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**Definition of Coloring**

- A graph $G = (V, E)$ is $k$-colorable iff:
  - $\exists f : V \mapsto \{1, \ldots, k\} : \forall (a, b) \in E, f(a) \neq f(b)$.
- The mapping $f$ is called **coloring** of $G$.
- $\chi(G)$ is the **chromatic number** $\chi(G)$ of $G$, iff
- $G$ is $\chi(G)$-colorable, but $G$ is not $(\chi(G) - 1)$-colorable.

**Definition**

Sei $G = (V, E)$ Graph.

$$\alpha(G) = \max\{ |V'| ; V' \subset V \land \forall a, b \in V' : (a, b) \notin E \}$$

$$\omega(G) = \max\{ |V'| ; V' \subset V \land \forall a, b \in V' : (a, b) \in E \}$$

$$\chi(G) = \min\{ k ; \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \notin E \}$$
Definition (Line-Graphs)

Let $G = (V, E)$ be an undirected graph. $L(G) = (E, E')$ is called line-graph of $G$, iff

$$E' = \{(e, e') \mid e, e' \in E \land e \cap e' \neq \emptyset\}.$$

A graph $H$ is called line-graph, iff a graph $G$ exists, with $L(G) = H$. 
Example 1

\[ \chi(G) \]

Line-Graph and Coloring (3:4)
Example 2

\[ \chi(G) \]

Line-Graph and Coloring (3:6)
Example 3

\[
\chi(G)
\]

Line-Graph and Coloring (3:8)
**Edge-Colouring I**

**Definition**

The Edge-Colouring-Problem for a graph $G$ corresponds to the node-colouring of $L(G)$:

$$\chi'(G) = \chi(L(G)).$$

**Theorem (Vizing 1965)**

$$\chi'(K_{2n}) = 2n - 1 \text{ and } \chi'(K_{2n+1}) = 2n + 1.$$  

**Theorem**

$$\chi'(G) \geq \omega(L(G)) \geq \Delta(G).$$

$$\Delta(G) = \max_{v \in V(G)} \{\deg(v)\}$$
Edge-Colouring II

Theorem (Holyer)

The $d$-Edge-Colouring-Problem is NP-complete for $d \geq 3$.

Theorem (König 1916)

Any bipartite graph with degree $\Delta$ is $\Delta$ edge-colourable (Running-Time $O(nm)$).

Theorem (Vizing 1964)

Any graph with degree $\Delta$ is $\Delta + 1$ edge-colourable (Running-Time $O(nm)$).
Proof I (Holyer)

- This component assembles a negation.
  - W.l.o.g. \((a, b)\) and \((h, i)\) are coloured the same and
  - \((c, d), (j, k), (g, l)\) use three different colours.

- We will use this to represent variables and
- will use an odd cycle to represent the clauses.
1. Case: \((h, i)\) and \((l, g)\) are coloured equal.

The colour \((i, e)\) and \((i, j)\) and show in the following:

\((a, b)\) and \((h, i)\) are coloured the same and

\((c, d), (j, k), (g, l)\) use three different colours.

2. Case: \((j, k)\) and \((l, g)\) are coloured the same.

In a same way we may proof:

\((c, d)\) and \((j, k)\) are coloured the same and

\((a, b), (h, i), (g, l)\) use three different colours.
Proof III (Holyer)

- **3.Case:** \((h, i)\) and \((j, k)\) are coloured the same and \((l, g)\) use an other colour.
- **Case 3a:** \((i, j)\) has the same colour as \((l, g)\)
- Show in the following:
- This case does not happen.
Proof IV (Holyer)

3. Case: \((h, i)\) and \((j, k)\) are coloured the same and \((l, g)\) use an other colour.

Case 3b: \((i, j)\) use the third colour.

Show in the following:

\((c, d)\) and \((j, k)\) are coloured the same and

\((a, b), (h, i), (g, l)\) use three different colours.
4. Case: \((h, i), (j, k)\) and \((l, g)\) are coloured with three different colours.

Show in the following:

- \((c, d)\) and \((j, k)\) are coloured the same and
- \((a, b), (h, i), (g, l)\) use three different colours.
Proof VI (Holyer)

- We will now merge two of these construction to create a more powerful one.
- This new construction has three “Exits” (pairs of dedicated edges).
- An exit has the value “false” iff both edges are colours the same (otherwise “true”).
- For this new component we have:
  - If the left [or right] exit is “false”, then all exits are “false”.
  - If the left [right] exit is “true”, then the right [left] exit is “true”.
Proof VI (Holyer)

- We combine now at least three components in a cyclic way, to represent a variable.
- This component has at least three “Exits” (pairs of dedicated edges).
- For this component holds:
- All exits have the same logical value.
Proof VII (Holyer)

- To verify a clause the exits [may be after an additional negation] of the corresponding literals are joined with an odd cycle.
- For this component we have:
- If all exits have the value "false", then we need four colours.
Proof (König)

Theorem (König)

Any bipartite graph with degree $\Delta$ is $\Delta$ edge-colourable (Running-Time $O(nm)$).

- Show how to colour an edge $(a, b)$ in $O(n)$ time.
- Let $c_a, c_b$ be the unused colours at the nodes $a, b$.
- If $c_a = c_b$, we may colour $(a, b)$ with $c_a$.
- Observe now the graph $H_{a,b}$, who consists only of edges coloured with $c_a, c_b$.
- $H_{a,b}$ consists of a disjoined set of paths and cycles.
- $a$ and $b$ are the endpoints of two different paths.
- Thus we may exchange the colours of one path.
- Running-Time: store for each node and colour the corresponding edge.
Proof (Vizing)

Theorem (Vizing)

Any graph with degree $\Delta$ is $\Delta + 1$ edge-colourable (Running-Time $O(nm)$).

- Proof by induction on the number of edges.
- Let $\Delta = \Delta(G)$ and $e = (x, y) \in E$.
- For $G - e$ exists an edge colouring $c : E \setminus \{e\} \mapsto \{1, 2, \cdots, \Delta + 1\}$.
- Note: At each node are $\Delta + 1 - \deg(v) \geq 1$ colours free.
- For $v \in V$ let $F_v$ be the set of free colours.
- If $F_x \cap F_y \neq \emptyset$ holds we may colour $(x, y)$.
- So assume for the following: $F_x \cap F_y = \emptyset$
Proof I (Vizing)

Construct a sequence \( \{y_1, y_2, \ldots, y_k\} \) of neighbours of \( x \) and \( \{b_1, b_2, \ldots, b_k\} \) of colours with:

- \( y_1 = y \) and
- \( b_j \in F_{y_j} \) and
- \( c((x, y_{j+1})) = b_j \) and
- \( \{y_1, y_2, \ldots, y_k\} \) are different.

If in round \( k \) the following hold:

- The edge \( (x, y_k) \) could be recoloured to colour \( f \in F_x \cap F_{y_k} \) with \( f \not\in \{b_1, b_2, \ldots, b_{k-1}\} \).

Then do the following:

- \( c((x, y_k)) = f \)
- \( c((x, y_i)) = b_i \) for \( 1 \leq i < k \).

We call this operation \( \text{Shift}(k, f) \).
We will now construct such a sequence.

What happens if the recolouring is not possible.

Then we have: \( y_{k+1} \in \{y_1, y_2, \ldots, y_k\} \),

I.e. \( y_{k+1} = y_i \) and \( b_k = b_{i-1} \).

Then we have \( i \neq 1 \) and \( i \neq k \).

Let \( a \in F_x \).

Consider \( H(a, b_k) \); the subgraph using the colours \( a \) and \( b_k \).

In each component of \( H(a, b_k) \) the colours may be exchanged.

At the node \( y_k \) starts a path \( P \) of \( H(a, b_k) \).

Let \( z \) be the other endpoint of path \( P \).
Proof III (Vizing)

- Recall $a \in F_x$.
- Recall $b_k \in F_{y_{i-1}}$.
- Note $P$ contains no edges of the form $(x, y_j)$ ($1 \leq j \leq k$) with the exception of $(x, y_i)$.
- If $z = x$ holds, we also have $(x, y_i)$ in $P$.
- We will now consider the following cases:
  - $z = y_{i-1}$
  - $z = x$
  - $z \notin (x, y_{i-1})$. I.e. $z \notin \{y_1, y_2, \ldots, y_k\}$

\[\text{edge-sequence } (y_1, \ldots, y_k) \quad y_1 = y, \quad b_j \in F_{y_j}, \quad c((x, y_{j+1})) = b_j\]
Proof IIIa (Vizing)

- Note: \( a \in F_x, b_k \in F_{y_{i-1}} \) and
- \( P \) contains no edges of the form \((x, y_j)\) \((j \in \{1, \ldots, k\\setminus\{i\}\})\)
- If \( z = x \) holds, we also have \((x, y_i)\) in \( P \).

Case: \( z = y_{i-1} \)

- Both edges at the ends of \( P \) are coloured with \( a \).
- Exchange the colours on \( P \).
- After this, the colour \( a \) is not used at \( y_{i-1} \).
- Do \( \text{Shift}(i - 1, a) \) as the final step.
Proof IIIb (Vizing)

• Note: $a \in F_x$, $b_k \in F_{y_{i-1}}$ and

$P$ contains no edges of the form $(x, y_j)$
$(j \in \{1, \ldots, k\\backslash\{i\}\})$

• If $z = x$ holds, we also have $(x, y_i)$ in $P$.

• Case: $z = x$
  
  • Exchange the colour on $P$.
  • Then the colour $b_k = b_{i-1}$ is not used at $x$.
  • Do $\text{Shift}(i - 1, b_{i-1})$ as the final step.
Proof IIIc (Vizing)

- Note: \( a \in F_x, b_k \in F_{y_{i-1}} \) and
- \( P \) contains no edges of the form \((x, y_j)\) \((j \in \{1, \ldots, k\}\setminus\{i\})\)
- If \( z = x \) holds, we also have \((x, y_i)\) in \( P \).

- Case: \( z \not\in (x, y_{i-1}) \)
  - Exchange the colours on the path \( P \) (if there are edges).
  - Then the colour \( a \) is not used at \( y_k \).
  - Do \( \text{Shift}(k, a) \) as the last step.
Some Bounds

**Note**

Let \( G = (V, E) \) be a graph. Then the following hold: \( \chi(G) \geq \omega(G) \).

**Note**

Let \( G = (V, E) \) be a graph with \( |V| = n \). Then we have: \( \chi(G) \geq n/\alpha(G) \).

**Theorem**

Let \( G = (V, E) \) be a graph with \( |E| = m \). Then: \( \chi(G)(\chi(G) - 1) \leq 2m \).

- Let \( k = \chi(G) \).
- There exist \( k \) independent sets \( I_i \) with \( i \in \{1, \ldots, k\} \).
- Between \( I_i \) and \( I_j \) \( (i \neq j) \) exists at least one edge.
- From which we get \( k \cdot (k - 1)/2 \) edges in total.
Colour with Greed

Let $G = (V, E)$ be a graph.

Choose an ordering of the nodes: $\sigma = (v_1, v_2, \ldots, v_n)$.

Algorithm: $\text{GreedyColour}(G, \sigma)$.

Let $V_i = \{v_1, v_2, \ldots, v_i\}$ and $G_i = G[V_i]$.

Colour: $c(v_1) := 1$.

Colour: $c(v_i) := \min\{k \in \mathbb{N} | k \neq c(u) \forall u \in \Gamma(v_i) \cap V_{i-1}\}$

Number of colours: $\text{GreedyColour}(G, \sigma) := |\{c(v) | v \in V\}|$.

We have: $\chi(G) \leq \text{GreedyColour}(G, \sigma) \leq \Delta(G) + 1$.

For odd cycles and cliques holds:

- $\chi(G) = \text{GreedyColour}(G, \sigma) = \Delta(G) + 1$.

Running time: $O(|V| + |E|)$
Analysis of the Error

1. Extreme case: $K_{1,\Delta}$.
2. Extreme case: $B_n$:
   - $B_n = (V_n, W_n, E_n)$
   - $V_n = \{v_1, v_2, v_3, \ldots, v_n\}$
   - $W_n = \{w_1, w_2, w_3, \ldots, w_n\}$
   - $E_n = \{\{v_i, w_j\} \mid v_i \in V_n, w_j \in W_n, i \neq j\}$

Note:
- $\text{GreedyColour}(B_n, (v_1, w_1, v_2, w_2, v_3, w_3, \ldots, v_n, w_n))$.
- $\text{GreedyColour}(B_n, (v_1, w_1, v_2, w_2, v_3, w_3, \ldots, v_n, w_n)) = n$.
- But $\chi(B_n) = 2$. 
Error-Analysis

Theorem

• Let $\varepsilon, \delta > 0$ and $c < 1$.
• For large enough $n$ exists graphs $G_n$ with:
  • $\chi(G_n) \leq n^\varepsilon$ and
  • on $o(n^{-\delta})$ orderings Greedy will use $c \cdot \frac{n}{\log n}$ colours.

Lemma

There is an ordering $\sigma^*$ with: $\text{GreedyColour}(G, \sigma^*) = \chi(G)$.

Lemma

$\min_{\sigma \in S_n} \text{GreedyColour}(G, \sigma) = \chi(G)$ hold.
Improvements

- Note: for $v_i$ are at most $d_{G_i}(v_i)$ colours unusable.
- Let $b(\sigma) = \max_{1 \leq i \leq n} d_{G_i}(v_i)$ with $\sigma = (v_1, v_2, \ldots, v_n)$.
- $\chi(G) \leq \min_{\sigma \in S_n} b(\sigma)$
- The ordering $\sigma$ which gives the minimum is constructable.
  - Choose $v_n$ with the minimal degree.
  - Recursively compute the ordering on $G - v_n$.
- Such an ordering is called: “smallest-last”
Lemma

Let $\sigma_{sl}$ be a smallest-last ordering. Then we have:

$$b(\sigma_{sl}) = \max_{H \subseteq G} \delta(H) = \min_{\sigma \in S_n} b(\sigma)$$

Proof

- $b(\sigma_{sl}) \leq \max_i \delta(G_i) \leq \max_{H \subseteq G} \delta(H)$
- Let $H^*$ be a subgraph of $G$ with: $\delta(H^*) = \max_{H \subseteq G} \delta(H)$.
- Let $j$ be the smallest index with: $H^*$ is a subgraph of $G_j$ for some permutation $\sigma$. Then we get:
  - $\max_{H \subseteq G} \delta(H) = \delta(H^*) \leq d_{H^*}(v_j) \leq d_{G_j}(v_j) \leq b(\sigma)$
  - Furthermore: $\max_{H \subseteq G} \delta(H) \leq \min_{\sigma \in S_n} b(\sigma)$.
  - The claim follows by: $\min_{\sigma \in S_n} b(\sigma) \leq b(\sigma_{sl})$. 
Implications I

Lemma

Let $G = (V, E)$ and $\sigma_{sl}$ smallest-last ordering. Then the following hold:

$$\chi(G) \leq GreedyColour(G, \sigma_{sl}) \leq 1 + \max_{H \subset G} \delta(H)$$

Running Time: $O(|V| + |E|)$. 
Lemma

Let $G = (V, E)$ connected and not $\Delta(G)$-regular. Then $\chi(G) \leq \Delta(G)$ holds.

- Let $v_1$ a node with $d(v_1) < \Delta(G)$.
- Choose ordering $\sigma = (v_1, v_2, v_3, \ldots, v_n)$ by breadth-first-search from $v_1$.
- Call $\text{GreedyColour}(G, \sigma^{-1})$. Then the following hold:
  - $d(v_1) < \Delta(G)$, d.h. $c(v_1) \leq \Delta(G)$
  - $v_i$ has a non-coloured neighbour, thus $c(v_i) \leq \Delta(G)$ holds.
Statements

Theorem (Brooks 1941)

Let $G = (V, E)$ be a connected Graph with at least three nodes. Let $G$ be no clique nor an odd cycle. Then the following holds:

$$\chi(G) \leq \Delta(G)$$

- If $G$ is not two-connected, consider block $B$:
  - If $B$ is regular, then $B$ is not $\Delta(G)$-regular.
  - If $B$ is not regular, colour the graph using the above algorithm.
  - In both cases we use at most $\Delta(G)$ colours.
- If $G$ two-connected and not regular, then colour again using the above algorithm.
- If $G$ two-connected and regular, continue as follows:
Proof

Theorem (Brooks 1941)

Let $G = (V, E)$ be a connected Graph with at least three nodes. Let $G$ be no clique nor an odd cycle. Then the following holds:

$$\chi(G) \leq \Delta(G)$$

- If $G$ is not two-connected (done)
- If $G$ is two-connected and not regular: (done)
- If $G$ is two-connected and regular, then continue:
  - Choose $v_1$ with neighbours $v_{n-1}$ and $v_n$, who are neighbours, such that $G - \{v_{n-1}, v_n\}$ is still connected.
  - Compute $v_2, v_3, \ldots, v_{n-2}$ using breadth-first-search from $v_1$ on $G - \{v_{n-1}, v_n\}$.
  - Colour with $\text{GreedyColour}(G, \sigma^{-1})$.
  - $v_{n-1}$ and $v_n$ get the same colour.
  - Thus at most $\Delta(G) - 1$ colours are not usable for $v_1$. 
Lemma

Let \( G = (V, E) \) two-connected, regular with at least three nodes. Let \( G \) be no clique nor a cycle. Then there exists \( x, y \in V \) with \( \text{dist}(x, y) = 2 \) and \( G - x - y \) is connected.

- Let \( v \in V \) with \( d(v) = \Delta(G) \).
- Then is \( H := G[\{v\} \cup \Gamma(v)] \) not complete.
- Thus there exists \( x', y' \) in \( \Gamma(v) \) with \( \text{dist}(x', y') = 2 \).
- If \( G - \{x', y'\} \) is connected, we are done!
- If not, is \( x', y' \) a minimal separator.
- We have \( \Delta(G) \geq 3 \) and \( d(v) \geq 3 \).
- Let \( C \) be the component in \( G - \{x', y'\} \), which contains \( v \).
Implications

- There exists $x$ in $C$ with $x$ is neighbored to $x'$ or $y'$.
- This hold for each component in $G - \{x', y'\}$.
- Thus there exists $y$ from some other component with $\text{dist}(x, y) = 2$.
- We will now show that $G - \{x, y\}$ is connected.
  - $x'$ and $y'$ are in $G - \{x, y\}$ connected.
  - Show: Each node in $G - \{x, y\}$ is connected with $x'$ or $y'$.
  - $G - x$ is connected.
  - Each node from $C - x$ is connected by a path $P$ with $x'$ or $y'$, without using $y$.
  - $G - y$ is connected.
  - Each node from $(V \setminus C) - y$ is connected by a path $P$ with $x'$ or $y'$, without using $x$.
  - Running time: $O(|V| + |E|)$. 

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**Running time:** $O(|V| + |E|)$. 

**Proof:**

- There exists $x$ in $C$ with $x$ is neighbored to $x'$ or $y'$.
- This hold for each component in $G - \{x', y'\}$.
- Thus there exists $y$ from some other component with $\text{dist}(x, y) = 2$.
- We will now show that $G - \{x, y\}$ is connected.
  - $x'$ and $y'$ are in $G - \{x, y\}$ connected.
  - Show: Each node in $G - \{x, y\}$ is connected with $x'$ or $y'$.
  - $G - x$ is connected.
  - Each node from $C - x$ is connected by a path $P$ with $x'$ or $y'$, without using $y$.
  - $G - y$ is connected.
  - Each node from $(V \setminus C) - y$ is connected by a path $P$ with $x'$ or $y'$, without using $x$.
  - Running time: $O(|V| + |E|)$. 

**Proof:**
Theorems

Theorem (Mycielski’s)

For each number \( k \) there is a graph \( G \) with:
1. \( \chi(G) = k \) and
2. \( \omega(G) = 2 \).

Theorem (Erdös)

For each numbers \( k \), \( l \) there is a graph \( G \) with:
1. \( \chi(G) = k \) and
2. The shortest cycle has length \( l \).

We will show only the first theorem:
1. \( M_i \) has no triangles.
2. \( \chi(M_i) = i \).
Proof (Construction)

- $M_3 = C_5$
- Let $v_1, v_2, \ldots, v_n$ be the nodes of $M_k$.
- $M_{k+1}$ has the following additional nodes $u_1, u_2, \ldots, u_n$ and $w$.
- Add the following edges:
  - $\{w, u_i\}$ for $1 \leq i \leq n$ and
  - $\{u_i, x\}$ iff $\{v_i, x\} \in E(M_k)$.
Proof (Construction)

- Note:
  - \{u_1, u_2, \ldots, u_n\} is a stable set.
  - \(\Gamma(v_i)\) is a stable set.
  - Thus there are no triangles in \(M_{k+1}\).
- \(\chi(M_{k+1}) \leq k + 1:\)
  - \(c(w) = k + 1\) and
  - \(c(u_i) = c(v_i)\).
Proof (Construction)

- If \( \chi(M_{k+1}) = k \), we have:
  - w.l.o.g.: \( c(w) = k \) and therefore:
    - \( \{c(v_i) \mid 1 \leq i \leq n\} = \{1, 2, \ldots, k\} \),
    - \( \{c(u_i) \mid 1 \leq i \leq n\} = \{1, \ldots, k-1\} \),

  - Choose a colouring \( c \) with
    \( |\{i \mid c(v_i) = k\}| \) minimal.

- If \( k \neq c(v_i) \neq c(u_i) \) for some \( i \),
  - change the colours: \( c(u_i) := c(v_i) \).

- Let \( v_j \) be a node with \( c(v_j) = k \).

- Then we have:
  - \( \{c(a) \mid a \in \Gamma(v_j)\} = \{1, \ldots, k-1\} \)
  - \( \{c(a) \mid a \in \Gamma(u_j)\} = \{1, \ldots, k\} \)

- Contradiction!
Computing the Colouring

Theorem (Widgerson 1983)

Let $G = (V, E)$ be a graph with $\chi(G) = 3$. Then we may efficiently compute a $O(\sqrt{n})$ colouring.

Proof:

- If $\chi(G) = 3$ holds, $\chi(G[\Gamma(v)]) \leq 2$ is true.
- We colour the nodes by checking their degree:
- As long as there is a node $v$ with $\deg_G(v) \geq \sqrt{n}$ colour $\Gamma(v)$ using two colours
- After at most $\sqrt{n}$ steps we get a subgraph with at most $\sqrt{n}$ nodes.
- Colour this subgraph with new colours.
- The number of colours is at most: $2 \cdot \sqrt{n} + \sqrt{n} = 3 \cdot \sqrt{n}$.
- Detailed analysis show: $\sqrt{8 \cdot n}$.
Computing the Colouring

**Theorem (Blum 1994)**

*Let* $G = (V, E)$ *be a graph with* $\chi(G) = 3$. *Then we may efficiently compute a* $O(n^{3/8})$ *colouring.*

**Theorem (Karger, Motwani, Sudan 1994)**

*Let* $G = (V, E)$ *be a graph with* $\chi(G) = 3$. *Then we may efficiently compute a* $O(n^{1/4})$ *colouring.*

**Theorem (Blum, Karger 1996)**

*Let* $G = (V, E)$ *be a graph with* $\chi(G) = 3$. *Then we may efficiently compute a* $O(n^{3/14})$ *colouring.*
Theorems

Theorem

The 3-colouring-problem is for graphs of degree $\leq 4$ NP-complete. The k-colouring-problem is NP-complete.

Theorem

Let $k \geq 3$ and $c = 1/(2 + 3 \cdot \log(k + 1))$. Then the $k$-colouring-problem on graphs with girth $\lceil c \log c \rceil$ is NP-complete.

Theorem

The colouring-problem could not be approximated by a constant factor (Assuming $P \neq NP$).

Theorem

To compute a 4-colouring for a 3-colourable graph is NP-hard.
Theorems

Lemma

If $\mathcal{P} \neq \overline{\mathcal{NP}}$, then there is no polynomial time algorithm with an approximation-factor of $4/3$ for the colouring-problem.

Theorem (Garry, Johnson 1976)

If $\mathcal{P} \neq \overline{\mathcal{NP}}$, then there is no polynomial time algorithm with an approximation-factor of $2$ for the colouring-problem.

Theorem (Land, Jannakakis 1993)

If $\mathcal{P} \neq \overline{\mathcal{NP}}$, then there is for any $\varepsilon > 0$ no polynomial time algorithm with an approximation-factor of $n^\varepsilon$ for the colouring-problem.

Theorem (Feige, Kilian 1996)

If $\mathcal{P} \neq \mathcal{ZPP}$, then there is for any $\varepsilon > 0$ no polynomial time algorithm with an approximation-factor of $n^{1-\varepsilon}$ for the colouring-problem.
**Theorems**

**Lemma**

*Let $0 < c \leq 1$ be a constant. There is a linear Algorithm, which approximates the colouring-problem with a factor of $\max(1, c \cdot n)$.***

- If $|V| \leq 2/c$ then just colour $G$:
  - Colour the graph by greedy algorithm using all permutations of the nodes.
  - Running time: $O((2/c)! \cdot \left(\frac{2}{c}\right)!)$.
  - Running time: $O(1)$ and error factor 1.

- If $|V| > 2/c$ then colour $G$:
  - Split $V(G)$ in $\lfloor c \cdot n \rfloor$ Parts of size $\left\lfloor \frac{n}{\lfloor c \cdot n \rfloor} \right\rfloor$ or $\left\lceil \frac{n}{\lfloor c \cdot n \rfloor} \right\rceil$.
  - Each part has size $\leq \frac{n}{cn-1} + 1 \leq \frac{2}{c} = O(1)$.
  - Each part may be coloured optimal in constant time.
  - Total number of colours: $\frac{|cn| \cdot \chi(G)}{\chi(G)} \leq cn$. 
Theorems

Theorem (Johnson 1974)

The colouring-problem could be approximated within a factor of $O(n/\log n)$ in time $O(nm)$.

Theorem

The colouring-problem could be efficiently approximated within a factor of $O(n(\log n) - 3(\log \log n)/2)$. 
Questions

- How hard is the edge-colouring-problem?
- How many colours needed to colour the edges of a clique?
- How could the edges of a bipartite graph be coloured?
- What is the upper bound for the number of colours for the edge-colouring?
- What is the idea of the proof of Vizing?
- How hard is the node-colouring-problem?
- What bounds are known?
- What error is possible when using greedy-colouring?
Legend

n : Not of relevance

g : implicitly used basics

i : idea of proof or algorithm

s : structure of proof or algorithm

w : Full knowledge