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**Embeddings**

**Definition**

Let $G = (V, E)$ and $H = (W, F)$ be graphs. An embedding (embedding-function) from $G$ into $H$ is: $f : V \mapsto W$. We use for embeddings the following cost-functions:

- $|W|/|V|$ (Expansion)
- $\max_{w \in W} |\{v \mid f(v) = w\}|$ (Load)
- $\max\{\text{dist}_H(f(a), f(b)) \mid \{a, b\} \in E\}$ (Dilation)

**Definition**

A routing for an embedding $f : V \mapsto W$ is a function: $r : E \mapsto \{\text{Paths in } H\}$ with: $r(\{a, b\})$ is a path from $f(a)$ to $f(b)$. Note the cost-functions:

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Example

- Load:
- Dilation:
- Congestion:
Example

- Load:
- Dilation:
- Congestion:
Example

- **Load:** 1
- **Dilation:**
- **Congestion:**
Example

- Load: 1
- Dilation: 5
- Congestion:
Example

- **Load**: 1
- **Dilation**: 5
- **Congestion**: 2
Example

- Load:
- Dilation:
- Congestion:
Example

- Load: 2
- Dilation:
- Congestion:
Example

- Load: 2
- Dilation: 1
- Congestion:
Example

- Load: 2
- Dilation: 1
- Congestion: 2
Example

- Load:
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Iterated Embeddings

Let $G_i = (V_i, E_i)$ be graphs for $i \in \{1, 2, 3\}$

- Let $G_1$ in $G_2$ with dilation $d$, load $l$ and congestion $c$ embeddable.
- Let $G_2$ in $G_3$ with dilation $d'$, load $l'$ and congestion $c'$ embeddable.
- Then is $G_1$ in $G_3$ embeddable with:
  - Dilation $d \cdot d'$,
  - Load $l \cdot l'$ and
  - Congestion $c \cdot c'$.

Proof obvious.
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Motivation

Definition (Embedding-Problem)

Given: $G, H$ graphs and $d, c, l \in \mathbb{N}$. Questions: Could $G$ be embedded into $H$ with dilation $d$, load $l$ and congestion $c$.

Theorem

*The embedding-problem is in $\mathcal{NPC}$.*

Proof:

- Let $d = c = l = 1$.
- Choose $G$ to be a cycle (or path) of length $|V(H)|$.
- We will investigate in the following some special networks.
- pathes, cycles, grids, ...
- trees and extended trees, ...
- hyper-cubes and related structures, ...
Definition (Embedding-Problem)

Given: $G, H$ graphs and $d, c, l \in \mathbb{N}$. Questions: Could $G$ be embedded into $H$ with dilation $d$, load $l$ and congestion $c$?

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Properties of the Networks to be considered

- **Number of nodes.**
- Number of edges.
- Degree.

- Length of paths in the network:
  - Diameter, i.e. the longest of all shortest paths.
  - Radius, i.e. the shortest of all longest paths.

- Connectivity, i.e. is there a bottle-neck.
  - Node-connectivity
  - Edge-connectivity

- Regularity,
  - May be all nodes look ‘similar’.
  - May be all edges look ‘similar’.

- Easy routing
  - May be the graph is based on some group-structure.
  - How many graphs are in some family of networks?
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Paths and cycles with $n$ nodes

**Path:**
\[
L(n) = (V_{L(n)}, E_{L(n)})
\]
\[
V_{L(n)} = \{0, 1, 2, \ldots, n - 1\}
\]
\[
E_{L(n)} = \{\{i, i + 1\} | 0 \leq i < n - 1\}
\]

$L(8)$:

**Cycle:**
\[
C(n) = (V_{C(n)}, E_{C(n)})
\]
\[
V_{C(n)} = \{0, 1, 2, \ldots, n - 1\}
\]
\[
E_{C(n)} = \{\{i, (i + 1) \mod n\} | 0 \leq i < n\}
\]

$C(8)$:
Paths and cycles with $n$ nodes

- **Path:**
  \[ L(n) = (V_{L(n)}, E_{L(n)}) \]
  \[ V_{L(n)} = \{0, 1, 2, \ldots, n-1\} \]
  \[ E_{L(n)} = \{\{i, i+1\} | 0 \leq i < n-1\} \]

  - Number of nodes: $n$
  - Degrees: $\{1, 2\}$
  - Number of edges: $n - 1$
  - Diameter: $n - 1$
  - Node-con.: 1
  - Edge-con.: 1

- **Cycle:**
  \[ C(n) = (V_{C(n)}, E_{C(n)}) \]
  \[ V_{C(n)} = \{0, 1, 2, \ldots, n-1\} \]
  \[ E_{C(n)} = \{\{i, (i+1) \mod n\} | 0 \leq i < n\} \]

- **$L(8)$:**
  \[ v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 \]

- **$C(8)$:**
  \[ v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_0 \]
Paths and cycles with \( n \) nodes

**Path:**
\[
L(n) = (V_{L(n)}, E_{L(n)})
\]
\[
V_{L(n)} = \{0, 1, 2, \ldots, n - 1\}
\]
\[
E_{L(n)} = \{\{i, i + 1\} \mid 0 \leq i < n - 1\}
\]
- Number of nodes: \( n \)
- Degrees: \( \{1, 2\} \)
- Number of edges: \( n - 1 \)
- Diameter: \( n - 1 \)
- Node-con.: 1
- Edge-con.: 1

**Cycle:**
\[
C(n) = (V_{C(n)}, E_{C(n)})
\]
\[
V_{C(n)} = \{0, 1, 2, \ldots, n - 1\}
\]
\[
E_{C(n)} = \{\{i, (i + 1) \mod n\} \mid 0 \leq i < n\}
\]
- Number of nodes: \( n \)
- Degree: 2
- Number of edges: \( n \)
- Diameter: \( \lfloor n / 2 \rfloor \)
- Node-con.: 2
- Edge-con.: 2
 Paths and cycles with \( n \) nodes

**Path:**

\[
L(n) = (V_{L(n)}, E_{L(n)})
\]
\[
V_{L(n)} = \{0, 1, 2, \ldots, n-1\}
\]
\[
E_{L(n)} = \{\{i, i+1\} \mid 0 \leq i < n-1\}
\]

Number of nodes: \( n \)  
Degrees: \( \{1, 2\} \)

Number of edges: \( n-1 \)  
Diameter: \( n-1 \)

Node-con.: 1  
Edge-con.: 1

\( L(8) \):

**Cycle:**

\[
C(n) = (V_{C(n)}, E_{C(n)})
\]
\[
V_{C(n)} = \{0, 1, 2, \ldots, n-1\}
\]
\[
E_{C(n)} = \{\{i, (i+1) \mod n\} \mid 0 \leq i < n\}
\]

Number of nodes: \( n \)  
Degree: 2

Number of edges: \( n \)  
Diameter: \( \lfloor n/2 \rfloor \)

Node-con.: 2  
Edge-con.: 2

\( C(8) \):
Paths and cycles with $n$ nodes

**Path:**

$L(n) = (V_{L(n)}, E_{L(n)})$

$V_{L(n)} = \{0, 1, 2, \cdots, n-1\}$

$E_{L(n)} = \{\{i, i+1\} \mid 0 \leq i < n-1\}$

Number of nodes: $n$

Degrees: $\{1, 2\}$

Number of edges: $n-1$

Diameter: $n-1$

Node-con.: 1

Edge-con.: 1

$L(8)$:

![Diagram of L(8)]

**Cycle:**

$C(n) = (V_{C(n)}, E_{C(n)})$

$V_{C(n)} = \{0, 1, 2, \cdots, n-1\}$

$E_{C(n)} = \{\{i, (i+1) \mod n\} \mid 0 \leq i < n\}$

Number of nodes: $n$

Degree: 2

Number of edges: $n$

Diameter: $\lfloor n/2 \rfloor$

Node-con.: 2

Edge-con.: 2

$C(8)$:

![Diagram of C(8)]
Product of Graphs

**Definition:**

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

- $G \times G' = (V \times V', E_1 \cup E_2)$.
- $E_1 = \{((a, a'), (b, b')) | a' = b' \land (a, b) \in E\}$.
- $E_2 = \{((a, a'), (b, b')) | a = b \land (a', b') \in E'\}$.

Example $L(10) \times C(4)$:
Product of Graphs

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Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

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**Example** $L(10) \times C(4)$:
**Product of Graphs**

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Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

- $G \times G' = (V \times V', E_1 \cup E_2)$.
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**Example $L(10) \times C(4)$:**

![Diagram of the product of graphs](image)
**Product of Graphs**

**Definition:**

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

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**Example** $L(10) \times C(4)$:
Product of Graphs

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Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

- $G \times G' = (V \times V', E_1 \cup E_2)$.
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- $E_2 = \{(a, a'), (b, b')\} | a = b \land (a', b') \in E'\}$.

Example $L(10) \times C(4)$:
Grid of dimension $d$

- Grids: $G(n_1, n_2, \cdots, n_d) = L(n_1) \times L(n_2) \times \cdots \times L(N_d)$ with $n_i > 1$

- Grid: $G(14, 4)$:
Grid of dimension \( d \)

- Grids: \( G(n_1, n_2, \ldots, n_d) = L(n_1) \times L(n_2) \times \cdots \times L(n_d) \) with \( n_i > 1 \)

  - Number of nodes: \( \prod_{i=1}^{d} n_i \)
  - Degrees: \( \{d, \ldots, 2 \cdot d\} \)

  - Number of edges: \( \sum_{i=1}^{d} (n_i - 1) \prod_{j=1, j \neq i}^{d} n_j \)
  - Diameter: \( \sum_{i=1}^{d} (n_i - 1) \)

- Node-con.: \( d \)
- Edge-con.: \( d \)

- Grid: \( G(14, 4) \):

  
  \[
  \begin{array}{ccccccccccccccc}
  0,3 & 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 & 7,3 & 8,3 & 9,3 & 10,3 & 11,3 & 12,3 & 13,3 \\
  0,2 & 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 & 9,2 & 10,2 & 11,2 & 12,2 & 13,2 \\
  0,1 & 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 & 10,1 & 11,1 & 12,1 & 13,1 \\
  0,0 & 1,0 & 2,0 & 3,0 & 4,0 & 5,0 & 6,0 & 7,0 & 8,0 & 9,0 & 10,0 & 11,0 & 12,0 & 13,0 \\
  \end{array}
  \]
Grid of dimension $d$

- Grids: $G(n_1, n_2, \ldots, n_d) = L(n_1) \times L(n_2) \times \cdots \times L(n_d)$ with $n_i > 1$

  Number of nodes: $\prod_{i=1}^{d} n_i$
  Degrees: $\{d, \ldots, 2 \cdot d\}$

  Number of edges: $\sum_{i=1}^{d} (n_i - 1) \prod_{j=1, j \neq i}^{d} n_j$
  Diameter: $\sum_{i=1}^{d} (n_i - 1)$

  Node-con.: $d$
  Edge-con.: $d$

- Grid: $G(14, 4)$:

```
0,0 1,0 2,0 3,0 4,0 5,0 6,0 7,0 8,0 9,0 10,0 11,0 12,0 13,0
0,1 1,1 2,1 3,1 4,1 5,1 6,1 7,1 8,1 9,1 10,1 11,1 12,1 13,1
0,2 1,2 2,2 3,2 4,2 5,2 6,2 7,2 8,2 9,2 10,2 11,2 12,2 13,2
0,3 1,3 2,3 3,3 4,3 5,3 6,3 7,3 8,3 9,3 10,3 11,3 12,3 13,3
```
Torus of dimension $d$

- Torus: $Tr(n_1, n_2, \cdots, n_d) = C(n_1) \times C(n_2) \times \cdots \times C(N_d)$ with $n_i > 1$

Torus: $Tr(14, 4)$:
Torus of dimension $d$

- Torus: $Tr(n_1, n_2, \cdots, n_d) = C(n_1) \times C(n_2) \times \cdots \times C(N_d)$ with $n_i > 1$
  
  Number of nodes: $\prod_{i=1}^{d} n_i$
  Degree: $2 \cdot d$
  Number of edges: $\prod_{i=1}^{d} n_i$
  Diameter: $\sum_{i=1}^{d} \lfloor n_i/2 \rfloor$
  Node-con.: $2 \cdot d$
  Edge-con.: $2 \cdot d$

- Torus: $Tr(14, 4)$:
Torus of dimension \( d \)

- Torus: \( Tr(n_1, n_2, \cdots, n_d) = C(n_1) \times C(n_2) \times \cdots \times C(N_d) \) with \( n_i > 1 \)
  - Number of nodes: \( \prod_{i=1}^{d} n_i \)
  - Degree: \( 2 \cdot d \)
  - Number of edges: \( \prod_{i=1}^{d} n_i \)
  - Diameter: \( \sum_{i=1}^{d} \lfloor n_i/2 \rfloor \)
  - Node-con.: \( 2 \cdot d \)
  - Edge-con.: \( 2 \cdot d \)

- Torus: \( Tr(14, 4) \):

![Torus Diagram]

\( \begin{array}{cccccccccccccccc}
0,3 & 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 & 7,3 & 8,3 & 9,3 & 10,3 & 11,3 & 12,3 & 13,3 \\
0,2 & 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 & 9,2 & 10,2 & 11,2 & 12,2 & 13,2 \\
0,1 & 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 & 10,1 & 11,1 & 12,1 & 13,1 \\
0,0 & 1,0 & 2,0 & 3,0 & 4,0 & 5,0 & 6,0 & 7,0 & 8,0 & 9,0 & 10,0 & 11,0 & 12,0 & 13,0 \\
\end{array} \)
Complete binary tree

\[
T(d) = (V_{T(d)}, E_{T(d)})
\]

\[
V_{T(d)} = \{ w \in \{0,1\}^* \mid |w| \leq d \}
\]

\[
E_{T(d)} = \{ \{w, wa\} \mid w, wa \in V, a \in \{0,1\} \}
\]
Complete binary tree

\[ T(d) = (V_{T(d)}, E_{T(d)}) \]
\[ V_{T(d)} = \{w \in \{0, 1\}^* \mid |w| \leq d\} \]
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Complete binary tree

\[ T(d) = (V_{T(d)}, E_{T(d)}) \]

\[ V_{T(d)} = \{ w \in \{0, 1\}^* \mid |w| \leq d \} \]

\[ E_{T(d)} = \{ \{w, wa\} \mid w, wa \in V, a \in \{0, 1\} \} \]

Number of nodes: \(2^{d+1} - 1\)
Number of edges: \(2^{d+1} - 2\)
Degrees: \(\{1, 2, 3\}\)
Diameter: \(2 \cdot d\)
Node-con.: 1
Edge-con.: 1
Complete $k$-nary tree

\[
T_k(d) = (V_{T_k(d)}, E_{T_k(d)})
\]

\[
V_{T_k(d)} = \{ w \in \{0, 1, \ldots, k-1\}^* | |w| \leq d \}
\]

\[
E_{T_k(d)} = \{ \{w, wa\} | w, wa \in V_{T_k(d)}, a \in \{0, 1, \ldots, k-1\} \}
\]
Complete $k$-nary tree

\[
T_k(d) = (V_{T_k(d)}, E_{T_k(d)})
\]
\[
V_{T_k(d)} = \{ w \in \{0, 1, \cdots, k - 1\}^* \mid |w| \leq d \}
\]
\[
E_{T_k(d)} = \{ \{w, wa\} \mid w, wa \in V_{T_k(d)}, a \in \{0, 1, \cdots, k - 1\} \}
\]
Complete $k$-nary tree

\[
T_k(d) = (V_{T_k(d)}, E_{T_k(d)})
\]

\[
V_{T_k(d)} = \{ w \in \{0, 1, \ldots, k - 1\}^* \mid |w| \leq d \}
\]

\[
E_{T_k(d)} = \{ \{w, wa\} \mid w, wa \in V_{T_k(d)}, a \in \{0, 1, \ldots, k - 1\} \}
\]
Complete $k$-nary tree

$$T_k(d) = (V_{T_k(d)}, E_{T_k(d)})$$

$$V_{T_k(d)} = \{w \in \{0, 1, \ldots, k-1\}^* \mid |w| \leq d\}$$

$$E_{T_k(d)} = \{\{w, wa\} \mid w, wa \in V_{T_k(d)}, a \in \{0, 1, \ldots, k-1\}\}$$
Complete \( k \)-nary tree

\[
T_k(d) = (V_{T_k(d)}, E_{T_k(d)})
\]

\[
V_{T_k(d)} = \{ w \in \{0, 1, \ldots, k - 1\}^* \mid |w| \leq d \}
\]

\[
E_{T_k(d)} = \{ \{w, wa\} \mid w, wa \in V_{T_k(d)}, a \in \{0, 1, \ldots, k - 1\} \}
\]
Complete $k$-nary tree

$$T_k(d) = (V_{T_k(d)}, E_{T_k(d)})$$

$$V_{T_k(d)} = \{w \in \{0, 1, \ldots, k-1\}^* \mid |w| \leq d\}$$

$$E_{T_k(d)} = \{\{w, wa\} \mid w, wa \in V_{T_k(d)}, a \in \{0, 1, \ldots, k-1\}\}$$

Number of nodes: $\sum_{i=0}^{d} k^i$

Degrees: $\{1, k, k + 1\}$

Number of edges: $\sum_{i=0}^{d} k^i - 1$

Diameter: $2 \cdot d$

Node-con.: 1

Edge-con.: 1
X-Tree

$$XT(d) = (V_{XT(d)}, E_{XT(d)}^1 \cup E_{XT(d)}^2)$$

$$V_{XT(d)} = \{w \in \{0, 1\}^* \mid |w| \leq d\}$$

$$E_{XT(d)}^1 = \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\}$$

$$E_{XT(d)}^2 = \{\{w, w'\} \mid w, w' \in V_{XT(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w')\}$$
$X_T(d) = (V_{X_T(d)}, E^1_{X_T(d)} \cup E^2_{X_T(d)})$

$V_{X_T(d)} = \{w \in \{0, 1\}^* \mid |w| \leq d\}$

$E^1_{X_T(d)} = \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\}$

$E^2_{X_T(d)} = \{\{w, w'\} \mid w, w' \in V_{X_T(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w')\}$
X-Tree

\[
\begin{align*}
XT(d) &= (V_{XT(d)}, E_{XT(d)}^1 \cup E_{XT(d)}^2) \\
V_{XT(d)} &= \{w \in \{0, 1\}^* \mid |w| \leq d\} \\
E_{XT(d)}^1 &= \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\} \\
E_{XT(d)}^2 &= \{\{w, w'\} \mid w, w' \in V_{XT(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w')\}
\end{align*}
\]
**X-Tree**

\[
\begin{align*}
XT(d) &= (V_{XT(d)}, E_{XT(d)}^1 \cup E_{XT(d)}^2) \\
V_{XT(d)} &= \{ w \in \{0, 1\}^* \mid |w| \leq d \} \\
E_{XT(d)}^1 &= \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\} \\
E_{XT(d)}^2 &= \{\{w, w'\} \mid w, w' \in V_{XT(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w')\}
\end{align*}
\]
**X-Tree**

\[
\begin{align*}
X_T(d) &= (V_{X_T(d)}, E^1_{X_T(d)} \cup E^2_{X_T(d)}) \\
V_{X_T(d)} &= \{ w \in \{0, 1\}^* \mid |w| \leq d \} \\
E^1_{X_T(d)} &= \{ \{w, wa\} \mid w, wa \in V, a \in \{0, 1\} \} \\
E^2_{X_T(d)} &= \{ \{w, w'\} \mid w, w' \in V_{X_T(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w') \}
\end{align*}
\]
$XT(d) = (V_{XT(d)}, E^1_{XT(d)} \cup E^2_{XT(d)})$

$V_{XT(d)} = \{w \in \{0, 1\}^* \mid |w| \leq d\}$

$E^1_{XT(d)} = \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\}$

$E^2_{XT(d)} = \{\{w, w'\} \mid w, w' \in V_{XT(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w')\}$
\[ XT(d) = (V_{XT(d)}, E_{XT(d)}^1 \cup E_{XT(d)}^2) \]
\[ V_{XT(d)} = \{ w \in \{0, 1\}^* \mid |w| \leq d \} \]
\[ E_{XT(d)}^1 = \{ \{ w, wa \} \mid w, wa \in V, a \in \{0, 1\} \} \]
\[ E_{XT(d)}^2 = \{ \{ w, w' \} \mid w, w' \in V_{XT(d)}, |w| = |w'|, \text{int}(w) + 1 = \text{int}(w') \} \]

Number of nodes: \( 2^{d+1} - 1 \)
Degrees: \{2, 3, 4, 5\}
Number of edges: \( 2^{d+2} - 4 - d \)
Diameter: \( 2 \cdot d - 1 \)
Node-con.: 2
Edge-con.: 2
Hypercube of dimension $d$

\[ HQ(d) = (V_{HQ(d)}, E_{HQ(d)}) \]
\[ V_{HQ(d)} = \{0, 1\}^d \]
\[ E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\} \]
Hypercube of dimension $d$

\[
\begin{align*}
HQ(d) & = (V_{HQ(d)}, E_{HQ(d)}) \\
V_{HQ(d)} & = \{0, 1\}^d \\
E_{HQ(d)} & = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}
\end{align*}
\]
Hypercube of dimension $d$

\[
\begin{align*}
HQ(d) &= (V_{HQ(d)}, E_{HQ(d)}) \\
V_{HQ(d)} &= \{0, 1\}^d \\
E_{HQ(d)} &= \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}
\end{align*}
\]
Hypercube of dimension $d$

$$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$$

$$V_{HQ(d)} = \{0, 1\}^d$$

$$E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}$$

Note the Gray-Code.
Hypercube of dimension $d$

\[
HQ(d) = (V_{HQ(d)}, E_{HQ(d)})
\]

\[
V_{HQ(d)} = \{0, 1\}^d
\]

\[
E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}
\]

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\[ HQ(d) = (V_{HQ(d)}, E_{HQ(d)}) \]
\[ V_{HQ(d)} = \{0, 1\}^d \]
\[ E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\} \]

Number of nodes: $2^d$
Degree: $d$
Node-con.: $d$
Number of edges: $d \cdot 2^{d-1}$
Diameter: $d$
Edge-con.: $d$

Note the Gray-Code.
Hypercube of dimension $d$ (alternative view)

\[
HQ(d) = (V_{HQ(d)}, E_{HQ(d)}) \\
V_{HQ(d)} = \{0, 1\}^d \\
E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}
\]
Hypercube of dimension $d$ (alternative view)

$$\begin{align*}
HQ(d) &= (V_{HQ(d)}, E_{HQ(d)}) \\
V_{HQ(d)} &= \{0, 1\}^d \\
E_{HQ(d)} &= \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}
\end{align*}$$
Hypercube of dimension $d$ (alternative view)

\[
HQ(d) = (V_{HQ(d)}, E_{HQ(d)})
\]

\[
V_{HQ(d)} = \{0, 1\}^d
\]

\[
E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}
\]
Hypercube of dimension $d$ (alternative view)

\[ HQ(d) = (V_{HQ(d)}, E_{HQ(d)}) \]
\[ V_{HQ(d)} = \{0, 1\}^d \]
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Hypercube of dimension $d$ (alternative view)

\begin{align*}
HQ(d) & = (V_{HQ(d)}, E_{HQ(d)}) \\
V_{HQ(d)} & = \{0, 1\}^d \\
E_{HQ(d)} & = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}
\end{align*}
Hypercube of dimension $d$ (alternative view)

$$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$$
$$V_{HQ(d)} = \{0, 1\}^d$$
$$E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}$$
Hypercube of dimension $d$ (alternative view)

$$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$$

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$$E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}$$
Hypercube of dimension $d$ (alternative view)

$$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$$
$$V_{HQ(d)} = \{0, 1\}^d$$
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Hypercube of dimension $d$ (alternative view)

\[ HQ(d) = (V_{HQ(d)}, E_{HQ(d)}) \]
\[ V_{HQ(d)} = \{0, 1\}^d \]
\[ E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\} \]
Hypercube of dimension $d$ (alternative view)

$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$

$V_{HQ(d)} = \{0, 1\}^d$

$E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}$
Hypercube of dimension $d$ (alternative view)

$$HQ(d) = (V_{HQ(d)}, E_{HQ(d)})$$

$$V_{HQ(d)} = \{0, 1\}^d$$

$$E_{HQ(d)} = \{\{w_0w', w_1w'\} \mid w_0w', w_1w' \in V_{HQ(d)}\}$$
Cube-Connected Cycles of dimension $d$

\[
CCC(d) = (V_{CCC(d)}, E^c_{CCC(d)} \cup E^h_{CCC(d)})
\]

\[
V_{CCC(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d
\]

\[
E^c_{CCC(d)} = \{(i, w), ((i + 1) \mod n, w)\} | w \in \{0, 1\}^d, 0 \leq i < n\}
\]

\[
E^h_{CCC(d)} = \{(i, w0w'), (i, w1w')\} | w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\}
\]
Cube-Connected Cycles of dimension $d$

\[
CCC(d) = (V_{CCC(d)}, E^c_{CCC(d)} \cup E^h_{CCC(d)})
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\[
E^h_{CCC(d)} = \{((i, w0w'), (i, w1w')) \mid w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\}
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Cube-Connected Cycles of dimension $d$

\[ CCC(d) = (V_{CCC(d)}, E^c_{CCC(d)} \cup E^h_{CCC(d)}) \]
\[ V_{CCC(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d \]
\[ E^c_{CCC(d)} = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\} \]
\[ E^h_{CCC(d)} = \{((i, w0w'), (i, w1w')) \mid w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\} \]
Cube-Connected Cycles of dimension $d$

$CCC(d) = (V_{CCC(d)}, E^c_{CCC(d)} \cup E^h_{CCC(d)})$

$V_{CCC(d)} = \{0, 1, \ldots, d-1\} \times \{0, 1\}^d$

$E^c_{CCC(d)} = \{(i, w), ((i + 1) \mod n, w)\} \mid w \in \{0, 1\}^d, 0 \leq i < n\}$

$E^h_{CCC(d)} = \{((i, w0w'), (i, w1w'))\} \mid w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\}$
Cube-Connected Cycles of dimension $d$

$$
\begin{align*}
CCC(d) &= (V_{CCC(d)}, E_{CCC}^c \cup E_{CCC}^h) \\
V_{CCC(d)} &= \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d \\
E_{CCC}^c &= \{\{(i, w), ((i + 1) \mod n, w)\} | w \in \{0, 1\}^d, 0 \leq i < n\} \\
E_{CCC}^h &= \{\{(i, w0w'), (i, w1w')\} | w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\}
\end{align*}
$$
Cube-Connected Cycles of dimension $d$

$$CCC(d) = (V_{CCC(d)}, E^c_{CCC(d)} \cup E^h_{CCC(d)})$$

$$V_{CCC(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d$$

$$E^c_{CCC(d)} = \{(i, w), ((i + 1) \mod n, w)\} \mid w \in \{0, 1\}^d, 0 \leq i < n\}$$

$$E^h_{CCC(d)} = \{(i, w0w'), (i, w1w')\} \mid w' \in \{0, 1\}^i, w \in \{0, 1\}^{n-i-1}\}$$

Number of nodes: $d \cdot 2^d$

Degree: 3

Number of edges: $3 \cdot d \cdot 2^{d-1}$

Diameter: $2 \cdot d - 2 + \lfloor d/2 \rfloor$

Node-con.: 3

Edge-con.: 3
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E^c_{BF(d)} \cup E^h_{CC(d)}) = \{(i, w0w'), (i, w1w') | w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$

$$V_{BF(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d$$

$$E^c_{BF(d)} = \{((i, w), ((i + 1) \mod n, w)) | w \in \{0, 1\}^d, 0 \leq i < n\}$$

$$E^h_{BF(d)} = \{((i, w0w'), ((i + 1) \mod n, w1w')) | w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$

\[
\begin{align*}
0,000 & \quad 0,001 & \quad 0,010 & \quad 0,011 & \quad 0,100 & \quad 0,101 & \quad 0,110 & \quad 0,111 \\
1,000 & \quad 1,001 & \quad 1,010 & \quad 1,011 & \quad 1,100 & \quad 1,101 & \quad 1,110 & \quad 1,111 \\
2,000 & \quad 2,001 & \quad 2,010 & \quad 2,011 & \quad 2,100 & \quad 2,101 & \quad 2,110 & \quad 2,111
\end{align*}
\]
Butterfly of dimension $d$

\[
BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h) = \{(i, w0w'), (i, w1w')\} \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\]

\[
V_{BF(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d
\]

\[
E_{BF(d)}^c = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\}
\]

\[
E_{BF(d)}^h = \{((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}
\]
Butterfly of dimension $d$

$$BF(d) = \left( V_{BF(d)}, E^c_{BF(d)} \cup E^h_{CCC(d)} \right) = \{ (i, w0w'), (i, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1} \}$$

$$V_{BF(d)} = \{0, 1, \cdots, d-1\} \times \{0, 1\}^d$$

$$E^c_{BF(d)} = \{ ((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n \}$$

$$E^h_{BF(d)} = \{ ((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1} \}$$
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h)$$

$$V_{BF(d)} = \{0, 1, \cdots, d - 1\} \times \{0, 1\}^d$$

$$E_{BF(d)}^c = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\}$$

$$E_{BF(d)}^h = \{((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^d, w' \in \{0, 1\}^{n-i-1}\}$$
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E^C_{BF(d)} \cup E^h_{CCC(d)}) = \{(i, w0w'), (i, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$

$$V_{BF(d)} = \{0, 1, \ldots, d-1\} \times \{0, 1\}^d$$

$$E^C_{BF(d)} = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\}$$

$$E^h_{BF(d)} = \{((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$
Butterfly of dimension $d$

\[
BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^{h}) = \{(i, w0w'), (i, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}
\]

\[
V_{BF(d)} = \{0, 1, \cdots, d - 1\} \times \{0, 1\}^d
\]

\[
E_{BF(d)}^c = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\}
\]

\[
E_{BF(d)}^{h} = \{((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}
\]
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E^c_{BF(d)} \cup E^h_{CCC(d)}) = \{(i, w0w'), (i, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$

$$V_{BF(d)} = \{0, 1, \ldots, d-1\} \times \{0, 1\}^d$$

$$E^c_{BF(d)} = \{(i, w), ((i+1) \mod n, w) \mid w \in \{0, 1\}^d, 0 \leq i < n\}$$

$$E^h_{BF(d)} = \{(i, w0w'), ((i+1) \mod n, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h)$$

- $V_{BF(d)} = \{0, 1, \cdots, d-1\} \times \{0, 1\}^d$
- $E_{BF(d)}^c = \{((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n\}$
- $E_{BF(d)}^h = \{((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$
Butterfly of dimension $d$

\[
BF(d) = \left( V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h \right) = \left\{ (i, w0w'), (i, w1w') \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1} \right\}
\]

\[
V_{BF(d)} = \{0, 1, \ldots, d-1\} \times \{0, 1\}^d
\]

\[
E_{BF(d)}^c = \left\{ ((i, w), ((i + 1) \mod n, w)) \mid w \in \{0, 1\}^d, 0 \leq i < n \right\}
\]

\[
E_{BF(d)}^h = \left\{ ((i, w0w'), ((i + 1) \mod n, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1} \right\}
\]

Number of nodes: $d \cdot 2^d$
Degree: 4

Number of edges: $d \cdot 2^{d+1}$
Diameter: $d + \lfloor d/2 \rfloor$

Node-con.: 4
Edge-con.: 4
DeBruijn network of dimension $d$

- **DeBruijn network:**

  \[ DB(d) = (V_{DB(d)}, E_{DB(d)}^s \cup E_{DB(d)}^{se}) \]

  \[ V_{DB(d)} = \{0, 1\}^d \]

  \[ E_{DB(d)}^s = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{DB(d)}\} \]

  \[ E_{DB(d)}^{se} = \{(aw, wb) | a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\} \]
DeBruijn network of dimension $d$

- DeBruijn network:
  \[
  DB(d) = (V_{DB(d)}, E^s_{DB(d)} \cup E^{se}_{DB(d)})
  \]
  \[
  V_{DB(d)} = \{0, 1\}^d
  \]
  \[
  E^s_{DB(d)} = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}
  \]
  \[
  E^{se}_{DB(d)} = \{(aw, wb) | a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \]
DeBruijn network of dimension $d$

- DeBruijn network:
  \[
  DB(d) = (V_{DB(d)}, E_{DB(d)}^s \cup E_{DB(d)}^{se})
  \]
  \[
  V_{DB(d)} = \{0, 1\}^d
  \]
  \[
  E_{DB(d)}^s = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}
  \]
  \[
  E_{DB(d)}^{se} = \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \]
DeBruijn network of dimension $d$

- DeBruijn network:
  
  $$DB(d) = (V_{DB(d)}, E^s_{DB(d)} \cup E^{se}_{DB(d)})$$
  
  $$V_{DB(d)} = \{0, 1\}^d$$
  
  $$E^s_{DB(d)} = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}$$
  
  $$E^{se}_{DB(d)} = \{(aw, wb) | a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}$$
DeBruijn network of dimension $d$

- DeBruijn network:
  
  \[
  DB(d) = \left( V_{DB(d)}, E_{DB(d)}^s \cup E_{DB(d)}^{se} \right)
  \]
  \[
  V_{DB(d)} = \{0, 1\}^d
  \]
  \[
  E_{DB(d)}^s = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}
  \]
  \[
  E_{DB(d)}^{se} = \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \]

  Number of nodes: $2^d$  
  Degree: $2 + 2$  
  Number of edges: $2^{d+1}$  
  Diameter: $d$
DeBruijn network of dimension $d$

**Undirected DeBruijn network:**

$$DB'(d) = (V_{DB(d)}, E_{DB(d)}^{ls} \cup E_{DB(d)}^{ise})$$

$$E_{DB(d)}^{ls} = \{ \{aw, wa\} \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)} \}$$

$$E_{DB(d)}^{ise} = \{ \{aw, wb\} \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)} \}$$

Number of nodes: $2^d$  
Degree: \{2, 3, 4\}  
Number of edges: $2^{d+1} - 3$  
Diameter: $d$
DeBruijn network of dimension $d$

- Undirected DeBruijn network:
  
  \[
  \begin{align*}
  DB'(d) &= (V_{DB(d)}, E'_{DB(d)} \cup E'_{se}_{DB(d)}) \\
  E'_{DB(d)} &= \{\{aw, wa\} \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\} \\
  E'_{se}_{DB(d)} &= \{\{aw, wb\} \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \end{align*}
  \]

  Number of nodes: $2^d$
  Degree: $\{2, 3, 4\}$
  Number of edges: $2^{d+1} - 3$
  Diameter: $d$
Shuffle-Exchange network of dimension $d$

- **Shuffle-Exchange network:**
  
  \[
  SE(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})
  \]

  \[
  V_{SE(d)} = \{0, 1\}^d
  \]

  \[
  E^s_{SE(d)} = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}
  \]

  \[
  E^e_{SE(d)} = \{(wa, wb) | a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}
  \]
Shuffle-Exchange network of dimension $d$

- **Shuffle-Exchange network:**
  
  $$SE(d) = (V_{SE(d)}, E^{s}_{SE(d)} \cup E^{e}_{SE(d)})$$
  
  $$V_{SE(d)} = \{0, 1\}^d$$
  
  $$E^{s}_{SE(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}$$
  
  $$E^{e}_{SE(d)} = \{(wa, wb) \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}$$
Shuffle-Exchange network of dimension $d$

- Shuffle-Exchange network:
  
  $SE(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})$
  
  $V_{SE(d)} = \{0, 1\}^d$
  
  $E^s_{SE(d)} = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}$
  
  $E^e_{SE(d)} = \{(wa, wb) | a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}$
Shuffle-Exchange network of dimension $d$

- **Shuffle-Exchange network:**
  \[
  SE(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})
  \]
  \[
  V_{SE(d)} = \{0, 1\}^d
  \]
  \[
  E^s_{SE(d)} = \{(aw, wa) | a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}
  \]
  \[
  E^e_{SE(d)} = \{(wa, wb) | a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}
  \]

Number of nodes: $2^d$  
Degree: $2 + 2$  
Number of edges: $2^{d+1}$  
Diameter: $2 \cdot d - 1$
Shuffle-Exchange network of dimension $d$

- **Undirected Shuffle-Exchange network:**
  
  $SE'(d) = (V_{SE(d)}, E'_{SE(d)} \cup E'_{SE(d)})$

  $E'_{SE(d)} = \{ \{aw, wa\} | a \in \{0, 1\}, aw, wa \in V_{SE(d)} \}$

  $E'_{SE(d)} = \{ \{wa, wb\} | a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)} \}$

  Number of nodes: $2^d$

  Number of edges: $2^{d+1}/3$

  Degree: $\{1, 2, 3\}$

  Diameter: $2 \cdot d - 1$
**Shuffle-Exchange network of dimension $d$**

- Undirected Shuffle-Exchange network:
  
  $$ SE'(d) = (V_{SE(d)}, E_{SE(d)}^s \cup E_{SE(d)}^e) $$
  
  $$ E_{SE(d)}^s = \{ \{aw, wa\} \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)} \} $$
  
  $$ E_{SE(d)}^e = \{ \{wa, wb\} \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)} \} $$

  Number of nodes: $2^d$
  Degree: $\{1, 2, 3\}$
  Number of edges: $2^{d+1}/3$
  Diameter: $2 \cdot d - 1$
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof:
**Lemma:**

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

**Proof:** Embed a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embed a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

\[ C(2^{d+1} - 1) \] may be embedded into \( T(d) \) with load 1 and dilation 3.

Proof: Embed a path recursively with dilation \( \leq 3 \) from the root to a son of the root.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

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$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embed a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

$C(2^{d+1} − 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embedd a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embed a path recursively with dilation $\leq 3$ from the root to a son of the root.
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$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embedd a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

\( C(2^{d+1} - 1) \) may be embedded into \( T(d) \) with load 1 and dilation 3.

Proof: Embed a path recursively with dilation \( \leq 3 \) from the root to a son of the root.
**Lemma:**

\( C(2^{d+1} - 1) \) may be embedded into \( T(d) \) with load 1 and dilation 3.

Proof: Embed a path recursively with dilation \( \leq 3 \) from the root to a son of the root.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $T(d)$ with load 1 and dilation 3.

Proof: Embed a path recursively with dilation $\leq 3$ from the root to a son of the root.
Lemma:

\( C(3 \cdot (2^d + 1) - 1) \) may be embedded into \( T(d) \) with load 3 and dilation 1.

Proof:

\[
C(n) \text{ into } T(d)
\]
Lemma:

\[C(3 \cdot (2^{d+1} - 1))\] may be embedded into \(T(d)\) with load 3 and dilation 1.

Proof: Use the in-order traversal through the tree.
**Lemma:**

\[ C(3 \cdot (2^{d+1} - 1)) \] may be embedded into \( T(d) \) with load 3 and dilation 1.

**Proof:** Use the in-order traversal through the tree.
Lemma:

$C(3 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 3 and dilation 1.

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Lemma:

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Lemma:

\[ C(3 \cdot (2^{d+1} - 1)) \] may be embedded into \( T(d) \) with load 3 and dilation 1.

Proof: Use the in-order traversal through the tree.
Lemma:

\(C(3 \cdot (2^{d+1} - 1))\) may be embedded into \(T(d)\) with load 3 and dilation 1.

Proof: Use the in-order traversal through the tree.

\[\begin{align*}
C(n) & \text{ into } T(d) \\
\end{align*}\]
$C(n)$ into $T(d)$

**Lemma:**

$C(3 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 3 and dilation 1.

**Proof:** Use the in-order traversal through the tree.
Lemma:

$C(3 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 3 and dilation 1.

Proof: Use the in-order traversal through the tree.
Lemma:

\[ C(2 \cdot (2^{d+1} - 1)) \] may be embedded into \( T(d) \) with load 2 and dilation 2.

Proof:
Lemma:

$C(2 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$C(2 \cdot (2^d + 1) - 1)$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$C(2 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$C(2 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$C(2 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order’ nodes.
Lemma:

\( C(2 \cdot (2^{d+1} - 1)) \) may be embedded into \( T(d) \) with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
C(n) into T(d)

Lemma:

$C(2 \cdot (2^d + 1) - 1)$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$C(2 \cdot (2^{d+1} - 1))$ may be embedded into $T(d)$ with load 2 and dilation 2.

Proof: Use the in-order traversal through the tree and jump the ‘in-order’ nodes.
C(n) into T(d)

**Lemma:**

\[ C(2 \cdot (2^{d+1} - 1)) \] may be embedded into \( T(d) \) with load 2 and dilation 2.

**Proof:** Use the in-order traversal through the tree and jump the ‘in-order” nodes.
Lemma:

$L(2^d + 1 - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof:
Lemma:

$L(2^{d+1} - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof: Place the path in levels through the tree.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof: Place the path in levels through the left part and through the right part and connect both to a cycle.
Lemma:

\[ C(2^{d+1} - 1) \] may be embedded into \( XT(d) \) with load 1 and dilation 1.

Proof: Place the path in levels through the left part and through the right part and connect both to a cycle.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof: Place the path in levels through the left part and through the right part and connect both to a cycle.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof: Place the path in levels through the left part and through the right part and connect both to a cycle.
Lemma:

$C(2^{d+1} - 1)$ may be embedded into $XT(d)$ with load 1 and dilation 1.

Proof: Place the path in levels through the left part and through the right part and connect both to a cycle.
Lemma:

$C(2^d)$ may be embedded into $HQ(d)$ with load 1 and dilation 1.

Proof: Gray-code.
Lemma:

$C(2^d)$ may be embedded into $HQ(d)$ with load 1 and dilation 1.

Proof: Gray-code.
**Lemma:**

\( C(2^d) \) may be embedded into \( HQ(d) \) with load 1 and dilation 1.

**Proof:** Gray-code.
Lemma:

If $2n \leq 2^d$ holds, then $C(2n)$ could be embedded into $HQ(d)$ with load 1 and dilation 1.

Proof: recursive structure of $HQ(d)$
Lemma:
If $2n \leq 2^d$ holds, then $C(2n)$ could be embedded into $HQ(d)$ with load 1 and dilation 1.

Proof: recursive structure of $HQ(d)$
Lemma:

If \( 2n \leq 2^d \) holds, then \( C(2n) \) could be embedded into \( HQ(d) \) with load 1 and dilation 1.

Proof: recursive structure of \( HQ(d) \)
Lemma:

If $2n \leq 2^d$ holds, then $C(2n)$ could be embedded into $HQ(d)$ with load 1 and dilation 1.

Proof: recursive structure of $HQ(d)$
Alternative proof: $G(2, 2^{d-1})$ is a sub-graph of $HQ(d)$. 
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof:
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof: Join cycles of length $d, 2d, 4d, ...$.
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, ... \).
**Lemma:**

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

**Proof:** Join cycles of length $d, 2d, 4d, ...$.
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, \ldots \)
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof: Join cycles of length $d, 2d, 4d, \ldots$. 

![Diagram showing the embedding of cycles into $BF(d)$](image-url)
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, \ldots \).
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof:
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof: Join cycles of length $d, 2d, 4d, ...$ (view using the gray-code).
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof: Join cycles of length $d, 2d, 4d, \ldots$ (view using the gray-code).
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, \ldots \) (view using the gray-code).

\[
\begin{align*}
0,000 & \quad 0,001 \\
1,000 & \quad 1,001 \\
2,000 & \quad 2,001 \\
0,000 & \quad 0,001 \\
0,010 & \quad 0,011 \\
1,010 & \quad 1,011 \\
2,010 & \quad 2,011 \\
0,100 & \quad 0,101 \\
1,100 & \quad 1,101 \\
2,100 & \quad 2,101 \\
0,110 & \quad 0,111 \\
1,110 & \quad 1,111 \\
2,110 & \quad 2,111 \\
0,110 & \quad 0,111
\end{align*}
\]
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, \ldots \) (view using the gray-code).
Lemma:

$C(d \cdot 2^d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.

Proof: Join cycles of length $d, 2d, 4d, \ldots$ (view using the gray-code).
Lemma:

\( C(d \cdot 2^d) \) may be embedded into \( BF(d) \) with load 1 and dilation 1.

Proof: Join cycles of length \( d, 2d, 4d, \ldots \) (view using the gray-code).
Lemma:

$C(d \cdot 2^d)$ may be embedded into $CCC(d)$ with load 1 and dilation 2.

Proof:
Lemma:

$C(d \cdot 2^d)$ may be embedded into $CCC(d)$ with load 1 and dilation 2.

Proof: Embed cycles in $BF(d)$ and embed $BF(d)$ in $CCC(d)$ with dilation 2.
Lemma:

$L(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Proof:
Lemma:

$L(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Proof: Place the path snake-wise through the grid.
Lemma:

$L(n_1 \cdot n_2 \cdot \ldots \cdot n_d)$ may be embedded into $G(n_1, n_2, \ldots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdot \ldots \cdot n_d)$ may be embedded into $G(n_1, n_2, \ldots, n_d)$ with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embed the path in the grid with dilation 1.
Lemma:

$L(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embedd the path in the grid with dilation 1.
Lemma:

\( L(n_1 \cdot n_2 \cdot \cdots \cdot n_d) \) may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1.

Lemma:

\( C(n) \) may be embedded into \( L(n) \) with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

\( C(n_1 \cdot n_2 \cdot \cdots \cdot n_d) \) may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embedd the path in the grid with dilation 1.
Lemma:

$L(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embed the path in the grid with dilation 1.
Lemma:

$L(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embed the path in the grid with dilation 1.
Lemma:

$L(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdot \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 2.

- Embedd cycle in the path with dilation 2.
- Embedd the path in the grid with dilation 1.
\[ L(n) \text{ into } G(n_1, n_2, \cdots, n_d) \]

**Lemma:**

\[ L(n_1 \cdot n_2 \cdot \cdots \cdot n_d) \] may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1.

**Lemma:**

\[ C(n) \] may be embedded into \( L(n) \) with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

**Lemma:**

\[ C(n_1 \cdot n_2 \cdot \cdots \cdot n_d) \] may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 2.

- Embed cycle in the path with dilation 2.
- Embed path in the grid with dilation 1.
Lemma:

$L(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1.

Lemma:

$C(n)$ may be embedded into $L(n)$ with load 1 and dilation 2.

- For each direction of the path, use every second node.
- Or: use the bandwidth 2 embedding of the cycle.

Lemma:

$C(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 2.

- Embedd cycle in the path with dilation 2.
- Embedd the path in the grid with dilation 1.
Lemma:

\( C(n_1 \cdot n_2 \cdots \cdot n_d) \) may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1, if at least one \( n_i \) is even.

Proof:

\[
\begin{array}{cccccccccccccc}
0,3 & 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 & 7,3 & 8,3 & 9,3 & 10,3 & 11,3 & 12,3 & 13,3 \\
0,2 & 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 & 9,2 & 10,2 & 11,2 & 12,2 & 13,2 \\
0,1 & 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 & 10,1 & 11,1 & 12,1 & 13,1 \\
0,0 & 1,0 & 2,0 & 3,0 & 4,0 & 5,0 & 6,0 & 7,0 & 8,0 & 9,0 & 10,0 & 11,0 & 12,0 & 13,0 \\
\end{array}
\]
Lemma:

$C(n_1 \cdot n_2 \cdots \cdot n_d)$ may be embedded into $G(n_1, n_2, \cdots, n_d)$ with load 1 and dilation 1, if at least one $n_i$ is even.

Proof: Place the path snake-wise through the grid.
**Lemma:**

\[ C(n_1 \cdot n_2 \cdots n_d) \] may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1, if at least one \( n_i \) is even.

**Lemma:**

\[ C(n_1 \cdot n_2 \cdots n_d) \] may not be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1, if all \( n_i \) are odd.

**Proof:**

\[
\begin{array}{cccccccccccccccc}
0,4 & 1,4 & 2,4 & 3,4 & 4,4 & 5,4 & 6,4 & 7,4 & 8,4 & 9,4 & 10,4 & 11,4 & 12,4 & 13,4 & 14,4 \\
0,3 & 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 & 7,3 & 8,3 & 9,3 & 10,3 & 11,3 & 12,3 & 13,3 & 14,3 \\
0,2 & 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 & 9,2 & 10,2 & 11,2 & 12,2 & 13,2 & 14,2 \\
0,1 & 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 & 10,1 & 11,1 & 12,1 & 13,1 & 14,1 \\
0,0 & 1,0 & 2,0 & 3,0 & 4,0 & 5,0 & 6,0 & 7,0 & 8,0 & 9,0 & 10,0 & 11,0 & 12,0 & 13,0 & 14,0 \\
\end{array}
\]
Lemma:

\( C(n_1 \cdot n_2 \cdots n_d) \) may be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1, if at least one \( n_i \) is even.

Lemma:

\( C(n_1 \cdot n_2 \cdots n_d) \) may not be embedded into \( G(n_1, n_2, \cdots, n_d) \) with load 1 and dilation 1, if all \( n_i \) are odd.

Proof: Consider the 2-colouring of the grid.
Lemma:

$T(d)$ may be embedded into $L(2^{d+1} - 1)$ with load 1 and dilation $\lceil 2^{d+1}/2d \rceil$.

Idea of Proof:

- Stretch the longest path of $T(d)$ on the path.
- Or use the bandwidth-embedding of the $T(d)$. 
Lemma:

$T(d)$ may be embedded into $L(2^{d+1} - 1)$ with load 1 and dilation $\lceil 2^{d+1}/2d \rceil$.

Idea of Proof:

- Stretch the longest path of $T(d)$ on the path.
- Or use the bandwidth-embedding of the $T(d)$.
Lemma:

$T(d)$ may be embedded into $L(2^{d+1} - 1)$ with load 1 and dilation $\lceil 2^{d+1}/2d \rceil$.

Idea of Proof:

- Stretch the longest path of $T(d)$ on the path.
- Or use the bandwidth-embedding of the $T(d)$. 

\begin{tikzpicture}
  \node (e) at (0,0) {$e$};
  \node (0) at (-1,-1) {0}
    child {node (00) {00} child {node (000) {000} child {node (0000) {0000}}}} child {node (001) {001} child {node (0011) {0011}}}
  \node (1) at (1,-1) {1}
    child {node (10) {10} child {node (100) {100} child {node (1000) {1000}}}} child {node (11) {11} child {node (111) {111}}};
\end{tikzpicture}
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

Proof:

- $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
- $f(w) = w10^{x(w)-1}$.
- Edges: $f((w, wa)) = f((w10^{x(w)-1}, wa10^{x(wa)-1}))$
- Dilation is 2.
Lemma:

\( T(d) \) may be embedded into \( HQ(d+1) \) with load 1 and dilation 2.

Proof:

- \( f : \{ w \in \{0, 1\}^* \mid |w| \leq d \} \mapsto \{ w \in \{0, 1\}^* \mid |w| = d + 1 \}. \)
- Add to \( w \) a bit-sequence of length \( x(w) = d + 1 - |w| \geq 1. \)
- \( f(w) = w10^{x(w)-1}. \)
- Edges: \( f((w, wa)) = f((w10^{x(w)-1}, wa10^{x(wa)-1})) \)
- Dilation is 2.

\[ E_{T(d)} = \{ \{ w, wa \} \mid w, wa \in V, a \in \{0, 1\} \} \text{ and } E_{HQ(d)} = \{ \{ w0w', w1w' \} \mid w0w', w1w' \in V_{HQ(d)} \} \]
**Lemma:**

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

**Proof:**

1. $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.
2. Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
3. $f(w) = w10^{x(w)-1}$.
4. Edges: $f((w, wa)) = f((w10^{x(w)-1}, wa10^{x(wa)-1}))$
5. Dilation is 2.
\(T(d)\) into \(HQ(d + 1)\)

\[E_{T(d)} = \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\} \text{ and } E_{HQ(d)} = \{\{w0w', w1w'\} \mid w0w', w1w' \in V_{HQ(d)}\}\]

**Lemma:**

\(T(d)\) may be embedded into \(HQ(d + 1)\) with load 1 and dilation 2.

**Proof:**

- \(f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}\).
- Add to \(w\) a bit-sequence of length \(x(w) = d + 1 - |w| \geq 1\).
- \(f(w) = w10^{x(w) - 1}\).
- **Edges:** \(f((w, wa)) = f((w10^{x(w) - 1}, wa10^{x(wa) - 1}))\)
- Dilation is 2.
**Lemma:**

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

**Proof:**

- $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
- $f(w) = w10^{x(w)-1}$.
- Edges: $f((w, wa)) = f((w10^{x(w)-1}, wa10^{x(wa)-1}))$.
- Dilation is 2.
**Lemma:**

*T(d)* may be embedded into *HQ(d + 1)* with load 1 and dilation 2.

**Proof:**

- \( f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}. \)
- Add to \( w \) a bit-sequence of length \( x(w) = d + 1 - |w| \geq 1 \).
- \( f(w) = w10^{x(w) - 1} \).
- Edges: \( f((w, wa)) = (w10^{x(w) - 1}, wa10^{x(wa) - 1}) \)
- Dilation is 2.
**XT(d) into HQ(d + 1)**

\[ E_{T(d)} = \{\{w, wa\} \mid w, wa \in V, a \in \{0, 1\}\} \text{ and } E_{HQ(d)} = \{\{w0', w1'\} \mid w0', w1' \in V_{HQ(d)}\} \]

**Lemma:**

\( XT(d) \) may be embedded into \( HQ(d + 1) \) with load 1 and dilation 2.

- \( f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}. \)
- Add to \( w \) a bit-sequence of length \( x(w) = d + 1 - |w| \geq 1. \)
- \( f(w) = \text{GrayCode}(w)10^{x(w)-1}. \)
- Edges: \( f((w, wa)) = f((\text{GrayCode}(w)10^{x(w)-1}, \text{GrayCode}(wa)10^{x(wa)-1})) \)
- Dilation is 2, because \( \text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b}. \)
Lemma:

$XT(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

- $f : \{ w \in \{0, 1\}^* \mid |w| \leq d \} \mapsto \{ w \in \{0, 1\}^* \mid |w| = d + 1 \}$.
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
- $f(w) = \text{GrayCode}(w)10^{x(w)-1}$.
- Edges: $f((w, wa)) = f((\text{GrayCode}(w)10^{x(w)-1}, \text{GrayCode}(wa)10^{x(wa)-1}))$
- Dilation is 2, because $\text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b}$. 
**Lemma:**

$XT(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

- $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.  
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.  
- $f(w) = \text{GrayCode}(w)10^{x(w)-1}$.  
- Edges: $f((w, wa)) = f((\text{GrayCode}(w)10^{x(w)-1}, \text{GrayCode}(wa)10^{x(wa)-1}))$.  
- Dilation is 2, because $\text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b}$.
**Lemma:**

$XT(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2.

- $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
- $f(w) = \text{GrayCode}(w)10^{x(w)-1}$.
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- Dilation is 2, because $\text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b}$.
**XT(d) into HQ(d + 1)**

\[ E_{T(d)} = \{ \{ w, wa \} \mid w, wa \in V, a \in \{0, 1\} \} \quad \text{and} \quad E_{HQ(d)} = \{ \{ w0w', w1w' \} \mid w0w', w1w' \in V_{HQ(d)} \} \]

**Lemma:**

**XT(d)** may be embedded into **HQ(d + 1)** with load 1 and dilation 2.

- \( f : \{ w \in \{0, 1\}^* \mid |w| \leq d \} \mapsto \{ w \in \{0, 1\}^* \mid |w| = d + 1 \} \).  
- Add to \( w \) a bit-sequence of length \( x(w) = d + 1 - |w| \geq 1 \).
- \( f(w) = \text{GrayCode}(w)10^{x(w) - 1} \).
- Edges: \( f((w, wa)) = f((\text{GrayCode}(w)10^{x(w) - 1}, \text{GrayCode}(wa)10^{x(wa) - 1})) \).
- Dilation is 2, because \( \text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b} \).
**Lemma:**

$\mathcal{X}_T(d)$ may be embedded into $\mathcal{H}_Q(d + 1)$ with load 1 and dilation 2.

- $f : \{w \in \{0, 1\}^* \mid |w| \leq d\} \mapsto \{w \in \{0, 1\}^* \mid |w| = d + 1\}$.
- Add to $w$ a bit-sequence of length $x(w) = d + 1 - |w| \geq 1$.
- $f(w) = \text{GrayCode}(w)10^{x(w) - 1}$.
- Edges: $f((w, wa)) = f((\text{GrayCode}(w)10^{x(w) - 1}, \text{GrayCode}(wa)10^{x(wa) - 1}))$
- Dilation is 2, because $\text{GrayCode}(wa) = \text{GrayCode}(w)a_{w,b}$.
Lemma:

$T(d)$ may not be embedded into $HQ(d + 1)$ for $d > 1$ with load 1 and dilation 1.

Proof:
Lemma:

$T(d)$ may not be embedded into $HQ(d + 1)$ for $d > 1$ with load 1 and dilation 1.

Proof: Consider the 2-colouring of $T(d)$ in $HQ(d + 1)$. 
Lemma:

*T(d)* may be embedded into *HQ(d + 1)* with load 1 and dilation 2, such that only one edge is stretched.

Proof:

![Diagram showing the embedding of T(d) into HQ(d + 1)]
**Lemma:**

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

**Proof:** Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 

![Diagram of the embedding process](image)
Lemma:

\( T(d) \) may be embedded into \( HQ(d + 1) \) with load 1 and dilation 2, such that only one edge is stretched.

Proof:
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 
**Lemma:**

\( T(d) \) may be embedded into \( HQ(d + 1) \) with load 1 and dilation 2, such that only one edge is stretched.

**Proof:** Recursive embedding of the double-rooted tree as a sub-graph of the \( HQ \).
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 
Lemma:

\( T(d) \) may be embedded into \( HQ(d + 1) \) with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the \( HQ \).
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 

![Diagram showing the embedding process](image-url)
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the HQ.
Lemma:

$T(d)$ may be embedded into $HQ(d+1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 

![Diagram of the embedding process](image-url)
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load 1 and dilation 2, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 

![Diagram showing the recursive embedding process](image-url)
Lemma:

$T(d)$ may be embedded into $HQ(d + 1)$ with load $1$ and dilation $2$, such that only one edge is stretched.

Proof: Recursive embedding of the double-rooted tree as a sub-graph of the $HQ$. 
Lemma:

$T(d)$ may be embedded into $DB(d + 1)$ with load 1 and dilation 1.

Proof: $f(w) \rightarrow 0^{d-|w|-1}1w$

- Show: Edge of the tree is placed to an edge of the DeBruijn.
- Edge of the tree: $w$ nach $wa$
- Placed to: $0^{n-|w|-1}1w$ and $0^{n-|w|-2}1wa$
- That is a shuffle or shuffle-exchange edge in the DeBruijn.
- Note: there is a second edge-disjoined tree in the DeBruijn.
Lemma:

$T(d)$ may be embedded into $DB(d + 1)$ with load 1 and dilation 1.

Proof: $f(w) \rightarrow 0^{d-|w|-1}1w$

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$T(d)$ into $DB(d + 1)$

**Lemma:**

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Lemma:

\( T(d) \) may be embedded into \( DB(d + 1) \) with load 1 and dilation 1.

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- Show: Edge of the tree is placed to an edge of the DeBruijn.
- Edge of the tree: \( w \) nach \( wa \)
- Placed to: \( 0^{n-|w|-1}1w \) and \( 0^{n-|w|-2}1wa \)
- That is a shuffle or shuffle-exchange edge in the DeBruijn.
- **Note:** there is a second edge-disjoined tree in the DeBruijn.
Lemma:

\[ T(d) \text{ into } DB(d + 1) \]

\[ T(d) \text{ may be embedded into } DB(d + 1) \text{ with load 1 and dilation 1.} \]

Proof: \( f(w) \rightarrow 0^{d-|w|-1}1w \)

- Show: Edge of the tree is placed to an edge of the DeBruijn.
- Edge of the tree: \( w \) nach \( wa \)
- Placed to: \( 0^{n-|w|-1}1w \) and \( 0^{n-|w|-2}1wa \)
- That is a shuffle or shuffle-exchange edge in the DeBruijn.
- Note: there is a second edge-disjoined tree in the DeBruijn.
Lemma:

$CCC(2d)$ may be embedded into $HQ(2d + \lceil \log 2d \rceil)$ with load 1 and dilation 1.

Proof:
Lemma:

$\text{CCC}(2d)$ may be embedded into $\text{HQ}(2d + \lceil \log 2d \rceil)$ with load 1 and dilation 1.

Proof: Embedd the cycles into sub-cubes.
CCC(4) into HQ (Example)
CCC(4) into HQ (Example)
CCC(4) into HQ (Example)
CCC(4) into HQ (Example)
CCC(4) into HQ (Example)
Steps of the Proof:

- **Embedd the cycles of length** $2d$ **into the** $HQ(\lceil\log_2 2d\rceil)$.

- Use the recursive embedding of the cycle of length $2^{\lceil\log_2 d\rceil}$.

**Note:**

- IF $G$ is embedded in $H$ with dilation $k$ and
- if $G'$ is embedded in $H'$ with dilation $k'$, then we may
- embed $G \times G'$ in $H \times H'$ with dilation $\max(k, k')$.
- Holds due to the definition of the product of graphs.

Furthermore we have: $CCC(2d)$ is a sub-graph of $C_{2d} \times HQ(2d)$.

Also we have: $C_{2d} \times HQ(2d)$ is sub-graph of $HQ(2d + \lceil\log_2 2d\rceil)$. 
CCC into HQ

Steps of the Proof:

- Embedd the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
- Use the recursive embedding of the cycle of length $2^{\lceil \log 2d \rceil}$.

Note:

- IF $G$ is embedded in $H$ with dilation $k$ and
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Steps of the Proof:

- Embed the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
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CCC into HQ

Steps of the Proof:

- Embed the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
- Use the recursive embedding of the cycle of length $2^{\lfloor \log 2d \rfloor}$.

Note:

- **IF** $G$ is embedded in $H$ with dilation $k$ and
- if $G'$ is embedded $H'$ with dilation $k'$, then we may
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Also we have: $C_{2d} \times HQ(2d)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$.
Steps of the Proof:

- Embed the cycles of length $2d$ into the $HQ([\log 2d])$.
- Use the recursive embedding of the cycle of length $2^{\lceil \log 2d \rceil}$.

Note:

- IF $G$ is embedded in $H$ with dilation $k$ and
- if $G'$ is embedded $H'$ with dilation $k'$, the we may
  - embed $G \times G'$ in $H \times H'$ with dilation $\max(k, k')$.
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Steps of the Proof:

- Embed the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
- Use the recursive embedding of the cycle of length $2^{\lceil \log 2d \rceil}$.

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- Embed the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
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CCC into HQ

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- Embed the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
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Steps of the Proof:

- Embedd the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
- Use the recursive embedding of the cycle of length $2 \lceil \log 2d \rceil$.

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- Embedd the cycles of length $2d$ into the $HQ(\lceil \log 2d \rceil)$.
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Furthermore we have: $CCC(2d)$ is a sub-graph of $C_{2d} \times HQ(2d)$.

Also we have: $C_{2d} \times HQ(2d)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$. 
CCC(3) into HQ (Example)
CCC(3) into HQ (Example)
CCC(3) into HQ (Example)
CCC(3) into HQ (Example)
CCC(3) into HQ (Example)
Lemma:

\( \text{CCC}(2d - 1) \) may be embedded into \( HQ(2d - 1 + \lceil \log 2d - 1 \rceil) \) with load 1 and dilation 2.

Proof:

- **Note:** \( \lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil \).
- We have: \( \text{CCC}(2d - 1) \) is sub-graph of \( C_{2d-1} \times HQ(2d - 1) \).
- Embedd \( C(2d - 1) \) with dilation 2 in \( C(2d) \).
- The we get: \( C_{2d-1} \times HQ(2d - 1) \) could be embedded with dilation 2 in \( C_{2d} \times HQ(2d - 1) \).
- Already known: \( C_{2d} \times HQ(2d) \) is sub-graph of \( HQ(2d + \lceil \log 2d \rceil) \).
- Thus we get: \( C_{2d} \times HQ(2d - 1) \) is sub-graph of \( HQ(2d + \lceil \log 2d \rceil) \).
Lemma:

$CCC(2d - 1)$ may be embedded into $HQ(2d - 1 + \lceil \log 2d - 1 \rceil)$ with load 1 and dilation 2.

Proof:

- Note: $\lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil$.
- We have: $CCC(2d - 1)$ is sub-graph of $C_{2d-1} \times HQ(2d - 1)$.
- Embedd $C(2d - 1)$ with dilation 2 in $C(2d)$.
- The we get: $C_{2d-1} \times HQ(2d - 1)$ could be embedded with dilation 2 in $C_{2d} \times HQ(2d - 1)$.
- Already known: $C_{2d} \times HQ(2d)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$.
- Thus we get: $C_{2d} \times HQ(2d - 1)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$.
Lemma:

$\text{CCC}(2d - 1)$ may be embedded into $\text{HQ}(2d - 1 + \lceil \log 2d - 1 \rceil)$ with load 1 and dilation 2.

Proof:

- Note: $\lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil$.
- We have: $\text{CCC}(2d - 1)$ is sub-graph of $C_{2d-1} \times \text{HQ}(2d - 1)$.
- Embedd $C(2d - 1)$ with dilation 2 in $C(2d)$.
- The we get: $C_{2d-1} \times \text{HQ}(2d - 1)$ could be embedded with dilation 2 in $C_{2d} \times \text{HQ}(2d - 1)$.
- Already known: $C_{2d} \times \text{HQ}(2d)$ is sub-graph of $\text{HQ}(2d + \lceil \log 2d \rceil)$.
- Thus we get: $C_{2d} \times \text{HQ}(2d - 1)$ is sub-graph of $\text{HQ}(2d + \lceil \log 2d \rceil)$. 
CCC into HQ

**Lemma:**

$CCC(2d - 1)$ may be embedded into $HQ(2d - 1 + \lceil \log 2d - 1 \rceil)$ with load 1 and dilation 2.

**Proof:**

- Note: $\lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil$.
- We have: $CCC(2d - 1)$ is sub-graph of $C_{2d-1} \times HQ(2d - 1)$.
- Embed $C(2d - 1)$ with dilation 2 in $C(2d)$.
- The we get: $C_{2d-1} \times HQ(2d - 1)$ could be embedded with dilation 2 in $C_{2d} \times HQ(2d - 1)$.
- Already known: $C_{2d} \times HQ(2d)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$.
- Thus we get: $C_{2d} \times HQ(2d - 1)$ is sub-graph of $HQ(2d + \lceil \log 2d \rceil)$. 
Lemma:

\[ \text{CCC}(2d - 1) \] may be embedded into \( \text{HQ}(2d - 1 + \lceil \log 2d - 1 \rceil) \) with load 1 and dilation 2.

Proof:

- Note: \( \lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil \).
- We have: \( \text{CCC}(2d - 1) \) is sub-graph of \( \text{C}_{2d-1} \times \text{HQ}(2d - 1) \).
- Embedd \( C(2d - 1) \) with dilation 2 in \( C(2d) \).
- The we get: \( C_{2d-1} \times \text{HQ}(2d - 1) \) could be embedded with dilation 2 in \( C_{2d} \times \text{HQ}(2d - 1) \).
- Already known: \( C_{2d} \times \text{HQ}(2d) \) is sub-graph of \( \text{HQ}(2d + \lceil \log 2d \rceil) \).
- Thus we get: \( C_{2d} \times \text{HQ}(2d - 1) \) is sub-graph of \( \text{HQ}(2d + \lceil \log 2d \rceil) \).
Lemma:

**CCC**\((2d - 1)\) may be embedded into **HQ**\((2d - 1 + \lceil \log 2d - 1 \rceil)\) with load 1 and dilation 2.

Proof:

- **Note:** \(\lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil\).
- **We have:** **CCC**\((2d - 1)\) is sub-graph of **C**\(_{2d-1} \times HQ(2d - 1)\).
- **Embedd** **C**\((2d - 1)\) with dilation 2 in **C**\((2d)\).
- **The we get:** **C**\(_{2d-1} \times HQ(2d - 1)\) could be embedded with dilation 2 in **C**\(_{2d} \times HQ(2d - 1)\).
- **Already known:** **C**\(_{2d} \times HQ(2d)\) is sub-graph of **HQ**\((2d + \lceil \log 2d \rceil)\).
- **Thus we get:** **C**\(_{2d} \times HQ(2d - 1)\) is sub-graph of **HQ**\((2d + \lceil \log 2d \rceil)\).
Lemma:

\( \text{CCC}(2d - 1) \) may be embedded into \( \text{HQ}(2d - 1 + \lceil \log 2d - 1 \rceil) \) with load 1 and dilation 2.

Proof:

- Note: \( \lceil \log 2d \rceil = \lceil \log 2d - 1 \rceil \).
- We have: \( \text{CCC}(2d - 1) \) is sub-graph of \( C_{2d-1} \times \text{HQ}(2d - 1) \).
- Embedd \( C(2d - 1) \) with dilation 2 in \( C(2d) \).
- The we get: \( C_{2d-1} \times \text{HQ}(2d - 1) \) could be embedded with dilation 2 in \( C_{2d} \times \text{HQ}(2d - 1) \).
- Already known: \( C_{2d} \times \text{HQ}(2d) \) is sub-graph of \( \text{HQ}(2d + \lceil \log 2d \rceil) \).
- Thus we get: \( C_{2d} \times \text{HQ}(2d - 1) \) is sub-graph of \( \text{HQ}(2d + \lceil \log 2d \rceil) \).
Lemma:

$BF(d)$ may be embedded into $HQ(d + \lceil \log d \rceil)$ with load 1 and dilation 2.

Proof:

- Embed $BF(d)$ in $CCC(d)$ with dilation 2 (trivial).
- Embed $CCC(d)$ in $HQ(d + \lceil \log d \rceil)$ with dilation 1.
**Lemma:**

$BF(d)$ may be embedded into $HQ(d + \lceil \log d \rceil)$ with load 1 and dilation 2.

**Proof:**

- Embed $BF(d)$ in $CCC(d)$ with dilation 2 (trivial).
- Embed $CCC(d)$ in $HQ(d + \lceil \log d \rceil)$ with dilation 1. (Not explained in image, but implied by the structure of the diagram.)
Lemma:

\( BF(d) \) may be embedded into \( HQ(d + \lceil \log d \rceil) \) with load 1 and dilation 2.

Proof:

- Embed \( BF(d) \) in \( CCC(d) \) with dilation 2 (trivial).
- Embed \( CCC(d) \) in \( HQ(d + \lceil \log d \rceil) \) with dilation 1.
Lemma:

$BF(2d)$ may be embedded into $HQ(2d + \lceil \log 2d \rceil)$ with load 1 and dilation 1.
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
Intro:

**BF(4) in HQ (Beispiel)**

```
0000 0001 0010 0011 0100 0101 0110 0111 1000 1001 1010 1011 1100 1101 1110 1111
```

Diagram with nodes labeled with binary numbers.
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
BF(4) in HQ (Beispiel)
Steps of the Proof:

- Embed cycle $C_{2d}$ into $HQ(\lceil \log 2d \rceil)$ as a subgraph by some function $f_C$.
- Embed $BF_{2d}$ into $HQ(2d + \lceil \log 2d \rceil)$:
  \[
  (i, w) \mapsto f_{2d}(i)w
  \]
- Assume that $(i, w)$ is now embedded onto $cw$ for $0 \leq i < 2d$ and $w \in \{0, 1\}^{2d}$.
- For $i$ from 0 to $2d - 1$ do the following:
  - Let $i' = (i + 1) \mod 2d$.
  - Exchange now node of the form $(i, w)$ with $(i', w)$ for $w = w'1w''$ with $|w'| = i$.
  - Let $t = f_{2d}(i) \oplus f_{2d}(i')$.
  - Let $cw'1w''$ be a node of the hypercube.
  - The move $cw'1w''$ to $(c \oplus t)w'1w''$.
  - Note, the dilation is not enlarged for any edge.
  - The edges of the form $\{(i, w'0w''), (i', w'1w'')\}$ have now a dilation of 1.
BF into HQ

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- Embed cycle $C_{2d}$ into $HQ(\lceil \log 2d \rceil)$ as a subgraph by some function $f_C$.
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Lemma:

$CCC(d)$ may be embedded into $BF(d)$ with load 1 and dilation 1.
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- Let \( P(w) := \#_{1}(w) \mod 2 \).
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Consider the edges on the cycles: \( \{(i, w), ((i + 1) \mod d, w)\} \):

- \( w_i \) has the \( i^{th} \) bit of \( w \) flipped.
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  - \( f(i, w) = ((i + 1) \mod d, w) \) if \( P(w) = 1 \).
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CCC into BF

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Lemma:

\( SE(d) \) may be embedded into \( DB(d) \) with load 1 and dilation 1.

Proof: Exercise
Lemma:

$DB(d)$ may be embedded into $HQ(d)$ with load 1 and dilation $\lceil d/4 \rceil$.

Proof:

- Consider edge in DB: $aw \leftrightarrow wb$.
- Split the node-strings into blocks: $awa'w' \leftrightarrow wbw'b'$ with $b = a'$.
- This makes small virtual DeBruijn within the original DeBruijn.
- Each virtual part is embedded in a hyper-cubes.
- The dilation sums up during this process.
- The proof is done by embedding the $DB(8)$ into the $HQ(8)$ with dilation 2.
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DB and HQ

**Lemma:**

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Lemma:

DB(d) may be embedded into HQ(d) with load 1 and dilation ⌈d/4⌉.

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Torus and Hypercube

Lemma:

\[ G(n_1, n_2, \cdots, n_t) \] may be embedded into \( HQ(d) \) with load 1 and dilation 1, iff
\[ d \geq \sum_{i=1}^{t} \lceil \log n_i \rceil. \]

Proof:

- Check the dimension-changes of the edges of the grid:
- In each square are precisely 2 dimensions.
- Thus each path of the form \( L(n_i) \) has to be embedded into a sub-cube.

Lemma:

\[ TR(n_1, n_2, \cdots, n_t) \] may be embedded into \( HQ(d) \) with load 1 and dilation 1, iff
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Theorem:
A binary tree may be embedded with dilation 3 and expansion 8 into the Hypercube.

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A binary tree may be embedded with dilation 7 and expansion 1 into the Hypercube.
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Caterpillars

**Definition:**

A binary tree is called caterpillar, iff all nodes with degree 3 are on a simple path.
The hair-length denotes the distance of the nodes to the path.

**Definition:**

A graph $G$ is called balanced, iff there exists a 2-colouring of $G$, which has as many red nodes as black nodes.
Caterpillars

Theorem:

Balanced caterpillars with hair-length 1 are sub-graphs of the hypercube.

Idea of proof: Cut the caterpillar in two balanced pieces.

Theorem:

Caterpillars with $4 \cdot n$ nodes may be embedded with congestion 1 and load 1 into $G(2, 2, n)$.

Proof: Embedd step by step 4 nodes of the caterpillar into the grid.
Caterpillars

Theorem:
Balanced caterpillars with hair-length 1 are sub-graphs of the hypercube.

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Caterpillars with $4 \cdot n$ nodes may be embedded with congestion 1 and load 1 into $G(2, 2, n)$.

Proof: Embedd step by step 4 nodes of the caterpillar into the grid.
Embedding-Problem

**Definition:**

Given: $G, H$ graphs and $d, c, l \in \mathbb{N}$. Questions: Could $G$ be embedded into $H$ with dilation $d$, load $l$ and congestion $c$. 
Embedding-Problem

Theorem:
The embedding-problem is NP-complete into the following cases:

- $G$ is a cycle, $d = c = l = 1$ and $H$ has the same number of nodes as $G$.
- $G, H$ arbitrary, $d$ a constant, $l = 1$, $c$ arbitrary.
- $G, H$ arbitrary, $c$ a constant, $l = 1$, $d$ arbitrary.
- $G, H$ arbitrary, $d, c, l$ constants.
- $G$ a balanced tree, $H$ a hyper-cube, $d = l = 1$.
- $G$ arbitrary, $H$ a path, $d$ a constant, $l = 1$, $c$ arbitrary.
- $G$ a tree, $H$ a path, $d$ a constant, $l = 1$, $c$ arbitrary.
- $G$ a caterpillar, $H$ a path, $d$ a constant, $l = 1$, $c$ arbitrary.
The Technic

- **Optical Fibers**
- Optical Sender
- Optical Receiver
- Optical Amplifiers
- Wavelengths: 1450–1650 nm (Nanometer)
- C-Band: 1530–1565 nm (currently used)
- L-Band: 1565–1625 nm (used soon)
- Width of a channel: about 10 GHz.
- Distance between channels: about 100 GHz.
- About 80 channels in the C-Band.
- With a channel-distance of 25 GHz about 200 channels in the C-Band
- Critical Angle: $\sin^{-1} \frac{\mu_2}{\mu_1}$
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Advantages and Disadvantages

- **High transfer-rate:**
  - Currently: 107 Gigabit per second.
  - Theoretical $50 \cdot 10^{12}$ bits per second.

- Low signal-loss: 0.2 db/km.
- Signal is not changed a lot (less jitter).
- Not so many optical Amplifiers are used.
- Less energy, space and less cost for the material.
- More channels per fiber.
- Less disturbance by other signals.
- Fast signal distribution.
- Low cost.

- Optical Devices are expensive (or not developed so far)
- Detour via electronic devices.
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- **Fast signal distribution.**
- **Low cost.**

- **Optical Devices are expensive (or not developed so far).**
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Advantages and Disadvantages

- High transfer-rate:
  - Currently: 107 Gigabit per second.
  - Theoretical $50 \cdot 10^{12}$ bits per second.
- Low signal-loss: 0.2 db/km.
- Signal is not changed a lot (less jitter).
- Not so many optical Amplifiers are used.
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Types of WDM and Problems

- **Types of WDM**
  - Wavelength-routed Networks: the receiver determines the wavelength statically.
  - Broadcasting Networks: Send with wavelength $\lambda$ to all. Only the receivers use $\lambda$ as input wavelength.
  - Static and dynamic optical paths.
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Optical Coupler

- Optical coupler has value $\alpha$.
- If input $l_i$ receives a signal of strength $P_i$,
  - then outputs $O_0 \alpha \cdot P_0$ and $O_1 (1 - \alpha) \cdot P_1$.
- Exists independent of the wavelength and dependent of the wavelength.

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Theorem

A crossbar is “wide-sense nonblocking”, i.e. any permutation and any extension to a sub-permutation is possible.
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**Theorem**

The Beneš Network is “nonblocking”, i.e. any permutation is possible.
The Beneš Network is nonblocking

- Each path $i$ has to traverse one of the sub-networks.
- Common inputs $2 \cdot i$ and $2 \cdot i - 1$ may not use the same sub-network.
- Common inputs $\pi(2 \cdot i)$ and $\pi(2 \cdot i - 1)$ may not use the same sub-network.
- The resulting conflict graph is bipartite (Sum of two Matchings).

Thus the paths may be placed on the two sub-networks.

The statement holds by a simple induction.
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Introduction

Input

- **Network**: \( G = (V, E) \)
- **Requests**: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
- **Routes**: \( \rho_1, \rho_2, \rho_3, \ldots \) paths from \( s_i \) to \( d_i \).

Routing

For the above input is a routing \( \mathcal{R} \):

- \( \mathcal{R} = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_q\} \) and
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is the colouring of the conflict-graph \( G^I_{\mathcal{R}} \):

- \( G^I_{\mathcal{R}} = (\mathcal{R}, F) \overset{\Delta}{=} (I, F) \) mit: \( F = \{\{\rho_i, \rho_j\} \mid \rho_i \cap \rho_j \cap E \neq \emptyset\} \)

- Each request is assigned a wavelength.
- If two request share an edge (in the same direction), then differ the wavelengths.
- \( w(G^I_{\mathcal{R}}) \) is the number of necessary wavelengths.
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- Requests: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
- Routing: \( \mathcal{R} = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_q\} \)

**Wavelength-Assignment**

is the colouring of the conflict-graph \( G^I_{\mathcal{R}} \):

- \( G^I_{\mathcal{R}} = (\mathcal{R}, F) \) mit: \( F = \{\{\rho_i, \rho_j\} \mid \rho_i \cap \rho_j \cap E \neq \emptyset\} \)
- Each request is assigned a wavelength.
- If two request share an edge (in the same direction), then differ the wavelengths.
- \( w(G^I_{\mathcal{R}}) \) is the number of necessary wavelengths.
Wavelength-Assignment

Input

- **Network**: \( G = (V, E) \)
- **Requests**: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
- **Routing**: \( \mathcal{R} = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_q\} \)

Wavelength-Assignment

is the colouring of the conflict-graph \( G^I_{\mathcal{R}} \):

- \( G^I_{\mathcal{R}} = (\mathcal{R}, F) \hat{=} (I, F) \) mit: \( F = \{\{\rho_i, \rho_j\} \mid \rho_i \cap \rho_j \cap E \neq \emptyset\} \)
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Wavelength-Assignment

**Input**
- Network: \( G = (V, E) \)
- Requests: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
- Routing: \( R = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_q\} \)

**Wavelength-Assignment**

is the colouring of the conflict-graph \( G^I_R \):
- \( G^I_R = (R, F) \hat{=} (I, F) \) mit: \( F = \{\{\rho_i, \rho_j\} \mid \rho_i \cap \rho_j \cap E \neq \emptyset\} \)
- Each request is assigned a wavelength.
- If two request share an edge (in the same direction), then differ the wavelengths.
- \( w(G^I_R) \) is the number of necessary wavelengths.
Definition

Given:

- Network: \( G = (V, E) \)
- Requests: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
- Routing: \( R = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_q\} \)

Then we define:

- The congestion of an edge \( e \) the number of routing-paths which use \( e \).
- \( c_e(G_R^I) = |\{r \in R \mid e \in r\}|. \)
- \( c(G_R^I) = \max_{e \in E} c_e(G_R^I). \)

Lemma

We have: \( c(G_R^I) \leq w(G_R^I). \)
Congestion

Definition

Given:
- Network: $G = (V, E)$
- Requests: $I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\}$
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Then we define:
- The congestion of an edge $e$ the number of routing-paths which use $e$.
  $$c_e(G^I_R) = |\{r \in R \mid e \in r\}|.$$
- $$c(G^I_R) = \max_{e \in E} c_e(G^I_R).$$

Lemma

We have: $c(G^I_R) \leq w(G^I_R)$. 
**Congestion**

**Definition**

Given:
- Network: \( G = (V, E) \)
- Requests: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
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Then we define:
- The congestion of an edge \( e \) the number of routing-paths which use \( e \).
- \( c_e(G^l_R) = |\{r \in R \mid e \in r\}|. \)
- \( c(G^l_R) = \max_{e \in E} c_e(G^l_R). \)

**Lemma**

We have: \( c(G^l_R) \leq w(G^l_R). \)
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**Definition**

Given:
- **Network**: \( G = (V, E) \)
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Then we define:
- The congestion of an edge \( e \) the number of routing-paths which use \( e \).
- \( c_e(G^I_\mathcal{R}) = |\{ r \in \mathcal{R} \mid e \in r \}| \).
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**Definition**

Given:

- Network: \( G = (V, E) \)
- Requests: \( I = \{(s_i, d_i) \mid s_i, d_i \in V \land 1 \leq i \leq q\} \)
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Then we define:

- The congestion of an edge \( e \) the number of routing-paths which use \( e \).
  - \( c_e(G^l_{\mathcal{R}}) = |\{r \in \mathcal{R} \mid e \in r\}| \).
- \( c(G^l_{\mathcal{R}}) = \max_{e \in E} c_e(G^l_{\mathcal{R}}) \).

**Lemma**

We have: \( c(G^l_{\mathcal{R}}) \leq w(G^l_{\mathcal{R}}) \).
Theorem

Let $L$ be the maximal length of a routing-path in $G^l_R$.

- Then we have: $w(G^l_R) \leq (c(G^l_R) - 1) \cdot L + 1$
- Is also the bound for the simple greedy algorithm.

Proof: The node degree in the conflict-graph is at most: $(c(G^l_R) - 1) \cdot L$. 
Greedy

**Theorem**

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Proof: The node degree in the conflict-graph is at most: $(c(G^I_\mathcal{R}) - 1) \cdot L$. 
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**Theorem**

Let $L$ be the maximal length of a routing-path in $G_{\mathcal{R}}^l$.

- Then we have: $w(G_{\mathcal{R}}^l) \leq (c(G_{\mathcal{R}}^l) - 1) \cdot L + 1$
- Is also the bound for the simple greedy algorithm.

Proof: The node degree in the conflict-graph is at most: $(c(G_{\mathcal{R}}^l) - 1) \cdot L$. 
Greedy improved

- Let $G^I_R$ be given.
- Let $\mathcal{R}_1$ be the paths of length $\geq \sqrt{|E|}$.
- Let $\mathcal{R}_2$ be the paths of length $< \sqrt{|E|}$.
- Colour each path in $\mathcal{R}_1$ with its own colour.
- Colour $\mathcal{R}_2$ with greed.

**Theorem**

We have: $w(G^I_R) \leq 2 \cdot \sqrt{|E|} \cdot c(G^I_R)$.

**Proof:**

- $|\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G^I_R)$, because otherwise we would have an edge $e$ with $c_e(G^I_R) > c(G^I_R)$.
- And $w(G^I_{\mathcal{R}_2}) \leq \sqrt{|E|} \cdot c(G^I_R)$ is easy.
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Greedy improved

- Let $G^l_R$ be given.
- Let $R_1$ be the paths of length $\geq \sqrt{|E|}$.
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We have: $w(G^l_R) \leq 2 \cdot \sqrt{|E|} \cdot c(G^l_R)$.

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- $|\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G^l_R)$, because
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- And $w(G^l_{R_2}) \leq \sqrt{|E|} \cdot c(G^l_R)$ is easy.
Greedy improved

- Let \( G_R^I \) be given.
- Let \( \mathcal{R}_1 \) be the paths of length \( \geq \sqrt{|E|} \).
- Let \( \mathcal{R}_2 \) be the paths of length \( < \sqrt{|E|} \).
- Colour each path in \( \mathcal{R}_1 \) with its own colour.
- Colour \( \mathcal{R}_2 \) with greed.

**Theorem**

We have: \( w(G_R^I) \leq 2 \cdot \sqrt{|E|} \cdot c(G_R^I) \).

**Proof:**

- \( |\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G_R^I) \), because
- otherwise we would have an edge \( e \) with \( c_e(G_R^I) > c(G_R^I) \).
- And \( w(G_{R_2}^I) \leq \sqrt{|E|} \cdot c(G_R^I) \) is easy.
Greedy improved

- Let $G^l_R$ be given.
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**Theorem**

*We have: $w(G^l_R) \leq 2 \cdot \sqrt{|E|} \cdot c(G^l_R)$.*

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- $|\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G^l_R)$, because
- otherwise we would have an edge $e$ with $c_e(G^l_R) > c(G^l_R)$.
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- Let $G^I_R$ be given.
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- $|\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G^l_R)$, because
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- And $w(G^l_{\mathcal{R}_2}) \leq \sqrt{|E|} \cdot c(G^l_R)$ is easy.
Greedy improved

- Let $G_R^I$ be given.
- Let $R_1$ be the paths of length $\geq \sqrt{|E|}$.
- Let $R_2$ be the paths of length $< \sqrt{|E|}$.
- Colour each path in $R_1$ with its own colour.
- Colour $R_2$ with greed.

**Theorem**

$We have: w(G_R^I) \leq 2 \cdot \sqrt{|E|} \cdot c(G_R^I).$

**Proof:**

- $|R_1| \leq \sqrt{|E|} \cdot c(G_R^I)$, because
- otherwise we would have an edge $e$ with $c_e(G_R^I) > c(G_R^I)$.
- And $w(G_R^I) \leq \sqrt{|E|} \cdot c(G_R^I)$ is easy.
Greedy improved

- Let $G^{l}_R$ be given.
- Let $\mathcal{R}_1$ be the paths of length $\geq \sqrt{|E|}$.
- Let $\mathcal{R}_2$ be the paths of length $< \sqrt{|E|}$.
- Colour each path in $\mathcal{R}_1$ with its own colour.
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**Theorem**

*We have: $w(G^{l}_R) \leq 2 \cdot \sqrt{|E|} \cdot c(G^{l}_R)$.***

**Proof:**

- $|\mathcal{R}_1| \leq \sqrt{|E|} \cdot c(G^{l}_R)$, because
- otherwise we would have an edge $e$ with $c_e(G^{l}_R) > c(G^{l}_R)$.
- And $w(G^{l}_{R_2}) \leq \sqrt{|E|} \cdot c(G^{l}_R)$ is easy.
**Theorem**

*If $G$ is a line, then we can compute $w(G^l_I)$ in polynomial time.*

**Proof:**

- Let $I_l$ be the requests going to the left.
- Let $I_r$ be the requests going to the right.
- $I_l$ and $I_r$ are independent.
- $w(G^l_I)$ corresponds to the colouring of an interval-graph.
- $w(G^l_I)$ corresponds to the colouring of an interval-graph.
Theorem

If $G$ is a line, then we can compute $w(G^l_R)$ in polynomial time.

Proof:

- Let $I_l$ be the requests going to the left.
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Line

**Theorem**

*If G is a line, then we can compute \( w(G_{IR}^l) \) in polynomial time.*

**Proof:**

- Let \( I_l \) be the requests going to the left.
- Let \( I_r \) be the requests going to the right.
- \( I_l \) and \( I_r \) are independent.
- \( w(G_{IR}^l) \) corresponds to the colouring of an interval-graph.
- \( w(G_{IR}^r) \) corresponds to the colouring of an interval-graph.
Theorem

If $G$ is a line, then we can compute $w(G^l_R)$ in polynomial time.

Proof:

- Let $I_l$ be the requests going to the left.
- Let $I_r$ be the requests going to the right.
- $I_l$ and $I_r$ are independent.
- $w(G^l_R)$ corresponds to the colouring of an interval-graph.
- $w(G^r_R)$ corresponds to the colouring of an interval-graph.
If $G$ is a cycle, then we can approximate $w(G^I_R)$ in polynomial time with a factor of 2.

Proof:

- Let $e$ be an edge in $G$.
- Let $I_1$ be the requests which use $e$ in the routing.
- Let $I_2$ be the requests which do not use $e$ in the routing.
- $w(G^I_R)$ corresponds to the colouring of an interval-graph.
- $w(G^b_R)$ corresponds to the colouring of an interval-graph.

If $G$ is a cycle, then the computation of $w(G^I_R)$ is NP-complete.

Proof:

- $w(G^I_R)$ corresponds to the colouring of an arc-graph.
Cycle

Theorem

If $G$ is a cycle, then we can approximate $w(G^l_R)$ in polynomial time with a factor of 2.

Proof:

- Let $e$ be an edge in $G$.
- Let $l_1$ be the requests which use $e$ in the routing.
- Let $l_2$ be the requests which do not use $e$ in the routing.
- $w(G^l_R)$ corresponds to the colouring of an interval-graph.
- $w(G^b_R)$ corresponds to the colouring of an interval-graph.

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**Theorem**

*If G is a cycle, then we can approximate \( w(G^{l}_{R}) \) in polynomial time with a factor of 2.*

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- Let \( I_{1} \) be the requests which use \( e \) in the routing.
- Let \( I_{2} \) be the requests which do not use \( e \) in the routing.
- \( w(G^{l}_{R}) \) corresponds to the colouring of an interval-graph.
- \( w(G^{b}_{R}) \) corresponds to the colouring of an interval-graph.

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- Let $e$ be an edge in $G$.
- Let $I_1$ be the requests which use $e$ in the routing.
- Let $I_2$ be the requests which do not use $e$ in the routing.
- $w(G^l_R)$ corresponds to the colouring of an interval-graph.
- $w(G^b_R)$ corresponds to the colouring of an interval-graph.

**Theorem**

If $G$ is a cycle, then the computation of $w(G^l_R)$ is NP-complete.

**Proof:**

- $w(G^l_R)$ corresponds to the colouring of an arc-graph.
**Theorem**

*If G is a cycle, then we can approximate \( w(G_R^I) \) in polynomial time with a factor of 2.*

**Proof:**

- Let \( e \) be an edge in \( G \).
- Let \( I_1 \) be the requests which use \( e \) in the routing.
- Let \( I_2 \) be the requests which do not use \( e \) in the routing.
- \( w(G^I_R) \) corresponds to the colouring of an interval-graph.
- \( w(G^b_R) \) corresponds to the colouring of an interval-graph.

**Theorem**

*If G is a cycle, then the computation of \( w(G_R^I) \) is NP-complete.*

**Proof:**

- \( w(G_R^I) \) corresponds to the colouring of an arc-graph.
Star

Theorem

*If G is a star, then we can compute w(G^I_R) in polynomial time.*

Proof:

- Let G = (\{0, 1, \ldots, n\}, E) be the star with central node 0.
- Let H = (\{s_1, s_2, \ldots s_n\}, \{d_1, d_2, \ldots d_n\}, F) be a bipartite graph,
- with: F = \{(s_i, d_j) \mid (i, j) \in I\}
- Computing of w(G^I_R) corresponds to the edge-colouring of H.
- Request of the form 0, i and i, 0 may be coloured later by greed.
Star

Theorem

*If G is a star, then we can compute $w(G_{IR}^I)$ in polynomial time.*

Proof:

- Let $G = (\{0, 1, \ldots, n\}, E)$ be the star with central node 0.
- Let $H = (\{s_1, s_2, \ldots s_n\}, \{d_1, d_2, \ldots d_n\}, F)$ be a bipartite graph,
- with: $F = \{(s_i, d_j) \mid (i, j) \in I\}$
- Computing of $w(G_{IR}^I)$ corresponds to the edge-colouring of $H$.
- Request of the form 0, $i$ and $i, 0$ may be coloured later by greed.
Theorem

If $G$ is a star, then we can compute $w(G^I_R)$ in polynomial time.

Proof:

- Let $G = (\{0, 1, \ldots, n\}, E)$ be the star with central node $0$.
- Let $H = (\{s_1, s_2, \ldots s_n\}, \{d_1, d_2, \ldots d_n\}, F)$ be a bipartite graph,
- with: $F = \{(s_i, d_j) \mid (i, j) \in I\}$
- Computing of $w(G^I_R)$ corresponds to the edge-colouring of $H$.
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**Theorem**

*If G is a star, then we can compute \( w(G^I_R) \) in polynomial time.*

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- Request of the form \( 0, i \) and \( i, 0 \) may be coloured later by greed.
Star

Theorem

If $G$ is a star, then we can compute $w(G^I_R)$ in polynomial time.

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- Let $G = (\{0, 1, \ldots, n\}, E)$ be the star with central node 0.
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- Computing of $w(G^I_R)$ corresponds to the edge-colouring of $H$.
- Request of the form 0, $i$ and $i, 0$ may be coloured later by greed.
**Theorem**

*If G is a star, then we can compute \( w(G^l_R) \) in polynomial time.*

**Proof:**

- Let \( G = (\{0, 1, \ldots, n\}, E) \) be the star with central node 0.
- Let \( H = (\{s_1, s_2, \ldots, s_n\}, \{d_1, d_2, \ldots, d_n\}, F) \) be a bipartite graph,
- with: \( F = \{(s_i, d_j) \mid (i, j) \in I\} \)
- Computing of \( w(G^l_R) \) corresponds to the edge-colouring of \( H \).
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Theorem

If $G$ is a star, then we can compute $w(G^I_R)$ in polynomial time.

Proof:

- Let $G = (\{0, 1, \ldots, n\}, E)$ be the star with central node 0.
- Let $H = (\{s_1, s_2, \ldots s_n\}, \{d_1, d_2, \ldots d_n\}, F)$ be a bipartite graph,
  with: $F = \{(s_i, d_j) \mid (i, j) \in I\}$
- Computing of $w(G^I_R)$ corresponds to the edge-colouring of $H$.
- Request of the form $0, i$ and $i, 0$ may be coloured later by greed.
Theorem

If $G$ is a spider-graph, then we can compute $w(G^l_R)$ in polynomial time.

Proof:

- Colour first the center star.
- Extend the colouring on each leg of the spider-graph by using the algorithm for paths.
**Theorem**

*If* \( G \) *is a spider-graph, then we can compute* \( w(G^I_R) \) *in polynomial time.*

**Proof:**

- **Colour first the center star.**
- **Extend the colouring on each leg of the spider-graph by using the algorithm for paths.**
Spider-Graph

Theorem

If $G$ is a spider-graph, then we can compute $w(G^l_R)$ in polynomial time.

Proof:

- Colour first the center star.

- Extend the colouring on each leg of the spider-graph by using the algorithm for paths.
Theorem

If $G$ is a spider-graph, then we can compute $w(G^l_R)$ in polynomial time.

Proof:

- Colour first the center star.
- Extend the colouring on each leg of the spider-graph by using the algorithm for paths.
Baum

**Theorem**

If $G$ is a tree, then the computation of $w(G^I_R)$ is NP-complete.

**Proof:**

- $w(G^I_R)$ corresponds to the colouring of an EPT-Graph.
Baum

Theorem

If $G$ is a tree, then the computation of $w(G^I_R)$ is NP-complete.

Proof:

- $w(G^I_R)$ corresponds to the colouring of an EPT-Graph.
Baum

Theorem

If $G$ is a tree, then the computation of $w(G^I_R)$ is NP-complete.

Proof:

- $w(G^I_R)$ corresponds to the colouring of an EPT-Graph.
If the requests are of type broadcast, then the wavelength-assignment becomes easy.

- We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).
- There are \( |V| - 1 \) nodes to be informed from \( v \).
- There have to be \( |V| - 1 \) paths starting in \( v \).
- Let \( d(w) \) be the out-degree of node \( w \in V \).
- Let \( d_{min}(G) = \min_{w \in V} d(w) \).
- At least \( (|V| - 1)/d(v) \) requests use the same edge of \( v \).
- Thus we have: \( w(G_R^I) \geq \lceil (|V| - 1)/d_{min}(G) \rceil \).
Broadcast

- If the requests are of type broadcast, then the wavelength-assignment becomes easy.

- **We have:** \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

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- Let \( d(w) \) be the out-degree of node \( w \in V \).

- Let \( d_{\text{min}}(G) = \min_{w \in V} d(w) \).

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- Thus we have: \( w(G^I_R) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil \).
If the requests are of type broadcast, then the wavelength-assignment becomes easy.

We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

There are \( |V| - 1 \) nodes to be informed from \( v \).

There have to be \( |V| - 1 \) paths starting in \( v \).

Let \( d(w) \) be the out-degree of node \( w \in V \).

Let \( d_{\text{min}}(G) = \min_{w \in V} d(w) \).

At least \( (|V| - 1)/d(v) \) requests use the same edge of \( v \).

Thus we have: \( w(G^l_{\mathcal{R}}) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil \).
If the requests are of type broadcast, then the wavelength-assignment becomes easy.

We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

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Let \( d(w) \) be the out-degree of node \( w \in V \).

Let \( d_{\min}(G) = \min_{w \in V} d(w) \).

At least \( (|V| - 1)/d(v) \) requests use the same edge of \( v \).

Thus we have: \( w(G^I_R) \geq \lceil (|V| - 1)/d_{\min}(G) \rceil \).
Broadcast

- If the requests are of type broadcast, then the wavelength-assignment becomes easy.
- We have: $I = \{(v, w) \mid w \in V\}$ for a start node $v$.
- There are $|V| - 1$ nodes to be informed from $v$.
- There have to be $|V| - 1$ paths starting in $v$.
- Let $d(w)$ be the out-degree of node $w \in V$.
- Let $d_{\text{min}}(G) = \min_{w \in V} d(w)$.
- At least $(|V| - 1)/d(v)$ requests use the same edge of $v$.
- Thus we have: $w(G^I_R) \geq \lceil(|V| - 1)/d_{\text{min}}(G)\rceil$. 
Broadcast

- If the requests are of type broadcast, then the wavelength-assignment becomes easy.

- We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

- There are \(|V| - 1\) nodes to be informed from \( v \).

- There have to be \(|V| - 1\) paths starting in \( v \).

- Let \( d(w) \) be the out-degree of node \( w \in V \).

- Let \( d_{\text{min}}(G) = \min_{w \in V} d(w) \).

- At least \((|V| - 1)/d(v)\) requests use the same edge of \( v \).

- Thus we have: \( w(G^I) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil \).
If the requests are of type broadcast, then the wavelength-assignment becomes easy.

We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

There are \( |V| - 1 \) nodes to be informed from \( v \).

There have to be \( |V| - 1 \) paths starting in \( v \).

Let \( d(w) \) be the out-degree of node \( w \in V \).

Let \( d_{\text{min}}(G) = \min_{w \in V} d(w) \).

At least \( (|V| - 1)/d(v) \) requests use the same edge of \( v \).

Thus we have: \( w(G^I_R) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil \).
Broadcast

- If the requests are of type broadcast, then the wavelength-assignment becomes easy.

- We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

- There are \( |V| - 1 \) nodes to be informed from \( v \).

- There have to be \( |V| - 1 \) paths starting in \( v \).

- Let \( d(w) \) be the out-degree of node \( w \in V \).

- Let \( d_{\text{min}}(G) = \min_{w \in V} d(w) \).

- At least \((|V| - 1)/d(v)\) requests use the same edge of \( v \).

Thus we have: \( w(G^I_R) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil \).
If the requests are of type broadcast, then the wavelength-assignment becomes easy.

We have: \( I = \{(v, w) \mid w \in V\} \) for a start node \( v \).

There are \(|V| - 1\) nodes to be informed from \( v \).

There have to be \(|V| - 1\) paths starting in \( v \).

Let \( d(w) \) be the out-degree of node \( w \in V \).

Let \( d_{\min}(G) = \min_{w \in V} d(w) \).

At least \(((|V| - 1)/d(v))\) requests use the same edge of \( v \).

Thus we have: \( w(G^I_C) \geq \lceil (|V| - 1)/d_{\min}(G) \rceil \).
Broadcast

**Theorem**

For an $k$ edge connected graph we have: $w(G^I_R) \leq \lceil (|V| - 1)/k \rceil$.

**Proof:**

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
- For each $i$ exist $k$ edge-disjoined paths from $v$ to $V_i$.
- Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Broadcast

**Theorem**

*For an $k$ edge connected graph we have: $w(G^l_R) \leq \lceil (|V| - 1)/k \rceil$.*

**Proof:**

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
  - For each $i$ exist $k$ edge-disjoined paths from $v$ to $V_i$.
  - Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Theorem

For an $k$ edge connected graph we have: $w(G^l_R) \leq \lceil((|V| - 1)/k)\rceil$.

Proof:

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil((|V| - 1)/k)\rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
  - For each $i$ exist $k$ edge-disjoined paths from $v$ to $V_i$.
  - Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil((|V| - 1)/k)\rceil$ colours used.
Theorem

For an $k$ edge connected graph we have: $w(G^t_R) \leq \lceil (|V| - 1)/k \rceil$.

Proof:

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
- For each $i$ exist $k$ edge-disjoined paths from $v$ to $V_i$.
- Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Broadcast

**Theorem**

For an \( k \) edge connected graph we have: \( w(\mathcal{G}_\mathcal{R}^l) \leq \lceil (|V| - 1)/k \rceil \).

**Proof:**

- Let \( v \) be the start-node.
- Split \( V \setminus \{v\} \) into \( s = \lceil (|V| - 1)/k \rceil \) subsets, with:
  - \( V_1, V_2, \ldots, V_s \) have a size of at most \( k \).
  - For each \( i \) exist \( k \) edge-disjoined paths from \( v \) to \( V_i \).
- Each \( V_i \) will be informed by using colour \( i \).
- In total are \( s = \lceil (|V| - 1)/k \rceil \) colours used.
**Theorem**

*For an $k$ edge connected graph we have: $w(G^I_k) \leq \lceil (|V| - 1)/k \rceil$.***

**Proof:**

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
  - For each $i$ exist $k$ edge-disjointed paths from $v$ to $V_i$.
  - Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Broadcast

**Theorem**

For an $k$ edge connected graph we have: $w(G^l_R) \leq \lceil (|V| - 1)/k \rceil$.

**Proof:**

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
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  - Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Broadcast

**Theorem**

For an $k$ edge connected graph we have: $w(G^I_R) \leq \lceil (|V| - 1)/k \rceil$.

**Proof:**

- Let $v$ be the start-node.
- Split $V \setminus \{v\}$ into $s = \lceil (|V| - 1)/k \rceil$ subsets, with:
  - $V_1, V_2, \ldots, V_s$ have a size of at most $k$.
  - For each $i$ exist $k$ edge-disjoined paths from $v$ to $V_i$.
  - Each $V_i$ will be informed by using colour $i$.
- In total are $s = \lceil (|V| - 1)/k \rceil$ colours used.
Broadcast

Theorem

For an $k$ edge connected graph we have: $w(G_{IR}^l) = \lceil (|V| - 1)/k \rceil$.

Proof:

- Known: $w(G_{IR}^l) \geq \lceil (|V| - 1)/d_{min}(G) \rceil$.
- Known: $w(G_{IR}^l) \leq \lceil (|V| - 1)/k \rceil$.
- Known: $k \leq d_{min} G$.
- Thus we have: $w(G_{IR}^l) = \lceil (|V| - 1)/k \rceil$. 
**Theorem**

For an $k$ edge connected graph we have: $w(G_R^i) = \lceil (|V| - 1)/k \rceil$.

**Proof:**

- **Known**: $w(G_R^i) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil$.
- **Known**: $w(G_R^i) \leq \lceil (|V| - 1)/k \rceil$.
- **Known**: $k \leq d_{\text{min}} G$.
- Thus we have: $w(G_R^i) = \lceil (|V| - 1)/k \rceil$. 
Broadcast

Theorem

For an $k$ edge connected graph we have: $w(G^I_R) = \lceil (|V| - 1)/k \rceil$.

Proof:

- Known: $w(G^I_R) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil$.
- Known: $w(G^I_R) \leq \lceil (|V| - 1)/k \rceil$.
- Known: $k \leq d_{\text{min}} G$.
- Thus we have: $w(G^I_R) = \lceil (|V| - 1)/k \rceil$. 
Broadcast

**Theorem**

*For an k edge connected graph we have: \( w(G^l_R) = \lceil (|V| - 1)/k \rceil. \)*

**Proof:**

- **Known:** \( w(G^l_R) \geq \lceil (|V| - 1)/d_{min}(G) \rceil. \)
- **Known:** \( w(G^l_R) \leq \lceil (|V| - 1)/k \rceil. \)
- **Known:** \( k \leq d_{min} G. \)
- Thus we have: \( w(G^l_R) = \lceil (|V| - 1)/k \rceil. \)
Broadcast

**Theorem**

*For an k edge connected graph we have:* \( w(G_{IR}^I) = \lceil (|V| - 1)/k \rceil. *

**Proof:**

- **Known:** \( w(G_{IR}^I) \geq \lceil (|V| - 1)/d_{\text{min}}(G) \rceil. 
- **Known:** \( w(G_{IR}^I) \leq \lceil (|V| - 1)/k \rceil. 
- **Known:** \( k \leq d_{\text{min}} G. 
- **Thus we have:** \( w(G_{IR}^I) = \lceil (|V| - 1)/k \rceil. 

Broadcast

**Theorem**

*For an $k$ edge connected graph we have: $w(G^l_R) = \lceil (|V| - 1)/k \rceil$.***

**Proof:**

- Known: $w(G^l_R) \geq \lceil (|V| - 1)/d_{min}(G) \rceil$.
- Known: $w(G^l_R) \leq \lceil (|V| - 1)/k \rceil$.
- Known: $k \leq d_{min} G$.
- Thus we have: $w(G^l_R) = \lceil (|V| - 1)/k \rceil$. 
More Results

Theorem

For the following graphs it is NP-complete to compute $w(G^{I}_{R_{min}})$:

- cycles,
- trees,
- binary trees and
- grids.
More Results

Theorem

For the following graphs it is NP-complete to compute $w(G_{R_{min}}^I)$:

- cycles,
- trees,
- binary trees and
- grids.
More Results

Theorem

For the following graphs it is NP-complete to compute $w(G_{\mathcal{R}_{\min}}^I)$:

- cycles,
- trees,
- binary trees and
- grids.
Theorem

For the following graphs it is NP-complete to compute $w(G^I_R_{min})$:

- cycles,
- trees,
- binary trees and
- grids.
More Results

Theorem

For the following graphs it is NP-complete to compute $w(G_{R_{\text{min}}})$:

- cycles,
- trees,
- binary trees and
- grids.
More Results

**Theorem**

Let $G^l_{\mathcal{R}_{\min}}$ given with $L = \max_{(x,y) \in I} \text{dist}(x,y)$. Then we have:

$$w(G^l_{\mathcal{R}}) = O(L \cdot c(G^l_{\mathcal{R}})).$$

**Theorem**

For each $L$ and $c$ there exists $G^l_{\mathcal{R}_{\min}}$ with: $L = \max_{(x,y) \in I} \text{dist}(x,y)$,

$$c = c(G^l_{\mathcal{R}_{\min}}),$$

$$w(G^l_{\mathcal{R}}) = \Omega(L \cdot c).$$

**Theorem**

Let $G^l_{\mathcal{R}_{\min}}$ given with $I$ is “one-to-many” communication. Then we have:

$$w(G^l_{\mathcal{R}}) = c(G^l_{\mathcal{R}}).$$
More Results

Theorem

Let \( G_{\mathcal{R}_{\text{min}}}^I \) given with \( L = \max_{(x,y) \in I} \text{dist}(x, y) \). Then we have:
\[
  w(G_{\mathcal{R}}^I) = O(L \cdot c(G_{\mathcal{R}}^I)).
\]

Theorem

For each \( L \) and \( c \) there exists \( G_{\mathcal{R}_{\text{min}}}^I \) with:
\[
  L = \max_{(x,y) \in I} \text{dist}(x, y),
  c = c(G_{\mathcal{R}_{\text{min}}}^I) \quad w(G_{\mathcal{R}}^I) = \Omega(L \cdot c).
\]

Theorem

Let \( G_{\mathcal{R}_{\text{min}}}^I \) given with \( I \) is “one-to-many” communication. Then we have:
\[
  w(G_{\mathcal{R}}^I) = c(G_{\mathcal{R}}^I).
\]
More Results

**Theorem**

Let $G^I_{\mathcal{R}_{\text{min}}}$ given with $L = \max_{(x,y)\in I} \text{dist}(x,y)$. Then we have:

$w(G^I_{\mathcal{R}}) = O(L \cdot c(G^I_{\mathcal{R}})).$

**Theorem**

For each $L$ and $c$ there exists $G^I_{\mathcal{R}_{\text{min}}}$ with: $L = \max_{(x,y)\in I} \text{dist}(x,y)$, $c = c(G^I_{\mathcal{R}_{\text{min}}})$ $w(G^I_{\mathcal{R}}) = \Omega(L \cdot c)$.

**Theorem**

Let $G^I_{\mathcal{R}_{\text{min}}}$ given with $I$ is “one-to-many” communication. Then we have:

$w(G^I_{\mathcal{R}}) = c(G^I_{\mathcal{R}}).$
Literature

Dissemination of Information in Optical Networks
From Technology to Algorithms
Questions

- Which problems are interesting for optical networks?
- For which is the Beneš Network used, what are it’s properties?
- What is the relation between wavelength-assignment and colouring a graph?
- How is the wavelength-assignment solved on the following graphs?
  - paths and cycles.
  - stars and spider-graphs.
- On which graphs is the wavelength-assignment hard?
- May the wavelength-assignment be solved if the connection structure is of type broadcast?
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- For which is the Beneš Network used, what are it’s properties?
- What is the relation between wavelength-assignment and colouring a graph?
- How is the wavelength-assignment solved on the following graphs?
  - paths and cycles.
  - stars and spider-graphs.
- On which graphs is the wavelength-assignment hard?
- May the wavelength-assignment be solved if the connection structure is of type broadcast?
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