(G - S) \leq |S|$. 

Proof ($\Leftarrow$) step of the induction $n \geq 4$:

- Choose $S$ maximal with $q(G - S) = |S|
- We show now that $G - S$ contains no even components.
- Let $U_1, U_2, \ldots, U_p$ be the odd components of $G - S$.
- We show now, that for $x_i \in V(U_i)$ the graph $U_i - \{x_i\}$ has a 1-factor.
- After this we will find a 1-factor in $G$. 
Proof I (Part 3a)

Theorem (Tutte 1947)

A graph \( G = (V, E) \) contains a 1-factor, iff for each \( S \subset V \) we have:
\[ q(G - S) \leq |S|. \]

Show: \( G - S \) contains no even components:

- Assume there is a even component \( V' \) and \( a \in V' \), then we get:

\[ |S| + 1 = 1 + q(G - S) \leq q(G - (S \cup \{a\})) \leq |S \cup \{a\}| = |S| + 1 \]

- This is a contradiction to the maximality of \( S \).
Proof I (Part 3b)

Show: For $x_i \in V(U_i)$ has the graph $U_i - \{x_i\}$ a 1-factor.

- Assume, $H = U_i - \{x_i\}$ has no 1-factor.
- There exists $S' \subset V(H)$ with $q(H - S') > |S'|$.

Intermediate Step:

- $|V(H)|$ is even and $q(H - S') - |S'|$ is also even.
- If $|S'|$ is odd, then is also $|V(H) - S'|$ and $q(H - S')$ odd.
- If $|S'|$ is even, then is also $|V(H) - S'|$ and $q(H - S')$ even.

Then we have: $q(H - S') \geq |S'| + 2$.

- $|S| + |S'| + 1 = |S \cup S' \cup \{x_i\}| \geq q(G - (S \cup S' \cup \{x_i\}))$
- $q(G - (S \cup S' \cup \{x_i\})) = q(G - S) - 1 + q(H - S')$
- $q(G - S) - 1 + q(H - S') \geq |S| - 1 + |S'| + 2 = |S| + |S'| + 1$

This is a contradiction to the maximality of $S$. 
Proof I (Part 3c)

Show: there is a 1-factor in $G$.

- Choose a matching $M$ with $|M| = p$ between $S$ and $U_1 \cup U_2 \cup \ldots \cup U_p$.
- Let: $U = \{U_1 \cup U_2 \cup \ldots \cup U_p\}$
- Let: $B = (U, S, \{\{U_i, s\} | \exists u_i \in V(U_i) : \{u_i, s\} \in E(G)\})$.
- Show that $B$ has a perfect matching.
- Let $X \subset U$ and $Y = \Gamma_B(X)$, then we have
  - $|X| \leq q(G - Y)$.
  - Put the above together: $|X| \leq q(G - Y) \leq |Y| = |\Gamma_B(X)|$.
- Thus $B$ has a perfect matching.
- Which is a 1-factor in $G$.!
Proof II

**Theorem (Petersen 1891)**

A Graph $G = (V, E)$ is the sum of $k$ 2-Factors, iff $G$ is $2k$-regular.

$\implies$ trivial.

$\impliedby$ Induction and using Eulerian graph property.

- If $k = 1$ hold, then consists $G$ of disjoint cycles.
- $G$ has w.l.o.g. a Eulerian cycle.
- Direct the edges by the order of the Eulerian cycle (directed node set is $F$).
- Let $V = \{x_1, x_2, \ldots, x_n\}$ and define:
  - $V_1 = \{x'_1, x'_2, \ldots, x'_n\}$ and $V_2 = \{x''_1, x''_2, \ldots, x''_n\}$.
- Thus $G' = (V_1, V_2, \{\{x'_i, x''_j\} \mid (x'_i, x''_j) \in F\}$) is a regular bipartite graph of degree $k$.
- This graph contains $k$ perfect matchings.
- These matchings give $k$ 2-factors in $G$. 
Proof III

Theorem (Petersen 1891)

Let \( G \) be a 3-regular 2-edge connected graph. Then is \( G \) the sum of a 1-factor and a 2-factor.

- Let \( A \subset V \).
- Let \( U_1, U_2, \cdots, U_p \) be the odd components in \( G - A \).
- For each component in \( U_i \) exists at least 2 edges in \( G \), who connect \( U_i \) and \( A \).
- Due to the 3-regularity are there at least 3 such edges.
- Thus there are at least \( 3 \cdot q(G - A) \) edges from \( G - A \) to \( A \).
- \( 3|A| = d_G(A) := \sum_{x \in A} d_G(x) \geq 3 \cdot q(G - A) \).
- \( q(G - A) \leq |A| \).
- The proof is finished by using the Theorem of Tutte.
**Posets**

**Definition**

Let $P$ be a finite set and $<$ be a transitive anti-reflexive relation.

The pair $(P, <)$ is called a **partly ordered set (poset)**.

A subset $A \subseteq P$ is called an **anti-chain**, iff $x < y$ implies $\{x, y\} \notin A$.

Furthermore, $C \subseteq P$ is called a **chain**, iff for all $x, y \in C$ holds either $x \leq y$ or $x > y$.

**Theorem (Dilworth)**

Let $P$ be a poset and $m$ is the cardinality of the largest anti-chain in $P$. Then $P$ is the union of $m$ chains.

**Theorem (Sperner)**

The cardinality of the maximal anti-chain in $Q^n$ is $\left(\binom{n}{\lfloor n/2 \rfloor}\right)$. 
Posets

Theorem (Leader 1995)

Let \( A, B \subseteq Q^n \) with \( |A| = \sum_{i=1}^{k} \binom{n}{i}, \ |B| = \sum_{i=1}^{l} \binom{n}{i} \) and \( k \leq l < n/2 \). Then we have:

- There are \( \binom{n}{k} \) edges connecting \( A \) with \( Q^n \setminus A \);
- There are \( \binom{n}{k} \) node disjoint paths from \( A \) to \( B \).


1. Golumbic M.C. Algorithmic Graph Theory and Perfect Graphs
5. Bollobás B.: Extremal Graph Theory, 1976
### Legend

- **n**: Not of relevance
- **g**: implicitly used basics
- **i**: idea of proof or algorithm
- **s**: structure of proof or algorithm
- **w**: Full knowledge