Contents I

1. Introduction
   - Problems
   - Types of Communication
   - Notations
   - Basics

2. Broadcast
   - Lower Bound
   - First Results
   - Trees

3. Complexity

4. Broadcast on Networks
   - Definition and first Results
   - Complexity
   - First Results
   - CCC
   - SE
   - BF
   - DB

5. Lower Bounds
   - Degree of the Nodes
Definition of a Broadcasts and Accumulation

Definition of Broadcast:

Given are $G = (V, E)$ and $v \in V$.

- $v$ has information $I(v)$ and
- no node from $V \setminus \{v\}$ knows $I(v)$.
- Each node of $V \setminus \{v\}$ has to receive information $I(v)$.

Definition of Accumulation:

Given are $G = (V, E)$ and $v \in V$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\bigcup_{w \in V} I(w)$.
Definition of a Gossip

Definition of Accumulation:

Given are $G = (V, E)$ and $v \in V$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\bigcup_{w \in V} I(w)$.

Definition (Gossip):

Given is $G = (V, E)$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Each node of $v \in V$ should receive the information $\bigcup_{w \in V} I(w)$.
Types of Communication

- Telegraph-Mode: Communication is directed.
  - Is also called one-way communication.
- Telephone-Mode: Information is exchanged.
  - Is also called two-way communication.
- Communication only between neighbours.
- Communication is done in rounds.
- In each round the active edges are a matching.
- Each round uses one time-unit.
Types of Communication

- In the broadcast-problem the information of one node is transferred to all others.
- The accumulation-problem is a “inverse” broadcast.
- A gossip distributes the sum of all informations to all nodes.
- In each round the communication is done by a matching.
- The communication on an edge may be one-way or two-way, depending on the mode.
- The size of send date is ignored.
**Definition**

- By $\text{comm}(A)$ we denote the complexity (number of rounds) of a communication-algorithm.

- $r(G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the gossip-problem on } G\}$

- $r_2(G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the gossip-problem on } G\}$

- $b(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the broadcast-problem on } G \text{ and } v\}$

- $b_2(v, G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the broadcast-problem on } G \text{ and } v\}$

- $a(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the accumulations-problem on } G \text{ and } v\}$

- $a_2(v, G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the accumulations-problem on } G \text{ and } v\}$
**Definition**

- \( b(G) = \max\{b(v, G) \mid v \in V\} \)
- \( b_2(G) = \max\{b_2(v, G) \mid v \in V\} \)
- \( a(G) = \max\{a(v, G) \mid v \in V\} \)
- \( a_2(G) = \max\{a_2(v, G) \mid v \in V\} \)
- \( \min b(G) = \min\{b(v, G) \mid v \in V\} \)
- \( \min a(G) = \min\{a(v, G) \mid v \in V\} \)
First Results

- For each graph \( G \) and \( v \in V \) we have:
  - \( a_2(v, G) = b_2(v, G) \)
  - \( a(v, G) = b(v, G) \)
  - \( a(G) = b(G) \)
  - \( \text{mina}(G) = \text{minb}(G) \)
  - \( b(v, G) = b_2(v, G) \)
  - \( b(G) = b_2(G) \)

- Note: reverse broadcast is accumulation.

- There exists a graph \( G \) with: \( r(G) = 2 \cdot r_2(G) \).

- Note: 2-clique or cycle of length four.

- The following holds: \( \text{minb}(G) \leq b(G) \leq r_2(G) \leq r(G) \leq 2 \cdot r_2(G) \).

- The inequalities result from the definitions.

- \( \text{minb}(L(n)) = \lceil n/2 \rceil \)

- Optimal broadcast on a line start in the center of the line.

- \( b(L(n)) = n - 1 \)

- A message from the left has to traverse all edges.
First Results II

Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
- Let $B = F_1, F_2, \cdots F_z$ be the corresponding one-way accumulation-algorithm.
- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
First Results III

Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

$\bullet \ b(G) = r(G)$

Proof (for $n = 8$):

Both broadcasts together are a gossip-algorithm.
First Results IV

- \( \text{rad}(G) \leq \minb(G) \).
- \( \text{rad}(G) \leq \text{diam}(G) \leq b(G) \).
- Let \( G = (V, E) \) and \( H = (V, F) \) with \( F \subseteq E \). Then we have:
  - \( b(G) \leq b(H) \).
  - \( \minb(G) \leq \minb(H) \).
  - \( r(G) \leq r(H) \).
  - \( r_2(G) \leq r_2(H) \).
- \( \minb(G) \leq (\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \).
- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G) \).
- \( r(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 2 \).
- \( r_2(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \).

\[
\begin{align*}
\text{diam}(G) &= \max\{\text{dist}(u, v) \mid u, v \in V\} \\
\text{rad}(v, G) &= \max\{\text{dist}(v, x) \mid x \in V\} \\
\text{rad}(G) &= \min\{\text{rad}(v, G) \mid v \in V\}
\end{align*}
\]
Lower Bound

Lemma

Let $G = (V, E)$ be a graph with $n$ nodes. Then we have:

- $b(G) \geq \min b(G) \geq \lceil \log n \rceil$

Proof:

- Let $A(t)$ be the number of informed nodes after $t$ rounds.
- $A(0) = 1$
- $A(t + 1) \leq 2 \cdot A(t)$
- $A(t) \leq 2^t$
- At the end $2^t \geq n$ must hold.
Optimal Broadcast-Tree

Each informed node has to send in each round the information to a non-informed node:

A tree $T_i$ is a broadcast-tree, iff

- the root of $T_i$ has $i$ successors $v_0, v_1, \cdots, v_{i-1}$ and
- $v_j$ is the root of a $T_j$. 
First Results

Lemma

We have:

1. \(\min b(K(n)) = b(K(n)) = \lceil \log n \rceil\) and
2. \(\min b(HQ(m)) = b(HQ(m)) = m\).

Proof \((K(n))\):

\[
\text{for } t = 1 \text{ to } \lceil \log n \rceil \text{ do } \\
\quad \text{for all } i \in \{0, 1, \cdots, 2^{t-1} - 1\} \text{ do in parallel } \\
\qquad \text{if } i + 2^{t-1} \leq n \text{ then } \\
\qquad \quad i \text{ sends to } i + 2^{t-1}
\]

Proof \((HQ(m))\):

\[
\text{for } i = 1 \text{ to } m \text{ do } \\
\quad \text{for all } a_1, a_2, \cdots, a_{i-1} \in \{0, 1\} \text{ do in parallel } \\
\qquad a_1a_2\cdots a_{i-1}00\cdots 0 \text{ sends to } a_1a_2\cdots a_{i-1}10\cdots 0
\]
First Results II

Lemma

For all $k, m \geq 2$ we have: $\min_b(T_k(m)) = k \cdot m$.

Idea of proof:

- $b(\varepsilon, T_k(m)) = k \cdot m$.
- $b(\varepsilon, T_k(m)) \leq b(\nu, T_k(m))$.
- Note that $\nu$ has to inform $\varepsilon$.
- and $\varepsilon$ has to inform the other successors.
Complexity

**Definition:**

The special Broadcast-Problem is:

- Given: $G = (V, E)$, $v \in V$ and $k \in \mathbb{N}$.
- Question: Does $b(v, G) \leq k$ hold?

**Definition:**

The Broadcast-Problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $b(G) \leq k$ hold?
Complexity

Theorem:
The special Broadcast-Problem on trees is in $\mathcal{P}$.

- The algorithm computes recursively the broadcast-time from a node (which we consider as root) in its subtree.
- For the leafs is this time 0.
- When all broadcast-times are computed for all successors of the root, we sort these times.
- After this we may compute the order of subtrees of the root in which we forward the information from the root.
- Example: 5 subtrees have broadcast-times 10, 10, 9, 9, 7. Then we inform these subtrees in the same order. The total broadcast-time from the root is $\max(10 + 1, 10 + 2, 9 + 3, 9 + 4, 7 + 5) = 13$.

Theorem:
The Broadcast-Problem on trees is in $\mathcal{P}$.
The special Broadcast-Problem is in $\mathcal{NP}$.

Proof: simple exercise (if we have the idea).

- IF a message from node $v$ has to be send to node $w$ and the remaining time is the same as the distance between $v$ and $w$, then we call this message critical.

- I.e. the messages has to be forwarded towards $w$ without any delay.

- If the shortest path between $v$ and $w$ unique, then we know precisely the way (times and places) the messages has to traverse towards $w$.

- If there exists an other node $w'$ with: $\text{dist}(v, w) = \text{dist}(v, w') + 1$ and the shortest path towards $w'$ splits from the path from $v$ to $w$, then is the message also critical on this path.
Idea of the Proof

Broadcast from $a_0$ in 9 rounds:

Thus each node $a_i, b_i$ has to be informed in round $i$. 
Idea of the Proof (Part A)

Broadcast from $a_0$ in 9 rounds:

May be extended to any number of “paths”.
Idea for the Variables

Consider the following situation:

- There are unique shortest paths from \( v \) to \( w, w', w'' \), which share the same splitting node.
- Assume that \( \text{dist}(v, w) - 2 = \text{dist}(v, w') = \text{dist}(v, w'') \) holds and that the message on the path from \( v \) towards \( w \) is critical.
- Then will be one of the other paths (i.e. from \( v \) to \( w' \)) critical.
- The other path (i.e. from \( v \) to \( w'' \)) is not critical:
  - We may delay the message on that path one time or
  - we may inform an additional node in the last step. informieren.
- We have now the idea for the “variable”: one path from \( v \) to \( w' \) is critical or the other path from \( v \) to \( w'' \) is critical.
Idea of the Proof (Part B)

Broadcast from \( a_0 \) in 9 rounds:

Thus we have a “Variable”.
3-SAT

Definition

A boolean formula $F$ is in 3-CNF (EXACT-3-CNF):

$$F(x_1, x_2, \ldots, x_r) = \bigwedge_{i=1}^{m} c_i$$

(clauses) $c_i = (l_1^i \lor l_2^i \lor l_3^i)$ \quad \forall 1 \leq i \leq m

(literals) $l_j^i = \begin{cases} \neg x_k \text{ oder } x_k \text{ für ein } k : 1 \leq k \leq r \end{cases}$ \quad \forall 1 \leq i \leq m, \forall 1 \leq j \leq 3

An assignment is a function $W : \{x_1, x_2, \ldots, x_r\} \mapsto \{0, 1\}$.
It is NP-complete to test, if there is an assignment which satisfies $F$. 
Idea of the Proof (Part C)

Thus we have many “variables”.
The last Step

- So far we are able to construct any number of variables.
- But the clauses are still missing.
- In 3-SAT a clause has to be satisfied by some variable.
- We may represent a clause by a node, which may only be informed the variables (paths), which are not critical (which represent the boolean value “true”). We have now the full idea for the reduction to 3-SAT.
Thus we have a "clause".
Idea of the Proof

- Consider a boolean formula $\mathcal{F}$ from $3-SAT$:
- Generate for each of the $n$ variables from $\mathcal{F}$ a critical path (Part A).
- Generate for each of the above critical paths an alternative (Part B).
- Thus we have now all literals.
- Generate for each literal $x$ paths, if the literal occurs in $\mathcal{F}$ $x$ times (Part C).
- Generate for each clause a construction given by Part D.
Complexity

Theorem:
The special broadcast-problem on graphs of degree 3 is in $\mathcal{NPC}$.

Proof: it is easy to build the above construction with nodes of degree $\leq 3$.

Theorem:
The special broadcast-problem on planar graphs of degree 3 is in $\mathcal{NPC}$.

Idea of proof: The planar 3-SAT is in $\mathcal{NPC}$. That is the dependency graph between clauses and variables is planar.

Definition:
Let $\mathcal{F}$ be a boolean formula in KNF. Let $V$ be the variables and $C$ be the clauses. The dependency graph is:

$$G_\mathcal{F} = (V, C, \{\{v, c\} \mid v \text{ is in } c\})$$
Complexity

Theorem:
The broadcast-problem on planar graphs of degree 3 is in \( \mathcal{NP} \).

Proof:
- Extend the above construction, such that there is a unique “hardest” node.
- Add to the above construction a very long path.
- Thus the broadcast from the start node of the long path is the hardest.
Complexity

Definition:

The gossip-problem is:

- Given: \( G = (V, E) \) and \( k \in \mathbb{N} \).
- Question: Does \( r_2(G) \leq k \) hold?

Theorem:

The gossip-problem is in \( \mathcal{NP} \).

Proof: Extend the above construction, such that there is a unique “hardest” node.
Definition:

The one-way gossip-problem is:
- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r(G) \leq k$ hold?

Theorem:

The one-way gossip-problem is in $\mathcal{NP}$.

Proof: Extend the above construction, such that there is a unique “hardest” node.
And prevent the blocking of critical messages.
Lemma

We have:

- \( b(\text{CCC}(k)) \leq 5k + O(1) \)
- \( b(\text{BF}(k)) \leq 4.5k + O(1) \)
- \( b(\text{SE}(k)) \leq 4k + O(1) \)
- \( b(\text{DB}(k)) \leq 3k + O(1) \)

Proof: Use the following statements:

- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G) \).
Theorem:

We have: \( \lceil \frac{5k}{2} \rceil - 2 \leq \min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil \frac{5k}{2} \rceil - 1. \)

The following parts are proven:

- \( \min b(\text{CCC}(k)) \geq \lceil \frac{5k}{2} \rceil - 2 \)
- Algorithm for \( \lceil \frac{5k}{2} \rceil - 1 \) will be presented.
CCC, Proof \( \text{minb}(\text{CCC}(k)) \geq \lceil 5 \cdot k/2 \rceil - 2 \)

- \( \text{diam}(\text{CCC}(k)) = \lceil 5/2 \cdot k \rceil - 2 \)
- The statement holds for even \( k \).
- Let \( k \) be odd.
- Let \((0,00 \cdots 0)\) be the origin of the message.
- The nodes \((\lfloor k/2 \rfloor, 11 \cdots 1)\) and \((\lceil k/2 \rceil + 1, 11 \cdots 1)\) are both in distance \( (\lceil 5 \cdot k/2 \rceil - 2) \).
- Thus we need one round more than the diameter.
- The statement holds, because the CCC is node-symmetric.
CCC, Algorithm for \( \left\lceil \frac{5 \cdot k}{2} \right\rceil - 1 \)

Algorithm \textsc{Broadcast-CCC}_k

\((0, 00\ldots 0)\) sends to \((0, 10\ldots 0)\);

for \(i = 0\) to \(k - 1\) do begin

\hspace{1em} for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel

\hspace{2em} \((i - 1, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \(i, a_0 \ldots a_{i-1}00 \ldots 0)\);

\hspace{1em} for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel

\hspace{2em} \((i, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \((i, a_0 \ldots a_{i-1}10 \ldots 0)\);

end;

for all \(\alpha \in \{0, 1\}^k\) do in parallel

\hspace{1em} Broadcast on cycle \(C_\alpha(k)\) starting from \((k - 1, \alpha)\);
Theorem:

We have: \( \min_b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5 \cdot k/2 \rceil - 2. \)

Idea of proof: Change the first phase and send in both directions.
Theorem:

We have: \( \min b(SE(k)) = b(SE(k)) = 2 \cdot k - 1 \)

Proof:

- The diameter provides the lower bound.
- Note \( SE(k) \) is not node-symmetric.
- We have to provide an algorithm for any node \( v \).
- Algorithm has to be without conflicts.
**SE, Proof**

For each $w = a_1 a_2 \ldots a_k \in \{0, 1\}^k$, let

- $w_1 = a_1$ and
- $w(t) = a_t a_{t+1} \ldots a_k$ (for $1 \leq t \leq k$)
- $w(k + 1) = \varepsilon$.

Let $\alpha = a_1 a_2 \ldots a_k$ in $SE_k$ be the origin.

$\alpha = a_1 a_2 \ldots a_{k-1} a_k$ sends to $a_1 a_2 \ldots a_{k-1} \bar{a}_k$ (exchange);

for $t = 1$ to $k - 1$ do

for all $\beta \in \{0, 1\}^t$ do in parallel

begin

if $\alpha(t) \notin \{\beta_1\}^+$

then $\alpha(t) \beta$ sends to $\alpha(t + 1) \beta a_t$ (shuffle);

$\alpha(t + 1) \beta a_t$ sends to $\alpha(t + 1) \beta \bar{a}_t$ (exchange)

end;
SE, Proof

\[ \alpha = a_1a_2 \ldots a_{k-1}a_k \text{ sends to } a_1a_2 \ldots a_{k-1}\bar{a}_k \text{ (exchange);} \]

for \( t = 1 \) to \( k - 1 \) do
  for all \( \beta \in \{0, 1\}^t \) do in parallel begin
    if \( \alpha(t) \notin \{\beta_1\}^+ \)
      then \( \alpha(t)\beta \) sends to \( \alpha(t + 1)\beta a_t \) (shuffle);
    \( \alpha(t + 1)\beta a_t \) sends to \( \alpha(t + 1)\beta\bar{a}_t \) (exchange) end;

Show: There are no conflicts!

- There is no conflict for the exchange-edges, because the last bit give a unique sender and receiver.
- Assume there is a conflict by the shuffle-edges.
- We have \( \alpha(t)\beta = \alpha(t + 1)\gamma a_t \) for some \( \beta, \gamma \in \{0, 1\}^t \).
- Then we have:
  \[ a_t\alpha(t + 1) = \alpha(t + 1)\gamma_1 \Rightarrow a_t = a_{t+1} = \ldots = a_k = \gamma_1 \Rightarrow \alpha(t) \in \{\gamma_1\}^+ . \]
- This is a contradiction: shuffle-edges for \( \alpha(t) \in \{\gamma_1\}^+ \) are not used.
SE, Proof

\[ \alpha = a_1a_2 \ldots a_{k-1}a_k \text{ sends to } a_1a_2 \ldots a_{k-1}\bar{a}_k \text{ (exchange)}; \]

\[ \text{for } t = 1 \text{ to } k - 1 \text{ do} \]

\[ \text{for all } \beta \in \{0, 1\}^t \text{ do in parallel begin} \]

\[ \text{if } \alpha(t) \not\in \{\beta_1\}^+ \]

\[ \text{then } \alpha(t)\beta \text{ sends to } \alpha(t + 1)\beta a_t \text{ (shuffle)}; \]

\[ \alpha(t + 1)\beta a_t \text{ sends to } \alpha(t + 1)\beta \bar{a}_t \text{ (exchange)} \text{ end}; \]

Show: All nodes are informed!

- Show by induction: After \(2 \cdot r + 1\) rounds are all nodes \(\alpha(r + 2)\beta, \beta \in \{0, 1\}^{r+1}\) informed.
- IS: \(r = 0\) is obvious.
- All nodes \(\alpha(r + 1) \not\in \{\beta_1\}^+, \beta \in \{0, 1\}^{r+1}\) will be informed, because all nodes \(\alpha(r + 2)\beta\) have already received the information.
- If \(\alpha(r + 1) \in \{\beta_1\}^+, \beta \in \{0, 1\}^{r+1}\) holds, then we have \(\alpha(r + 2)\beta a_{r+1} = \alpha(r + 1)\beta_1\beta a_{r+1}\).
- This node has been informed before.
Theorem:

We have: $\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$

- The diameter gives the lower bound.
- Algorithm will be provided in the following.
BF (Idea of proof)

- Distribute the information in two ways:
  - Prefer in the first strategy the cycle-edges.
  - Prefer in the second strategy the cross-edges.
- Split the butterfly into two isomorph parts.
- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

\[
\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
BF (Proof I)

Splitting of $BF(m)$ in $F_0$ and $F_1$:

- $F_0$ has nodes: $\{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_1$ has nodes: $\{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_0$ and $F_1$ are isomorph.

$\#_0(w)$ denotes the number of 0’en in $w$.
$\#_1(w)$ denotes the number of 1’en in $w$. 

$\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$
Consider $F_0$: from node $v_0 = (0,00\cdots00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

Consider $F_1$: from node $v_1 = (m - 1, 00\cdots01)$ exists a unique path of length $m - 1$ to $w_1 = (0, \alpha 1)$ for $\alpha \in \{0, 1\}^{m-1}$.

First step of the algorithm $v_0$ informs $v_1$.

Then we use in $F_0$ and $F_1$ two different strategies.
BF (Proof III)

- **Aim:** Inform in \(\lfloor 3m/2 \rfloor\) steps the nodes \(w_0 = (m - 1, \alpha 0)\) and \(w_1 = (0, \alpha 1)\) for \(\alpha \in \{0, 1\}^{m^{-1}}\).

- If a node \(w_0 = (m - 1, \alpha 0)\) gets informed, then it informs in the next step \(w_1 = (0, \alpha 1)\) (if necessary).

- If a node \(w_1 = (0, \alpha 1)\) gets informed, then it informs in the next step \(w_0 = (m - 1, \alpha 0)\) (if necessary).

\[\lfloor 3m/2 \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m\]
BF (Proof IV)

- In $F_0$ a informed node $(l, \alpha 0)$ sends first to $(l + 1, \alpha 0)$ and then to $(l + 1, \alpha(l)0)$. \([\alpha(l) = \alpha_1 \ldots \bar{\alpha}_l \ldots]\)
- In $F_1$ a informed node $(l, \alpha 1)$ sends first to $(l + 1, \alpha(l)1)$ and then to $(l + 1, \alpha 1)$.
- The time to inform from $v_0 = (0, 00 \cdots 00)$ a node $w_0 = (m - 1, \alpha 0)$ is: $1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha)$.
- The time to inform from $v_1 = (m - 1, 00 \cdots 01)$ a node $w_1 = (0, \alpha 1)$ is: $1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha)$.
BF (Proof V)

**Case 1: m is odd:**

- **Case 1.1: \( \#_1(\alpha) < (m - 1)/2: \)**
  Node \( w_0 \) will be informed from \( v_0 \) at time
  \( m + \#_1(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor. \)

  After this \( w_0 \) sends to \( w_1 \).

  \( w_1 \) is informed at time \( \lfloor 3m/2 \rfloor. \)

- **Case 1.2: \( \#_0(\alpha) < (m - 1)/2: \)**
  node \( w_1 \) will be informed from \( v_0 \) at time
  \( m + \#_0(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor. \)

  \( w_0 \) will be informed from \( w_1 \) at time \( \lfloor 3m/2 \rfloor. \)

- **Case 1.3: \( \#_0(\alpha) = \#_1(\alpha) = (m - 1)/2: \)**
  \( w_0 \) is informed at time
  \( m + \#_1(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor. \)

  \( w_1 \) is informed at time \( m + \#_0(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor. \)
BF (Proof V)

**Case 2:** $m$ is even:

- **Case 2.1:** $\#_1(\alpha) \leq (m - 2)/2$:
  node $w_0$ will be informed from $v_0$ at time $m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_1$ will be informed at time $\lfloor 3m/2 \rfloor$.

- **Case 2.2:** $\#_0(\alpha) \leq (m - 2)/2$:
  node $w_1$ will be informed from $v_0$ at time $m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_0$ will be informed at time $\lfloor 3m/2 \rfloor$.

In the last phase we distribute the information on the cycles.

- Running time is: $\lceil m/2 \rceil$ rounds.
- Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
Theorem:

We have: \( d \leq \min b(DB(d)) = b(DB(d)) \leq \lfloor 3/2 \cdot (d + 1) \rfloor. \)

Proof:

- Idea \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \overline{y_1})\).
- The order is given by the parity.
- Let \(\alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2.\)
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \overline{\alpha})\).
- \((0011000)\) informs first \((0110000)\) and then \((0110001).\)
DB (Proof)

- For $k \in \{0, 1\}$ consider the path $P_k$
  from $(y_1, y_2, \ldots, y_d)$ to $(z_1, z_2, \ldots, z_{d-1}, z_d)$.

  $$(y_1, \ldots, y_d), (y_2, \ldots, y_d, k), (y_3, \ldots, y_d, k, z_1), (y_4, \ldots, y_d, k, z_1, z_2), \ldots$$
  $$\ldots, (y_d, k, z_1, \ldots, z_{d-2}), (k, z_1, \ldots, z_{d-1}), (z_1, \ldots, z_{d-1}, z_d))$$

- Let $v_{0i} = (y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2})$ the i-th node on $P_0$.
- Let $v_{1i} = (y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2})$ the i-th node on $P_1$.
- We have different times (1 or 2) for sending:
  - $(y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 0, z_1, \ldots, z_{i-2}, z_{i-1})$
  - $(y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 1, z_1, \ldots, z_{i-2}, z_{i-1})$.
- Thus the sum of running times is on $P_0$ and $P_1$: $3(d + 1)$.
- Thus the running time for the broadcast is: $\lceil 3(d + 1)/2 \rceil$. 

Degree of the Nodes

Theorem:

Let $n \geq 5$ and $G = (V, E)$ be a graph with $n$ nodes:

- If $\Delta(G) = 3$ holds, we have: $b(G) \geq \min b(G) \geq 1.4404 \log(n) - 3$.
- If $\Delta(G) = 4$ holds, we have: $b(G) \geq \min b(G) \geq 1.1374 \log(n) - 2$.

Proof:

- Let $A$ be a broadcast-algorithm.
- Let $\text{Broad}_i^A(v_0)$ be the set of nodes, which are informed from $v_0$ by $A$ in $i$ rounds.
- Let $\text{Rec}_i^A(v_0) = \text{Broad}_i^A(v_0) \setminus \text{Broad}_{i-1}^A(v_0)$.
- Let $\text{Rec}_0^A(v_0) = \{v_0\}$.
- We have: $|\text{Broad}_i^A(v_0)| = \sum_{s=0}^{i} |\text{Rec}_s^A(v_0)|$. 
Building the Idea

We consider here only the case $\Delta(G) = 3$. The case $\Delta(G) = 4$ is similar.

- The initial node may send at most three times.
- The initial node sends only in rounds 1, 2, 3.
- Any other nodes will be informed at time $t$ via an edge $e$.
- No further node may be informed via $e$.
- Thus any other node may send at most two times.
- If a node $v$ is informed in round $t$ by $w$, then did $w$ receive the information at round $t - 1$ or $t - 2$.
- Thus the number of newly informed nodes in round $t > 3$, is at most the number of nodes which got informed in rounds $t - 1$ and $t - 2$. 
Proof

- Let $A(i) = |Rec_i^A(v_0)|$.
- $A(0) = 1$
- $A(1) = 1$
- $A(2) = 2$
- $A(3) = 4$
- $A(i) = A(i - 1) + A(i - 2)$ für $i \geq 4$.
- Show by induction: $A(i) \leq 1.61804^i$ for $i \geq 0$. 
Proof

- $A(0) = 1 \leq 1 = 1.61804^0$
- $A(1) = 1 \leq 1.61804 = 1.61804^1$
- $A(2) = 2 \leq 2.61805 = 1.61804^2$
- $A(3) = 4 \leq 4.23612 = 1.61804^3$

Induction step ($i \geq 4$):

- We have: $A(j) \leq 1.61804^j$ for any $j \leq i - 1$.
- $A(i) = A(i - 1) + A(i - 2) \leq 1.61804^{i-1} + 1.61804^{i-2} \leq 1.61804^i$
- Note for this: $1.61804 + 1 \leq 1.61804^2$.

Thus we have: $n \leq |\text{Broad}_t^A(v_0)| = \sum_{i=0}^{t} |\text{Rec}_i^A(v_0)| \leq \sum_{i=0}^{t} A(i) \leq \sum_{i=0}^{t} 1.61804^i = \frac{1.61804^{t+1} - 1}{1.61804 - 1} \leq 3 \cdot 1.61804^t$

- $t \geq 1.4404 \cdot \log_2 n - 3$.
- Proof of the second statement may be done in the same way.
More Results

Consequence:
\[ b(DB_k) \geq \min b(DB_k) \geq 1.1374 \cdot k - 2 \]

Theorem:
\[ b(BF_m) = \min b(BF_m) > 1.7396m \text{ for large enough } m. \]
Idea of Proof: Check the number of nodes in distance \( k \).

Theorem:
\[ b(DB_m) > 1.3042m \text{ for large enough } m. \]
Idea of Proof: Check the number of nodes in distance \( k \).
# Overview

| Graph  | |V| | Diameter | Lower Bound | Upper Bound |
|--------|---|-----|----------|-------------|-------------|
| $K_n$  | $n$ | 1   | $\lceil \log_2 n \rceil$ | $\lceil \log_2 n \rceil$ |
| $HQ_k$ | $2^k$ | $k$ | $k$ | $k$ |
| $CCC_k$ | $k \cdot 2^k$ | $\lceil 5k/2 \rceil - 2$ | $\lceil 5k/2 \rceil - 2$ | $\lceil 5k/2 \rceil - 2$ |
| $SE_k$ | $2^k$ | $2k - 1$ | $2k - 1$ | $2k - 1$ |
| $DB_k$ | $2^k$ | $k$ | $1.4404k$ | $\frac{3}{2}(k + 1)$ |
| $BF_k$ | $k \cdot 2^k$ | $\lceil 3k/2 \rceil$ | $1.7609k$ | $2k - \frac{1}{2} \log \log k + c$ |
Literature

J. Hromkovič, et al.:
Dissemination of Information in Communication Networks:
Broadcasting, Gossiping, Leader Election, and Fault-Tolerance.
Questions

- Give the idea for the NP-completeness proof for the broadcast problem?
- Give the idea for the broadcast on the following networks
  - CCC
  - BF
  - SE
  - DB
- What are the ideas for the lower bounds for the broadcast problem?
Legend

\n \n n : Not of relevance  
g : implicitly used basics  
i : idea of proof or algorithm  
s : structure of proof or algorithm  
w : Full knowledge