Exercise 1 \hspace{1cm} (2+2+2 points)

For a graph \( G = (V, E) \), let \( mps(G) \) denote the maximum number of edges in a planar subgraph of \( G \).

- (a) Prove that \( cr(G) \geq |E| - mps(G) \).
- (b) Use the result in (a) to determine the crossing number of the Petersen graph.
- (c) Use the result in (a) to determine the crossing number of \( K_{4,4} \).

Exercise 2 \hspace{1cm} (4 points)

Prove or disprove: A graph \( G = (V, E) \) satisfies \( cr(G) \leq 1 \), if and only if there exists an edge \( e \in E \) so that \( G - e \) is planar.

Exercise 3 \hspace{1cm} (3+1 points)

- (a) Prove that for every \( n \geq 1 \), there exist \( 2n^3 \) points and \( n^3 \) lines in the plane, so that there are at least \( n^4 \) incidences between these points and lines. Hint: Consider the points \( (x,y) \) with \( x \in \{1, \ldots, n\} \) and \( y \in \{1, \ldots, 2n^2\} \), and the lines of the form \( y = kx + d \) with \( k \in \{1, \ldots, n\} \) and \( d \in \{1, \ldots, n^2\} \).
- (b) Deduce from (a) that the Szemerédi-Trotter bound is asymptotically tight.

Exercise 4 \hspace{1cm} (4 points)

Let \( P \) be a set of \( n \) points in the Euclidean plane, and let \( C \) be a set of \( n \) circles of unit radius. Prove that there are at most \( O(n^{4/3}) \) incidences between \( P \) and \( C \).