Motivation

1. There are limits to the computing power of a single Computer
2. Computers become cheaper
3. Specialized computers are expensive
4. There are tasks with large data
5. Many problems are very complex
   1. Weather and other Simulations
   2. Crash tests
   3. Military applications
   4. Large data: (SETI, ...)
   5. More similar problems
6. Thus there is the need for computers with more than one CPU
7. or a quantum computer?
Pipeline: (systolic array)

- There is a sequence of processors \((P_i)\) \(1 \leq i \leq n\).
- Processor \(P_1\) receives the input.
- Output of \(P_1\) will be passed as the input of \(P_2\).
- Output of \(P_i\) will be passed as the input of \(P_{i+1}\) \(1 \leq i < n\).
- Processor \(P_n\) delivers the final output.
- Processors may be different.
- Processors may run different programs.
- Intermediate outputs may be buffered.
- Pipelining is one important type of parallel system (in practice).
Systolic Arrays

- Idea: use more than one data stream.
- Data streams may interact with each other.
- Each processor is the same.
- There is a global synchronization.
- Processors may run simple programs.
- Advantage: really fast (for special applications).
Systolic Array with three data streams
Vector Computer

- Vector of processes.
- Each processor has different data.
- But each processor executes the same program.
- Addition of two vectors:
  1. Read vector $A$
  2. Read vector $B$
  3. Add (each processor)
  4. Output the summ

- Single Instruction Multiple Data SIMD-Computer.
- Aim: Multiple Instruction Multiple Data MIMD-Computer.
- I.e. Fast processors with fast connections.
Example: Transputer

- **Advantage:** very flexible, any fixed network of degree 4 possible.
- **Disadvantage:** long wires may be necessary, only a fixed network possible.
Beispiel: Transputer II
Parallele Computer I

- Advantage: “normal” CPUs.
- Advantage: fast links possible.
- Advantage: no special hardware.
- Advantage: variable network, may change during execution.
- Advantage: very large networks may be possible.
- Disadvantage: still a limited degree for the network.
- Disadvantage: large network are complicated.
- Problem: cooling large systems.
- Problem: fault tolerance.
- Problem: construct such a system.
- Problem: generate good data throughput with constant degree network.
- Problem: do the program structures fit the structure of the network.
Parallel Computer II (Goodput)

- Look for good networks.
- Trees, Grids, Pyramids, ...
- $HQ(n)$, $CCC(n)$, $BF(n)$, $SE(n)$, $DB(n)$, ...
- Pancake Network and Burned Pancake Network.
- Problem: Physical placement of the processors.
- Problem: Length of wires.
- Problem: Has the network a nice structure.
- If the network becomes too large, we may use efficiency.
- Solution: choose a mixed network structure.
Parallel Computer IV (Network)
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

   ![Diagram of CPU and memory as a single logical unit]

2. CPUs and memory are connected by a network:

   ![Diagram of CPUs and memory connected by a network]

The difference is more on the practical side.
PRAM (theoretical model)

- Ignore/unify the costs for each computation step.
- Ignore/unify the costs for each communication step.
Definition RAM

- RAM: Random Access Machine
- CPU may access any memory cell
- Memory is unlimited
- Complexity measurements
  - uniform: each operation cost one unit
  - logarithmic: cost are measured according to the size of the numbers
Idea of PRAM

- Many processes
- Common program
- Program may select single processors
- Common memory
Definition PRAM

- Consists of processors $P_i$ with $1 \leq i \leq p$ (prozessor has id $i$).
- Consists of registers $R_j$ with $1 \leq j \leq m$.
- Each processor has some local registers.
- Each processor $P_i$ may access each register $R_j$.
- Each processor executes the same program.
- The program is synchronized, thus each processor executes the same instructions.
- A selection is possible by using the processor id.
- The input of length $n$ is written to registers $R_j$ with $1 \leq j \leq n$.
- The output is placed in some known registers.
- The registers contain words (numbers) in the uniform cost measurement.
- The registers contain bits in the logarithmic cost measurement.
Definition PRAM

- The following instructions are possible:
  - 1. processor $P_i$ reads register $R_j$: $R_j \rightarrow P_i(x)$.
  - 2. processor $P_i$ writes value of $x$ into register $R_j$: $P_i(x) \rightarrow R_j$.
  - 3. processor may do some local computation using local registers:
      \[ x := y \ast 5. \]

- For the access to the register we have the following variations:
  - EREW Exclusive Read Exclusive Write
  - CREW Concurrent Read Exclusive Write
  - CRCW Concurrent Read Concurrent Write
  - ERCW Exclusive Read Concurrent Write

- Write conflicts may be solved using the following rules:
  - Arbitrary: any processor gets access to the register.
  - Common: all processors writing to the same register have to write the same value.
  - Priority: the processor with the smallest id gets access to the register.
Computation of an “Or” (Idea)

\[ x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \]

\[ 0 \lor 1 \lor 0 \lor 0 \lor 1 \lor 0 \lor 0 \lor 1 \rightarrow 1 \]
Computing an “Or”

- **Task:** Compute \( x = \bigvee_{i=1}^{n} x_i \).
- **Input:** \( x_i \) is in register \( R_i \) (\( 1 \leq i \leq n \)).
- **Output computed in** \( R_{n+1} \).
- **Model:** CRCW Arbitrary, Common oder Priority.
- **Programm:** Or
  
  ```plaintext
  for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel
  
  \( R_i \rightarrow P_i(x) \)
  
  if \( x = \text{true} \) then \( P_i(x) \rightarrow R_{n+1} \)
  ```

- **Running time:** \( O(1) \) (exact 2 steps).
- **Number of processors:** \( n \).
- **Memory:** \( n + 1 \).
- **Possible models:** ERCW (Arbitrary, Common oder Priority).
Computing an “Or” (EREW)

- **Problem:**
  no writing of two processors
to the same register
at the same time.

- **Idea:** combine pairwise the results

- **With this idea,** computing the sum is also possible.

- **Thus computing the “Or”** is just a special case of computing a sum.
Computing the Sum (Idea)
Computing the Sum (Idea)

\[
\begin{array}{cccc}
103 & 45 & 30 & 15 \\
\end{array}
\]

\[
\begin{array}{cccc}
P_1 & P_2 & P_3 & P_4 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
12 & 6 & 34 & 5 & 7 & 23 & 4 & 11 \\
\end{array}
\]
Computing the sum (EREW)

- Task: compute \( x = \sum_{i=1}^{n} x_i \) with \( n = 2^k \).
- Input: \( x_i \) is in register \( R_i \) (\( 1 \leq i \leq n \)).
- Output: should be in \( R_1 \) (input may be overwritten).
- Model: EREW.

Programm: Summe

\[
\text{for all } P_i \text{ where } 1 \leq i \leq n/2 \text{ do in parallel }
\]
\[
R_{2 \cdot i - 1} \rightarrow P_i(x)
\]
\[
\text{for } j = 1 \text{ to } k \text{ do }
\]
\[
\text{if } (i - 1) \equiv 0 \pmod{2^{j-1}} \text{ then }
\]
\[
R_{2 \cdot i - 1 + 2^{j-1}} \rightarrow P_i(y)
\]
\[
x := x + y
\]
\[
P_i(x) \rightarrow R_{2 \cdot i - 1}
\]

- Running time: \( O(k) = O(\log n) \) (precise \( 3 \cdot k + 1 \) steps).
- Number of processors: \( n/2 \).
- Size of memory: \( n \).
Addition of Matrices

- Let $A, B$ two $(n \times n)$-Matrices.
- Sum $A + B$ is computable with $n^2$ processors in Zeit $O(1)$ on a EREW PRAM.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2.n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2.n^2}$ bis $R_{3.n^2}$.
- Programm: MatSumme
  
  for all $P_i$ where $1 \leq i \leq n^2$ do in parallel
  
  $R_i \rightarrow P_i(x)$
  $R_{i+n^2} \rightarrow P_i(y)$
  $x := x + y$
  $P_i(x) \rightarrow R_{i+2.n^2}$

- Running time: $O(1)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$.

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

$\top$
Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$.

Let $A, B$ be two $(n \times n)$-Matrices.

- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$

- Register $A_{i,j} = R_{(i-1)\cdot n+j}$ ($1 \leq i, j \leq n$).
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ ($1 \leq i, j \leq n$).
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ ($1 \leq i, j \leq n$).
- Processor $P_{i,j} = P_{(i-1)\cdot n+j}$ ($1 \leq i, j \leq n$).

Use the above notation to simplify the algorithm.

Each processor has to do some hidden local computation to implement the above expressions.
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices

- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

  - Programm: MatrProd 1
    - for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
      - $h = 0$
      - for $l = 1$ to $n$ do
        - $A_{i,l} \rightarrow P_{i,j}(a)$
        - $B_{l,j} \rightarrow P_{i,j}(b)$
        - $h = h + a \cdot b$
        - $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$.
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

Programm: MatrProd 2

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

$h = 0$

for $l = 1$ to $n$ do

$A_{i,l} \rightarrow P_{i,j}(a)$

$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

\[
\begin{align*}
A_{i,j} &= R_{(i-1) \cdot n + j} \\
B_{i,j} &= R_{(i-1) \cdot n + j + n^2} \\
C_{i,j} &= R_{(i-1) \cdot n + j + 2 \cdot n^2} \\
P_{i,j} &= P_{(i-1) \cdot n + j}
\end{align*}
\]
Compute the Prefixsum

Problem:

- Task: Compute $s_i = \sum_{j=1}^{i} x_j$ for $1 \leq i \leq n$.
- Input: $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- Output: $s_i$ should be in register $R_i$ for $1 \leq i \leq n$. 
Computing Prefixsum (Idea)
Computing the Prefixsum

- **Task:** Compute $s_i = \sum_{j=1}^{i} x_j$ for $1 \leq i \leq n$.
- **Input:** $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- **Output:** $s_i$ should be in register $R_i$ for $1 \leq i \leq n$.
- **Model:** EREW

**Programm: Summe**

```plaintext
for all $P_i$ where $1 \leq i \leq n$ do in parallel

$R_i \rightarrow P_i(x)$

for $j = 1$ to $k$
do

if $i > 2^{j-1}$ then

$R_{i-2^{j-1}} \rightarrow P_i(y)$

$x := x + y$

$P_i(x) \rightarrow R_i$
```

- **Running time:** $O(k) = O(\log n)$ (precisely $3 \cdot k + 1$ steps).
- **Number of processors:** $n$.
- **Size of memory:** $n$. 
Compute the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- Input: \( x_j \) is in register \( R_j \) (\( 1 \leq j \leq n \)).
- Output: \( m \) should be in register \( R_{n+1} \).
- Possible with \( n \) processors in time \( O(\log n) \) using a EREW PRAM.
- Question: could it be done faster? (i.e. on a ERCW PRAM).
- A maximum is larger or equal than all other values.
- Idea: compare all pairs of numbers.
- The maximum will always win.
Compute the Maximum (Idea)

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### Compute the Maximum (Idea)

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Values: 0, 1
Computing the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- Input: \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
- Output: \( m \) in register \( R_{n+1} \).
- Model: CRCW.

Programm: Maximum

```
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
   \( P_{i,1}(1) \rightarrow W_i \)
for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
   \( R_i \rightarrow P_{i,j}(a) \)
   \( R_j \rightarrow P_{i,j}(b) \)
   if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
   \( W_i \rightarrow P_{i,1}(h) \)
   if \( h = 1 \) then
      \( R_i \rightarrow P_{i,1}(h) \)
      \( P_{i,1}(h) \rightarrow R_{n+1} \)
```
Computing the Maximum

*: Programm: Maximum

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel

\[ P_{i,1}(1) \rightarrow W_i \]

for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel

\[ R_i \rightarrow P_{i,j}(a) \]
\[ R_j \rightarrow P_{i,j}(b) \]
\[ \text{if } a < b \text{ then } P_{i,j}(0) \rightarrow W_i \]

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel

\[ W_i \rightarrow P_{i,1}(h) \]
\[ \text{if } h = 1 \text{ then } \]
\[ R_i \rightarrow P_{i,1}(h) \]
\[ P_{i,1}(h) \rightarrow R_{n+1} \]

*: Running time: \( O(1) \).

*: Number of processors: \( O(n^2) \).

*: Memory: \( O(n) \).
Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till \( n \)
- Input: Father of node \( i \) is written in register \( R_i \).
- For the roots \( i \) we have: in register \( R_i \) is written \( i \).

Programm: Ranking

\[
\text{for all } P_i \text{ where } 1 \leq i \leq n \text{ do in parallel} \\
\text{for } j = 1 \text{ to } \lceil \log n \rceil \text{ do} \\
R_i \to P_i(h) \\
R_h \to P_i(h) \\
P_i(h) \to R_i
\]

Running time: \( O(\log n) \).

Number of processors: \( O(n) \).

Memory: \( O(n) \).

Model: CREW.
### Short Summary

<table>
<thead>
<tr>
<th>Problem</th>
<th>processors</th>
<th>memory</th>
<th>time</th>
<th>model</th>
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</thead>
<tbody>
<tr>
<td>Or</td>
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<td>(O(n))</td>
<td>(O(t))</td>
<td>ERCW</td>
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<tr>
<td>Or</td>
<td>(O(n/\log n))</td>
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</tr>
<tr>
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<td>(O(t))</td>
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<td>(O(n))</td>
<td>(O(\log n))</td>
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<td>(O(\log n))</td>
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**Question:** May we save some processors?  
May we do this saving in any situation?  
How do we estimate the efficiency of a parallel algorithm?
Cost Measurement

Let $A$ be any parallel algorithm, we denote:

- $T_A(n)$ the running time of $A$.
- $P_A(n)$ the number of processors used by $A$.
- $R_A(n)$ the number of registers used by $A$.
- $W_A(n)$ the number of accesses to registers done by $A$.
- $ST(n)$ the running time of the best [known] sequential algorithm.
- $\text{Eff}_A(n) := \frac{ST(n)}{P_A(n) \cdot T_A(n)}$ the efficiency of $A$.
- $\text{AEff}_A(n) := \frac{W_A(n)}{P_A(n) \cdot T_A(n)}$ the usage efficiency of $A$. 
## Efficiency

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<th>time</th>
<th>$W(n)$</th>
<th>$AEff$</th>
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Task: Compute the \( k \)-th (\( k \)-smallest) element in a unsorted sequence \( S = \{s_1, \ldots, s_n\} \).

Lower bound: \( n - 1 \) comparisons

Start with a nice sequential algorithm

Programm: Select\( (k, S) \)

\[\text{if } |S| \leq 50 \text{ then return } \text{\( k \)-th number in } S\]

Split \( S \) in \( \lceil n/5 \rceil \) sub-sequences \( H_i \) of size \( \leq 5 \)

Sort each \( H_i \)

Let \( M \) be the sequence of the middle elements in \( H_i \)

\[m := \text{Select}(\lceil |M|/2 \rceil, M)\]

\[S_1 := \{s \in S \mid s < m\}\]

\[S_2 := \{s \in S \mid s = m\}\]

\[S_3 := \{s \in S \mid s > m\}\]

\[\text{if } |S_1| \geq k \text{ then return } \text{Select}(k, S_1)\]

\[\text{if } |S_1| + |S_2| \geq k \text{ then return } m\]

return \( \text{Select}(k - |S_1| - |S_2|, S_3) \)
### Example for the k-th Element (Slow Motion)

**Input/Data:**

| 61 | 76 | 58 | 37 | 96 | 31 | 49 | 50 | 9  | 70 | 18 | 29 | 18 | 30 | 68 | 4  | 48 | 82 | 11 | 46 | 34 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 78 | 40 | 38 | 24 | 33 | 58 | 82 | 35 | 33 | 90 | 54 | 10 | 44 | 46 | 49 | 38 | 30 | 14 | 29 | 18 | 6  |
| 26 | 7  | 40 | 91 | 45 | 64 | 21 | 30 | 94 | 30 | 41 | 67 | 2  | 79 | 42 | 28 | 15 | 9  | 44 | 88 | 50 |
| 94 | 8  | 77 | 47 | 32 | 79 | 65 | 24 | 4  | 15 | 74 | 23 | 41 | 66 | 82 | 59 | 63 | 52 | 64 | 70 |
| 6  | 2  | 44 | 48 | 72 | 82 | 12 | 16 | 51 | 2  | 22 | 42 | 41 | 56 | 55 | 72 | 42 | 67 | 64 | 61 | 90 |

**M:**

| 61 | 8  | 44 | 47 | 45 | 64 | 49 | 30 | 33 | 30 | 22 | 42 | 23 | 46 | 55 | 38 | 42 | 63 | 44 | 61 | 50 |

**sorted M:**

| 8  | 22 | 23 | 30 | 30 | 33 | 38 | 42 | 42 | 44 | 44 | 45 | 46 | 47 | 49 | 50 | 55 | 61 | 61 | 63 | 64 |
## Example for the k-th Element

**Input/Data:**

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<th>28</th>
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</table>

**M:**

| 41 | 56 | 72 | 28 | 56 | 56 | 54 | 22 | 52 | 65 | 28 | 16 | 86 | 51 | 40 | 26 | 37 | 57 | 55 | 55 | 39 |

**sorted M:**

| 16 | 22 | 26 | 28 | 28 | 37 | 39 | 40 | 41 | 51 | 52 | 54 | 55 | 55 | 56 | 56 | 56 | 57 | 65 | 72 | 86 |
**Example for the k-th Element (Worst Case)**

**Input/Data:**

```
60  76  56  75  51  94  86  78  78  84  63  51  93  63  88  54  76  82  87  63  93
82  69  56  62  50  85  94  90  60  73  84  94  90  93  85  67  81  66  60  85  88
44  31  0   17  13  0   18  33  39  34  11  75  65  84  68  62  78  85  51  75  88
38  17  22  2   21  16  25  24  21  0   16  87  84  94  51  73  78  67  66  58  66
33  8   21  4   6   20  12  11  14  36  38  66  64  92  50  57  57  67  64  63  66
```

**M:**

```
44  31  22  17  21  20  25  33  39  36  38  75  84  92  68  62  78  67  64  63  88
```

**sorted M:**

```
17  20  21  22  25  31  33  36  38  39  44  62  63  64  67  68  75  78  84  88  92
```
Running Time

- For some constants $c, d$ we get:
  - $T(n) \leq d \cdot n$ for $n \leq 50$
  - $T(n) \leq c \cdot n + T(n/5) + T(3n/4)$

<table>
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<tr>
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</tbody>
</table>
Claim: $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.

Proof:

$n = 50$:

$$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$

$n \geq 50$:

$$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$

$$T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n$$

Running time $T(n)$ is in $O(n)$. 
Parallel k-Select

- Input $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
- Each $P_i$ knows $n, P(n)$.
- Each $P_i$ works on $\lceil n^x \rceil$ elements.
- We will now create a parallel version of the program Select(k, S).
- We will get a parallel recursive program.

1. Easy solution for small $S$.
2. Split $S$ into small sub-sequences for the processors.
3. Compute parallel the median of the sub-sequences.
4. Compute parallel and recursive the median of medians.
5. Compute the splitting into the three sub-sequences.
6. Do the final recursion.
Example for the k-th Element

Input/Data:

| 43 22 69 12 21 9 83 26 45 66 42 78 27 96 85 46 53 36 41 48 93 61 52 59 2 72 24 33 28 77 87 89 36 43 42 |
| 79 95 97 25 1 13 93 61 3 28 95 32 78 55 88 92 50 71 97 85 1 29 76 47 96 81 34 90 20 45 26 40 18 17 32 |
| 9 94 29 27 81 60 81 8 34 89 65 91 97 82 48 64 39 82 63 94 23 33 51 38 10 41 14 61 84 37 66 58 34 95 48 |
| 41 92 19 71 69 2 88 67 39 5 68 62 7 19 40 27 45 54 68 66 10 35 54 31 92 17 24 29 8 0 12 91 22 64 32 |
| 31 14 28 34 89 88 45 57 77 51 31 52 85 42 29 33 21 7 32 66 50 24 66 21 39 73 57 93 6 26 2 91 30 81 45 |
| 51 69 83 74 12 31 89 11 36 42 2 92 69 1 5 83 43 68 69 84 90 96 96 46 78 59 69 11 79 18 7 85 83 72 66 |
| 76 94 19 12 60 60 87 93 7 81 86 16 87 58 57 34 83 89 87 5 82 9 20 66 81 14 13 5 18 45 15 48 60 69 56 |
| 7 60 89 36 59 18 58 55 20 61 71 36 88 26 80 86 17 84 21 32 52 66 97 33 9 46 13 87 31 12 69 67 55 44 92 |
| 50 49 67 35 32 70 77 17 24 70 64 84 40 37 5 26 62 74 84 46 28 20 92 66 64 76 64 23 28 3 30 95 37 51 72 |
| 76 13 62 73 67 25 41 16 81 53 63 32 41 39 46 49 31 90 86 79 20 29 8 13 47 18 59 30 54 30 20 38 84 18 42 |
| 88 42 92 78 44 89 91 50 83 39 78 92 68 64 37 41 95 24 63 90 29 33 80 5 1 44 21 66 29 24 36 |
| 44 10 64 13 69 80 9 12 87 37 74 49 20 28 51 64 1 22 76 23 31 79 58 56 86 69 88 78 69 7 69 23 20 88 63 |
| 87 12 56 72 24 54 10 20 39 68 77 77 87 44 25 70 34 3 23 75 97 53 23 20 19 58 18 86 97 60 3 49 44 47 18 |
| 76 54 73 35 60 1 94 38 97 7 55 69 12 37 68 87 89 24 75 21 89 35 51 16 1 93 92 54 63 72 6 69 57 2 39 |
| 23 74 72 74 28 33 51 0 56 77 54 0 90 33 31 11 70 1 78 31 17 22 39 16 24 95 63 74 23 38 40 59 2 79 64 |

\[ P_1 \ P_2 \ P_3 \ P_4 \ P_5 \ P_6 \ P_7 \ P_8 \ P_9 \ P_{10} \ P_{11} \ P_{12} \ P_{13} \ P_{14} \ P_{15} \ P_{16} \ P_{17} \ P_{18} \ P_{19} \ P_{20} \ P_{21} \ P_{22} \ P_{23} \ P_{24} \ P_{25} \ P_{26} \ P_{27} \ P_{28} \ P_{29} \ P_{30} \ P_{31} \ P_{32} \ P_{33} \ P_{34} \ P_{35} \]

\[ M: \]

| 50 54 67 35 59 33 81 26 39 53 65 52 69 39 46 64 45 54 69 48 31 35 54 38 39 59 57 54 28 37 21 66 36 51 45 |

sorted \( M: \)

| 21 26 28 31 33 35 35 36 37 38 39 39 39 45 45 46 48 50 51 52 53 54 54 54 54 57 59 59 64 65 66 67 69 69 81 |
Parallel k-Select

Programm: ParSelect(k,S)
1: 
   \textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \ \textbf{returns} \ Select(k, S).
2: 
   S \ \text{is split into } \lceil |S|^{1-x} \rceil \ \text{sub-sequences } S_i \ \text{with } |S_i| \leq \lceil n^x \rceil
   P_i \ \text{stores the start-address of } S_i.
3: 
   \textbf{for all} \ P_i \ \text{where } 1 \leq i \leq \lceil n^{1-x} \rceil \ \textbf{do in parallel} \\
   m_i := Select(\lceil |S_i|/2 \rceil, S_i) \\
   P_i(m_1) \rightarrow R_i.
   \text{Assume in the following that } M \ \text{is the sequence of these values.}
4: 
   m := ParSelect(\lceil |M|/2 \rceil, M).
5: More to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:

Distribute \( m \) via broadcast to all \( P_i \).

\[ \text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel} \]

\[ L_i := \{ s \in S_i \mid s < m \} \]
\[ E_i := \{ s \in S_i \mid s = m \} \]
\[ G_i := \{ s \in S_i \mid s > m \} \]

5.2:

Compute with Parallel Prefix:

\[ l_i := \sum_{j=1}^{i} |L_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \]
\[ e_i := \sum_{j=1}^{i} |E_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \]
\[ g_i := \sum_{j=1}^{i} |G_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \]
Let: \( l_0 = e_0 = g_0 = 0 \)

5.3:

Even more to come!
Parallel k-Select

Programm: ParSelect(k, S) Steps 5+6

5.3:
Compute $L = \{s \in S \mid s < m\}$, $E = \{s \in S \mid s = m\}$
and $G = \{s \in S \mid s > m\}$ as follows:

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

- $P_i$ writes $L_i$ in $R_{l_i-1+1}, \ldots, R_{l_i}$.
- $P_i$ writes $E_i$ in $R_{e_i-1+1}, \ldots, R_{e_i}$.
- $P_i$ writes $G_i$ in $R_{g_i-1+1}, \ldots, R_{g_i}$.

6:

if $|L| \geq k$ then return $\text{ParSelect}(k, L)$
if $|L| + |E| \geq k$ then return $m$
return $\text{Select}(k - |L| - |E|, G)$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: \(O(1)\)
   \[\text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } Select(k, S).\]

2: \(O(\log_2(|S|^{1-x}))\) thus we have \(O(\log n)\)
   \(S\) is split into \(\lceil|S|^{1-x}\rceil\) sub-sequences \(S_i\) with \(|S_i| \leq \lceil n^x \rceil\)
   \(P_i\) stores the start-address of \(S_i\).

3: \(O(n^x)\)
   
   \[\text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel}\]
   \[m_i := Select(\lceil|S_i|/2\rceil, S_i)\]
   \[P_i(m_1) \rightarrow R_i.\]
   Assume in the following that \(M\) is the sequence of these values

4: \(T_{\text{ParSelect}}(n^{1-x})\)
   \[m := \text{ParSelect}(\lceil|M|/2\rceil, M).\]
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$

Distribute $m$ via broadcast to all $P_i$.

5.1b: $O(|S_i|)$ thus we have $O(n^x)$

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{s \in S_i \mid s < m\}$
$E_i := \{s \in S_i \mid s = m\}$
$G_i := \{s \in S_i \mid s > m\}$

5.2: $O(\log_2(n^{1-x}))$

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$
Parallel $k$-Select (Running Time)

Programm: $\text{ParSelect}(k,S)$ Steps 5+6

5.3: $O(n^x)$
Compute $L = \{s \in S \mid s < m\}$, $E = \{s \in S \mid s = m\}$
and $G = \{s \in S \mid s > m\}$ as follows:

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$P_i$ writes $L_i$ in $R_{l_{i-1}+1}, \ldots, R_{l_i}$.
$P_i$ writes $E_i$ in $R_{e_{i-1}+1}, \ldots, R_{e_i}$.
$P_i$ writes $G_i$ in $R_{g_{i-1}+1}, \ldots, R_{g_i}$.

6: $T_{\text{ParSelect}}(3 \cdot n/4)$

if $|L| \geq k$ then return $\text{ParSelect}(k, L)$
if $|L| + |E| \geq k$ then return $m$
return $\text{Select}(k - |L| - |E|, G)$
Parallel k-Select (Running Time)

Adding all up we get:

\[ T_{\text{ParSelect}}(n) = c_1 \log n + c_2 \cdot n^x + T_{\text{ParSelect}}(n^{1-x}) + T_{\text{ParSelect}}(3/4 \cdot n). \]

\[ T_{\text{ParSelect}}(n) = O(n^x) \text{ with } P_{\text{ParSelect}}(n) = O(n^{1-x}). \]

\[
\text{Eff}_{\text{ParSelect}}(n) = \frac{O(n)}{O(n^x) \cdot O(n^{1-x})} = O(1)
\]
Sequential Merging

- **Input:**
  \( A = (a_1, a_2, \ldots, a_r) \) and \( B = (b_1, b_2, \ldots, b_s) \) two sorted sequences

- **Output:**
  \( C = (c_1, c_2, \ldots, c_n) \) sorted sequence of \( A \) and \( B \) with \( n = r + s \).

- **Programm: Merge**
  
  \[
  \begin{align*}
  i &:= 1; j := 1; n := r + s \\
  \text{for } k := 1 \text{ to } n \text{ do} \\
  &\quad \text{if } a_i < b_j \\
  &\quad \quad \text{then } c_k := a_i; i := i + 1; \\
  &\quad \quad \text{else } c_k := b_j; j := j + 1;
  \end{align*}
  \]

- Algorithm does not care about special cases.

- Running time: at most \( r + s \) comparisons, i.e. \( O(n) \).

- Lower bound on the number of comparisons is \( r + s \), i.e. \( \Omega(n) \).
The border lines may not intersect each other.
Thus we may separate the two sequences into disjoint blocks.
Let $A_i$ the $i$ block of size $\lceil r/p \rceil$.
Let $\hat{B}_i$ block in $B$ which should be merged with $A_i$.
Thus we may uses a PRAM easily (in this case).
Idea for Parallel Merging (CREW)

Let $A_i$ [resp. $B_i$] the $i$ block of size $\lceil r/p \rceil$ [resp. $\lceil s/p \rceil$].

Let $\hat{B}_i$ [resp. $A_i$] block in $B$ [resp. $A$] which should be merged with $A_i$ [resp. $B_i$].

$P_i$ cares about $A_i$ and $\hat{B}_i$ if $|\hat{B}_i| \leq \lceil r/p \rceil$.

Let $C$ be those where one $P_j$ takes already care of.

$P_i$ cares about $A_i \setminus C$ and $\hat{B}_i \setminus C$. 
Parallel Merging (CREW)

1. Use $P(n)$ processors.
2. Each processor $P_i$ computes for $A$ [$B$] its part of size $r/P(n)$ [$s/P(n)$].
3. Each processor $P_i$ computes the part from $B$ [$A$] which should be merged with its $A$-block [$B$-block].
4. Each processor computes its $A$ or $B$ block, where only he is responsible for.
5. This block has size $O(n/P(n))$.
6. Each processor merges its block into the resulting sequence.
7. Time: $O(\log n + n/P(n))$.
8. Efficiency
   \[
   \frac{n}{O(P(n)) \cdot O(\log n + n/P(n))}.
   \]
9. Efficiency is 1 for $P(n) \leq n/\log n$. 
Do some splitting into pairs of blocks of the same size.
- Rekursive splitting into pairs of blocks of the same size.
- Thus we may avoid read conflicts.
Merging (EREW)

1. Use $P(n)$ processors.
2. Compute the median $m$ of the sequences $A$ and $B$.
3. Split the sequences $A$ and $B$ in two sub-sequences each of the “same” size ($-1 \leq |A| - |B| \leq 1$).
4. Continue recursively, till all sub-sequences are smaller than $n/P(n)$.
5. Do the merging in the same way as before.

Remaining problem: Find the median of two sequences.
Example for the Median for two Sorted Sequences

- Sequences $A$ and $B$ are sorted.
- Compute median $a$ of $A$ and median $b$ of $B$. 
Median for two Sorted Sequences

1. Sequences $A$ and $B$ are sorted.
2. Compute median $a$ of $A$ and median $b$ of $B$.
4. The median of $A$ and $B$ is in one block-pair of the four blocks.
5. Search recursively for the median.

Running time: $O(\log n)$
Running Time for Merging (EREW)

1. Use \( P(n) \) processors.

2. Compute the median \( m \) of the sequences \( A \) and \( B \). \( O(\log n) \)

3. Split the sequences \( A \) and \( B \) in two sub-sequences each.

4. Continue recursively, till all sub-sequences are smaller than \( n/P(n) \).
   \( O(\log n \cdot \log(P(n))) \)

5. Merge in the same way as before. \( O(n/P(n)) \)

- Running time: \( O(n/P(n) + \log(n)^2) \).
- Efficiency

\[
\frac{O(n)}{O(P(n)) \cdot O(n/P(n) + \log(n)^2)} = \frac{O(n)}{O(n + P(n) \cdot \log(n)^2)}.
\]

Efficiency is 1 for \( P(n) < \frac{n}{(\log n)^2} \).
Questions

- Explain the motivation behind parallel systems.
- Describe the different models of a PRAM.
- Describe idea of the k-select algorithm.
- For which problems do the running time of CWCR and EWCR algorithms differ?
Legend

- : Not of relevance
- : implicitly used basics
- : idea of proof or algorithm
- : structure of proof or algorithm
- : Full knowledge