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Motivation

- Shows the quality of any algorithm.
- Interesting property of any problem.
- Interesting techniques to prove lower bounds.
  - No assumption about the used algorithms
  - Have to show a property for all algorithms and some inputs.
  - For all algorithms there is an input, such that the running time is at least....
  - Typically more complicated than upper bounds.
- Here we start with lower bounds for coloring cycles.
Ideas

- Model distributed computers, connected in a cycle.
- No assumption about structure of the algorithm.
- Assume the running time is $t$ on a cycle of length $n$.
- Step one: Normalize the behavior of the algorithm.
- Step two: Extend the possible inputs for the algorithms, such that the algorithm works still correct.
- Step three: find some contradiction.
Step one: Normalize the behavior of the algorithm

- After $t$ steps a node may know the identifiers of $2t + 1$ nodes. Let

$$W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n \land x_i = x_j \Rightarrow i = j\}$$

be the set of possible surroundings.

- It is not necessary to color any node before step $t$:
  - Each node may simulate the behavior of the $2t + 1$ nodes in the surrounding.
  - Or each nodes sends also the history of colors.

- Thus after $t$ rounds node $v$ has the topological information $\zeta(v)$:

$$\zeta(v) = (x_1, x_2, \ldots, x_s) \in W_{s,n} \text{ with } s = 2t + 1.$$

- Any algorithm will use some deterministic strategy $\pi$ to find a coloring:

$$c(v) \leftarrow \Phi_\pi(\zeta(v)) \text{ with } \Phi_\pi : W_{s,n} \mapsto \{1, 2, \ldots, c_{\max}\}.$$
Step two: Extend the possible inputs

- The set of nodes is $W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\}$.
- The set of edges is $E_{s,n}$. They contain any possible edge is any cycle:
  \[E_{s,n} = \{( (x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1}) ) \mid x_1 \neq x_{s+1}\}\]
- This graph $B_{s,n} = (W_{s,n}, E_{s,n})$ has $\binom{n}{s}$ nodes of degree $n - s$. Thus it has $(n - s)\binom{n}{s} s!$ edges.

Theorem (Coloring $B_{s,n}$)

If an algorithm $\pi_t$ colors any cycle of length $n$ with $c$ colors in $t$ steps, then it will define a legal coloring of $B_{s,n}$.
Step two: Extend the possible inputs

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \quad \text{and} \quad E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

**Theorem (Coloring \( B_{s,n} \))**

*If an algorithm \( \pi_t \) colors any cycle of length \( n \) with \( c \) colors in \( t \) steps, then it will define a legal coloring of \( B_{s,n} \).*

- Assume algorithm \( \pi_t \) colors cycle of length \( n \) correct, but not the \( B_{s,n} \).
- Thus there is an edge \( e = ((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \in E_{s,n} \) which is not colored correctly.
- Take this edge and extend it to a cycle of length \( n \) using the missing identifiers.
- This cycle with this order is not colored correctly.
- Contradiction.
Lower Bound for even length cycle

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \quad \text{and} \quad E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

**Theorem (Distributed Coloring \( C_{2n} \))**

*Any deterministic distributed algorithm uses \( n - 1 \) rounds to color a cycle of length \( 2n \) with 2 colors.*

- Assume the algorithm runs in time \( t \leq n - 2 \).
- Then this algorithm will color the graph \( B_{2t+1,2n} \) with 2 colors.
- \( B_{2t+1,2n} \) is bipartite for \( t \leq n - 2 \).
- We will now construct the following cycle:

\[
\begin{align*}
(1, 2, 3, \ldots, 2t + 1) & \rightarrow (2, 3, 4, \ldots, 2t + 2) \rightarrow \\
\rightarrow (3, 4, 5, \ldots, 2t + 3) & \rightarrow (4, \ldots, 2t + 3, 1) \rightarrow \\
\rightarrow \ldots & \rightarrow (2t + 2, 2t + 3, 1, 2, \ldots, 2t - 1) \rightarrow \\
\rightarrow (2t + 3, 1, 2, \ldots, 2t) & \rightarrow (1, 2, 3, \ldots, 2t + 1)
\end{align*}
\]
Lower Bound for even length cycle

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \text{ and } E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_s+1\} \]

**Theorem (Parallel Coloring \( C_{2n} \))**

Any deterministic parallel algorithm uses \( \log n \) rounds to color a cycle of length \( 2n \) with 2 colors.

- Assume the algorithm runs in time \( t \leq \log n \).
- The best way to collect information is doubling (see lower bound for broadcast/accumulation).
- Then we may use its strategy to construct a distributed version running in \( t \) time.
- Contradiction.
Step four: find some contradiction

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \text{ and } E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

- We want a lower bound for the 3-coloring of cycles.
- Step a) Show \( \chi(B_{2t+1,n}) \geq \log^2 t \cdot n \).
- Step b) Show \( \chi(\tilde{B}_s,n) \leq \chi(B_s,n) \).
- Step c) Use the line-graph construction.
- Step d) Show property for coloring a line-graph.
- Step e) Put everything together.
Construction of $\tilde{B}_{s,n}$

$W_{s,n} = \{(x_1, x_2, ..., x_s) \mid 1 \leq x_i \leq n\}$ and $E_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$

- Remember:
  - $W_{s,n} = \{(x_1, x_2, ..., x_s) \mid 1 \leq x_i \leq n \land x_i = x_j \Rightarrow i = j\}$
  - $E_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$
  - $B_{s,n} = (W_{s,n}, E_{s,n})$

- Construct now:
  - $\tilde{W}_{s,n} = \{(x_1, x_2, ..., x_s) \mid 1 \leq x_1 < x_2 < ... < x_s \leq n\}$
  - $\tilde{E}_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$
  - $\tilde{B}_{s,n} = (\tilde{W}_{s,n}, \tilde{E}_{s,n})$

- Thus $\tilde{B}_{s,n}$ is by definition a non-directed sub-graph of $B_{s,n}$.

**Lemma**

We have: $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$. 
Line-Graphs

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < \ldots < x_s\}, \quad \tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1})) \mid x_1 \neq x_{s+1}\}, \quad \chi(\tilde{B}_s, n) \leq \chi(B_s, n) \]

**Definition (Line-Graphs)**

Let \( G = (V, E) \) be a directed graph. \( DL(G) = (E, E') \) is called line-graph of \( G \), iff

\[ E' = \{(e, e') \mid e, e' \in E \land e \cap e' \neq \emptyset\}. \]

A graph \( H \) is called directed line-graph, iff a graph \( G \) exists, with \( DL(G) = H \).
Line-Graphs

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < \ldots < x_s\}, \quad \tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1})) \mid x_1 \neq x_{s+1}\}, \quad \chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n}) \]

Definition (Line-Graphs)

Let \( G = (V, E) \) be an undirected graph. \( L(G) = (E, E') \) is called line-graph of \( G \), iff

\[
E' = \{\{e, e'\} \mid e, e' \in E \land e \cap e' \neq \emptyset\}.
\]

A graph \( H \) is called line-graph, iff a graph \( G \) exists, with \( L(G) = H \).
Example 1

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < \ldots < x_s\}, \tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1}) \mid x_1 \neq x_{s+1}\} \]

\[ \chi(\tilde{B}_s, n) \leq \chi(B_s, n) \]
Example 2

\[ \tilde{W}_s, n = \{ (x_1, \ldots, x_s) | x_1 < \ldots < x_s \}, \tilde{E}_s, n = \{ (x_1, x_2, \ldots, x_s, (x_2, \ldots, x_{s+1}) | x_1 \neq x_{s+1} \}, \chi(\tilde{B}_s, n) \leq \chi(B_s, n) \]
Example 3

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < \ldots < x_s\}, \quad \tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1})) \mid x_1 \neq x_{s+1}\}, \quad \chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n}) \]
DeBruijn network of dimension $d$

$\tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < \ldots < x_s\}$, $\tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1})) \mid x_1 \neq x_{s+1}\}$, $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$

- DeBruijn network:
  \[
  \begin{align*}
  DB(d) &= (V_{DB(d)}, E_{DB(d)}^s \cup E_{DB(d)}^{se}) \\
  V_{DB(d)} &= \{0, 1\}^d \\
  E_{DB(d)}^s &= \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\} \\
  E_{DB(d)}^{se} &= \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \end{align*}
  \]

  Number of nodes: $2^d$  
  Degree: $2 + 2$  
  Number of edges: $2^{d+1}$  
  Diameter: $d$

**Lemma**

We have: $DB(d + 1) = DL(DB(d))$ for $d \geq 1$. 
Line-Graph Properties of $\tilde{B}_{s,n}$

$\tilde{W}_{s,n} = \{(x_1, \ldots, x_s) | x_1 < \ldots < x_s\}$, $\tilde{E}_{s,n} = \{\{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1}\}$, $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$

Lemma

1. $\tilde{B}_{1,n}$ is the complete directed graph of $n$ nodes.
2. We have $\tilde{B}_{s+1,n} = LG(\tilde{B}_{s,n})$ for $s \geq 1$.

Proof:

1. By definition: $\tilde{E}_{s,n} = \{\{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1}\}$.

2. By construction:

   - In $\tilde{B}_{s,n}$: $(x_1, x_2, \ldots, x_s) \rightarrow (x_2, x_3, \ldots, x_{s+1})$ and
     $(x_2, x_3, \ldots, x_{s+1}) \rightarrow (x_3, x_4, \ldots, x_{s+2})$.
   - In $V(DL(\tilde{B}_{s+1,n}))$: $((x_1, x_2, \ldots, x_s), (x_2, x_3, \ldots, x_{s+1}))$ and
     $((x_2, x_3, \ldots, x_{s+1}), (x_3, x_4, \ldots, x_{s+2}))$.
   - In $V(DL(\tilde{B}_{s+1,n}))$: $(x_1, x_2, \ldots x_s, x_{s+1})$ and $(x_2, x_3, \ldots x_{s+1}, x_{s+2})$ (simplified).
   - In $E(DL(\tilde{B}_{s+1,n}))$: $((x_1, x_2, \ldots x_s, x_{s+1}), (x_2, x_3, \ldots x_{s+1}, x_{s+2}))$. 
Bounds for Coloring Line-Graphs

\[ \bar{W}_{s, n} = \{ (x_1, \ldots, x_s) \mid x_1 < \ldots < x_s \}, \quad \bar{E}_{s, n} = \{ (x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1}) \mid x_1 \neq x_{s+1} \}, \quad \chi(\bar{B}_{s, n}) \leq \chi(B_{s, n}) \]

**Lemma**

Let \( H \) be any directed graph, then we have \( \chi(DL(H)) \geq \log(\chi(H)) \).

**Proof:**

- Let \( k = \chi(DL(H)) \), thus we can color the nodes from \( DL(H) \) with \( k \) colors.
- Thus we may color the edges from \( H \) with \( k \) colors: \( \chi'(H) \leq k \).
- For any edge \( e = (v, w) \) of \( H \) let \( c'(e) \) be the color of \( e \).
- Define now a coloring of the nodes \( v \) of \( H \):
  \[ c(v) = \bigcup_{e \in c'} c'(e). \]
- This is a correct \( 2^k \) node-coloring of \( H \).
- Thus \( \chi(H) \leq 2^k = 2^{\chi(DL(H))} \).
- Thus \( \log(\chi(H)) \leq \chi(DL(H)) \).
Results

Lemma

We have $\chi(\tilde{B}_s, n) \geq \log^{(s-1)} n$.

Proof:

- $\tilde{B}_{1,n}$ is the complete directed graph of $n$ nodes.
- $\chi(\tilde{B}_{1,n}) = n$.
- We have $\tilde{B}_{s+1,n} = LG(\tilde{B}_s, n)$ for $s \geq 1$.
- We have already: $\chi(DL(H)) \geq \log(\chi(H))$.
- Thus we get $\chi(\tilde{B}_{s+1,n}) \geq \log(\chi(\tilde{B}_s, n))$.
- Thus we get $\chi(\tilde{B}_s, n) \geq \log^{(s-1)}(\chi(\tilde{B}_1, n))$.
- Thus we get $\chi(\tilde{B}_s, n) \geq \log^{(s-1)}(n)$. 
**Theorem**

Any deterministic distributed algorithm needs at least \( \frac{1}{2}(\log^* n - 1) \) rounds to color a cycle of length \( n \) with 3 colors.

**Proof:**

- We have already: \( \chi(\tilde{B}_s,n) \geq \log^{(s-1)} n \), resp.:
- We have already: \( \chi(\tilde{B}_{2t+1},n) \geq \log^{(2t)} n \).
- We also have: \( \chi(\tilde{B}_{2t+1},n) \leq 3 \).
- Thus we get: \( \log^{(2t)} n \leq 3 \) and finally
- \( 2t \geq \log^* n - 1 \).
Comparison with NP-complete

- NP-hard: the “most complicated” problems for the class $\mathcal{NP}$.

- Theory of NP-complete problems was developed, to “explain” that for many problems no polynomial time deterministic algorithm is known.

- A problem is NP-hard $\iff$
  - It is possible in polynomial time to reduce any other problem from NP to a NP-hard problem.
  - First NP-hard problem: Does a non-deterministic TM stop in polynomial time?
    - All other NP-hard problems were reduced from this.

- We assume (proof is still missing), that for these NP-hard problem no deterministic polynomial time algorithms exit.

- Thus we may assume, that for NP-complete problems no polynomial time deterministic parallel algorithm will be known using a polynomial number of processors.
Some Observations about Problems from $\mathcal{P}$

- Any problem from $\mathcal{P}$ is a candidate for a parallel algorithm.
- A problem is well to parallelize, if there is a parallel deterministic algorithm
  - which uses a polynomial number of processors
  - and runs in poly-logarithmic time.
- These class is called $\mathcal{NC}$ (Nick’s Class).
- We have by definition: $\mathcal{NC} \subseteq \mathcal{P}$.
- Important Question: $\mathcal{NC} \cong \mathcal{P}$?
- It is assumed, $\mathcal{NC} \neq \mathcal{P}$
- Thus the theory of $\mathcal{P}$-complete problems was developed.
- And it follows just the technique of $\mathcal{NPC}$. 
Recall the situation for $\mathcal{NPC}$ (try to separate $\mathcal{NP}$ from $\mathcal{P}$):

- Hard problem: stops a non-deterministic TM in polynomial time?
- Reduction: runs deterministic in polynomial time.

Or in other words:

- Hard problem: a nice candidate from the “hard class”.
- Reduction by computation within the “easy class”.

Uses the analogous technique for $\mathcal{P}$ (try to separate $\mathcal{P}$ from $\mathcal{NC}$):

- Hard problem: stops a deterministic TM in polynomial time?
- Reduction: runs deterministic in time poly-logarithmic time.
- Analog reduction: using poly-logarithmic memory.
Poly-Logarithmic Time versus Memory

- We had:
  - Reduction: runs deterministic in time poly-logarithmic time.
  - Analog reduction: using poly-logarithmic memory.

- We will transform an algorithm running deterministic in time poly-logarithmic time into one using poly-logarithmic memory.
  - From the parallel algorithm running deterministic in time poly-logarithmic we build a circuit network.
  - This has poly-logarithmic depth and polynomial width.
  - To compute any value within this circuit network we only need to store the values on a path towards the input.
  - Thus we have poly-logarithmic memory (and do not care about the running time).
A problem $X$ is called $\mathcal{P}$-complete, iff:

- $X$ is in $\mathcal{P}$.
- Any problem $Y$ from $\mathcal{P}$ could be reduced to $X$ with poly-logarithmic memory.
- I.e.
  - there is a function $f$ computable with poly-logarithmic memory, such that
  - $\forall w \in \Sigma^* : w \in X \iff f(w) \in Y$
Definition (Generability)
- Input: Set $X$ with binary operator $\odot$, $T \subset X$ and $s \in X$.
- Output: Is $s$ in the closure of $T$ in terms of $\odot$.

Let $S \odot S := \{a \odot b \mid a, b \in S\}$.

Algorithm for Generability($X, \odot, S, s$) in $\mathcal{P}$:
- while $S \neq S \odot S$ do $S = S \odot S$
- return $s \in S$.

We will first show $\mathcal{P}$-completeness for a ternary operation.
- i.e. $\odot$ will be substituted by next($u, v, w$).
- Reduction from the halting problem of a deterministic TM.
First Reduction

**Definition (Generability')**
- Input: Set $X$ with ternary operator $\text{next}(u, v, w)$, $T \subset X$ and $s \in X$.
- Output: Is $s$ in the closure of $T$ in terms of $\circ$.

**Definition (TM)**
- Input band with positions $0, 1, 2, \cdots T(n) + 1$.
- By $c(i, j) \in \Sigma$ we denote the contents at position $i$ at time $j$.
- Let $c(0, j) = c(T(n) + 1, j) = \$\,$ for all time points $j$.
- The function $\text{trans}$ defines the transitions for the TM.
- I.e. $c(p, t + 1) = \text{trans}(c(p - 1, t), c(p, t), c(p + 1, t))$.
- Input given at positions $c(p, 0)$ ($\forall p : 1 \leq p \leq T(n)$).
- Output placed at $c(1, T(n))$ where $\#$ encodes a “true”.
First Reduction (Generability’)

Theorem:
Generability’ is \( P \)-complete.

Proof:
- A TM may be transformed in \( NC \) into the above form.
- The triple \((t, p, sym)\) encodes that the contents at position \( p \) and time \( t \) is \( sym \).
- We will now compute the input for Generability’ from the above TM:
  - \( X = \{0, 1, \ldots, T(n)\} \times \{0, 1, \ldots, T(n) + 1\} \times \Sigma. \)
  - \( T = \{(0, i, c(0, i)) \mid 0 \leq i \leq T(n) + 1\} \)
  - \( s = (T(n), 1, \#) \)
  - \( next = trans \)
- This can be done in \( NC \).
- \( s \) is in the closure of \( next \) iff TM stops with “True”.
First Reduction (Generability)

Theorem:

Generability ist \( P \)-complete.

Proof:

- Reduktion von Generability’
- \( X' := X \cup X^2 \) (\( X \) form above)
- \( T = \{(0, i, c(0, i)) \mid 0 \leq i \leq T(n) + 1\} \)
- \( s = (T(n), 1, \#) \)
- It remains to define \( \text{next} \) as a binary Operator \( \odot \).
- \( u \odot v := (u, v) \) and
- \( (u, v) \odot w := \text{next}(u, v, w) \)
Lemma:

If $\circ$ is associative, then is the corresponding Generability-Problem in $\mathcal{NC}$.

Proof:

- We transform this problem into the reachability problem on a graph $G$.
- If $x \circ z = y$ then generate an edge $(x, y)$ with label $z$.
- $G = (X, E)$ with $E = \{(x, y) \mid \exists z \in X : x \circ z = y\}$
- and $\forall(x, y) \in E : l(x, y) := \{z \in X \mid x \circ z = y\}$.
- If there is a path from $a \in T$ to $s$ using edges with labels $b, c, d, \cdots$, then we may generate $s$ by $((\cdots(a \circ b) \circ c) \circ d) \cdots$.
- If $s$ may be generated by using elements from $T$ with $\circ$, then we may have also the form $((\cdots(a \circ b) \circ c) \circ d) \cdots$.
- This will give us a path in the above constructed graph $G$. 
Reduktion (CVP)

Definition (CVP)

- Input: A boolean circuit with some input.
- Output: Is the output value $true$.

Theorem:

The problem CVP is $\mathcal{P}$-complete.

Proof

- Reduction form the Generability Problem.
- The elements from $T$ are the inputs for the circuit with value $true$.
- The output will be the element $s$. 
Details for the Reduction (CVP)

- For each element \( x \) from \( X \setminus T \) do:
- Compute pairs from \( X \times X \) which will give \( x \):
  \[
  (y_1, z_1), (y_2, z_2), (y_3, z_3), \ldots, (y_{k_x}, z_{k_x})
  \]
- I.e. \( y_i \odot z_i = x \) for all \( 1 \leq i \leq k_x \).
- This is one part of the circuit:
  \[
  x = \bigvee_{i=1}^{k_x} y_i \land z_i
  \]
- Thus \( x \) will have the value \( true \) iff \( x \) may be generated.
- Thus \( s \) will have the value \( true \) iff \( s \) may be generated.
- This construction is in \( \mathcal{NC} \).
Reduktion (MCVP)

**Definition (MCVP)**

- **Input:** A boolean circuit with some input and only operators $\lor$ und $\land$.
- **Output:** Is the output value $true$.

**Theorem:**

The MCVP is $\mathcal{P}$-complete.

**Proof:**

- Similar proof to the CVP problem.
Reduktion (TSMCVP)

Definition (TSMCVP)

- Input: A boolean circuit with some input and only operators $\lor$, $\land$ and a topological sorting of the values.
- Output: Is the output value true.

Theorem:
The TSMCVP is $\mathcal{P}$-complete.

Proof:
- Similar proof to the CVP problem.
- Note: the proof for Generability’ did contain a topological sorting.
- This was the lexicographical order of the elements $(t, p, sym)$.
- This order could easily be preserved during the following step of the reduction.
**Definition (CFE)**

- **Input:** a context-free grammar $G$.
- **Output:** will $G$ generate the empty word $\varepsilon$.

**Theorem:**

The CFE is $\mathcal{P}$-complete.

**Proof (Reduktion from Generability Problem):**

- Let $(X, T, \odot, s)$ be the input for the Generability problem.
- Let $X$ be the non-terminals of $G$.
- Let $s$ be the start symbol.
- For each $x \in T$ generate the rule: $x \rightarrow \varepsilon$.
- If $y \odot z = x$ generate the rule: $x \rightarrow yz$.
- Note: If $G$ contains no $\varepsilon$-rules, then is CFE in $\mathcal{NC}$.
Definition (LFMIS)

- Input: non-directed graph \( G = (V, E) \).
- Output: lexicographical first maximum independent set (IS) of \( G \).

Theorem:
The LFMIS is \( \mathcal{P} \)-complete.

Proof (Reduction from MCVP problem)

1. Consider the greedy-strategy for the LFMIS problem.
2. Let \( V = \{v_1, v_2, \cdots, v_n\} \) nodes for the MCVP Problems in their topological sorting.
3. Let \( \{v_1, v_2, \cdots, v_e\} \) be the input nodes and \( v_n \) be the output node.
4. We construct \( G = (V', E') \) as input for LFMIS.
Continuation of the Reduction (LFMIS)

Let \( V' = \{v'_1, v'_2, v'_3, \ldots, v'_n, v''_n\} \) be numbered from 1 till 2\( n \).

- The numbers of \( v'_i, v''_i \) are exchanged, if
  - \( v_i \) is an or-node or
  - \( v_i \) is an input node with the value \text{false}.

For all \( 1 \leq i \leq n \) generate an edge \( \{v'_i, v''_i\} \).

Thus only one of the nodes \( v'_i, v''_i \) is in the IS.

- If \( v \) is an and-node \( G \) with input \( u \) and \( w \), then add the edges \( \{v', u''\} \) and \( \{v', w''\} \).

Thus \( v' \) will be in the IS iff non of the nodes \( u'', w'' \) are in the IS.

- If \( v \) is an or-node \( G \) with inputs \( u \) and \( w \), then add the edges \( \{v'', u'\} \) and \( \{v'', w'\} \).

Thus \( v'' \) will be in the IS iff if non of the nodes \( u', w' \) are in the IS.

Thus LFMIS is simulating correctly the boolean circuit.
Reduction (LFMC)

Definition (LFMC)

- Input: non-directed graph $G = (V, E)$.
- Output: lexicographical first maximum clique of $G$.

Theorem:

Das LFMC is $P$-complete.

Proof

- Reduction from LFMIS problem.
- Let $G = (V, E)$ be the input for LFMIS problem.
- Then $G = (V, \overline{E})$ will be input for the LFMC problem.
Given $G = (V, E)$

Procedure DFS(v)

if $DFI(v) = 0$ then
    counter := counter + 1
    $DFI(v) :=$ counter
    forall $w \in V : (v, w) \in E$ do
        DFS(w)
Reduction (DFS)

**Definition (DFS)**

- **Input**: directed graph $G = (V, E)$ and $v \in V$.
- **Output**: The values $DFI(w)$ of the call $DFS(v)$ for all $w \in V$.

**Theorem:**

The DFS is \(\mathcal{P}\)-complete.

**Proof**

- Reduction from CVP problem with \(\odot := \overline{x} \lor \overline{y} = \overline{x} \land \overline{y}\)
- It is easy to see, that this version of CVP Problem is also \(\mathcal{P}\)-complete.
- Idea: for each value of \(v\) in the input of CVP will be in $G = (V, E)$ two nodes $s$ and $t$, with $v$ is true iff $DFI(s) < DFI(t)$. 
Continuation of the Reduction (DFS)

- Let $v_1, v_2, \cdots, v_n$ be the nodes of the circuit.
- For each $v_i$ we will build a sub-graph $G_i$.
- These sub-graphs $G_i$ will be edge-disjoint, but not node-disjoint.
- $G_i$ and $G_j$ ($i < j$) may have common nodes $i \neq j$.
- $v_i$ has $v_{i_1}$ and $v_{i_2}$ as input nodes
- and the nodes $v_{o_1}, v_{o_2}, v_{o_3}, \cdots, v_{o_k}$ use $v_i$ as input.
- Then has $G_i$ for $k = 3$ the following structure.
- We indicate the order of the edges in the adjacency list by the number of arrow heads.
- If $v_i$ is an input node in the circuit and the nodes $v_{o_1}, v_{o_2}, v_{o_3}, \cdots, v_{o_k}$ use $v_i$ as input, then we will have a simplified graph $G_i$. This is seen as the second one.
Continuation of the Reduction (DFS)

\[last(i - 1)\rightarrow first(i)\rightarrow s(i)\]

- \[i_1 \neq i\]
- \[i_2 \neq i\]

\[v_i \text{ ist intern}\]

\[last(i)\rightarrow t(i)\]

- \[i \neq o_1\]
- \[i \neq o_2\]
- \[i \neq o_3\]
Continuation of the Reduction (DFS)

$$last(i - 1)$$

$$v_i$$ ist Input

$$first(i)$$

$$s(i)$$

$$last(i)$$

$$t(i)$$

$$i \# o_1$$

$$i \# o_2$$

$$i \# o_3$$
Continuation of the Reduction (DFS)

- The DFS run starts at $first(1)$.
- After $last(i)$ will be the next visited node $first(i + 1)$.
- The order how $s(i)$ and $t(i)$ in $G_i$ are visited, will be given by the value of $v_i$. 
- After $last(n)$ is visited, is each graph $G_i$ is also visited, excluding some minor parts.
Lemma

We consider a DFS-run in $G$ stating in node $\text{first}(1)$:

- If $v_i$ has the value $\text{true}$, then $s(i)$ will be visited before $t(i)$ and the nodes $i\#o_1, i\#o_2, \cdots, i\#o_k$ are visited after $\text{first}(i)$ and before $\text{last}(i)$.

- If $v_i$ has the value $\text{false}$, then the node $t(i)$ will be visited before $s(i)$ and none of the nodes $i\#o_1, i\#o_2, \cdots, i\#o_k$ will be visited in the interval between $\text{first}(i)$ and $\text{last}(i)$ visits.

Proof:

- By induction:

- Start of induction, consider all input-nodes.

- Induction-step, Assume above statement holds for all graphs $G_j$ ($1 \leq j < i$).
Continuation of the Reduction (Start of Induction)

- If \( v_i \) has the value \textit{true}, then we visit \( s(i) \) before \( t(i) \) and the nodes \( i \# o_1, i \# o_2, \ldots, i \# o_k \) are visited after \textit{first}(i)\) and before \textit{last}(i)\).

\[
\text{First}(i) \quad \downarrow
\text{last}(i - 1) \quad \downarrow
\text{last}(i) \quad \downarrow
\text{vi ist Input} \quad \downarrow
\text{vi has the value true, then we visit s(i) before t(i)} \quad \downarrow
\text{and the nodes i\#o_1, i\#o_2, \ldots, i\#o_k} \quad \downarrow
\text{are visited after first(i) and before last(i)}.
\]
Continuation of the Reduction (Induction-Step)

- If \( v_i \) has the value \textit{true}, then \( s(i) \) will be visited before \( t(i) \) and the nodes \( i \# o_1, i \# o_2, \ldots, i \# o_k \) are visited after \textit{first}(i) \) and before \textit{last}(i).
- Then the nodes \( v_{i_1} \) and \( v_{i_2} \) have the value \textit{false}.
Continuation of the Reduction (Induction-Step)

- If $v_i$ has the value $false$, then the node $t(i)$ will be visited before $s(i)$ and none of the nodes $i\#o_1, i\#o_2, \cdots, i\#o_k$ will be visited in the interval between $first(i)$ and $last(i)$ visits.
- Then one of the nodes $v_{i_1}$ or $v_{i_2}$ has the value $true$.

```
last(i - 1)

first(i)

\node (first) at (0,0) [circle,fill,inner sep=2pt] {$first(i)$};
\node (last) at (0,-1) [circle,fill,inner sep=2pt] {$last(i)$};
\node (ti) at (1,1) [circle,fill,inner sep=2pt] {$t(i)$};
\node (si) at (1,2) [circle,fill,inner sep=2pt] {$s(i)$};
\node (vi) at (0,1) [circle,fill,inner sep=2pt] {$v_i$ ist intern};
\draw[->] (first) -- (ti);
\draw[->] (last) -- (si);
\draw[->] (first) -- (si);
\draw[->] (last) -- (ti);
\draw[->] (vi) -- (first);
\draw[->] (vi) -- (last);
\draw[->] (vi) -- (ti);
\draw[->] (vi) -- (si);
```
Continuation of the Reduction (DFS)

- The construction is a NC-Reduction.
- The construction is the direct simulation of the operations of the circuit.
- The construction may be also given for non-directed graphs.
Reduction (MAXFLOW)

Definition (MAXFLOW)

- Input: directed graph $G = (V, E)$, $s, t \in V$ and capacity function $c : E \mapsto \mathbb{N}$.
- Output: Maximal flow from $s$ to $t$, i.e. function $f : E \mapsto \mathbb{N}$.
  - with: $\forall e \in E : f(e) \leq c(e)$
  - and: $\forall v \in V \setminus \{s, t\}: \sum_{e=(a, v) \in E} f(e) = \sum_{e=(v, a) \in E} f(e)$

Theorem:

The MAXFLOW problem is $\mathcal{P}$-complete.

Proof:

- Reduction from the problem CVP.
- Show, even to compute the parity of a flow (PMAXFLOW), is $\mathcal{P}$-complete.
Continuation of the Reduction (MAXFLOW)

- W.l.o.g. out-degree of a input node 1.
- W.l.o.g. out-degree of a node is at most 2.
- W.l.o.g. circuit is revers topological sorted, i.e. $v_0$ is the output node.
- W.l.o.g. $v_0$ is an or.
- Given is the circuit graph $G = (V, E)$.
- Input for PMAXFLOW: $G' = (V \cup \{s, t\}, E')$.
- $E \subseteq E'$.
- $E' \subseteq E \cup \{(s, v), (v, t) \mid v \in V\}$
Continuation of the Reduction (MAXFLOW)

- \( \forall(i, j) \in E : c((i, j)) = 2^i \).
- If the value of \( v_i \) is true then let: \( f((i, j)) = 2^i \ (\forall(i, j) \in E) \).
- If the value of \( v_i \) is false then let: \( f((i, j)) = 0 \ (\forall(i, j) \in E) \).
- Let \( d(0) = 1 \) and otherwise let \( d(i) \) be the out-degree of \( v_i \).
- Let \( (k, i), (j, i) \in E \), and let \( \text{surplus}(i) := 2^k + 2^j - d(i)2^i \).
- \( \forall i \in V : c(s, i) = 2^i \) if the value of \( v_i \) is true.
- \( \forall i \in V : c(s, i) = 0 \) if the value of \( v_i \) is false.
- \( \forall i \in V : c(i, t) = \text{surplus}(i) \) if \( v_i \) is an and-node.
- \( \forall i \in V : c(i, s) = \text{surplus}(i) \) if \( v_i \) is an or-node.
- \( c(0, t) = 1 \).
Continuation of the Reduction (MAXFLOW)

- $\forall i \in V : f(s, i) = c(s, i)$.
- $\forall i \in V : f(i, j) = c(i, j)$ if $v_i$ is an input-node.
- $\forall (i, j) \in E : f(i, j) = c(i, j) = 2^i$ if the value of $v_i$ is $true$.
- $\forall (i, j) \in E : f(i, j) = 0$ if the value of $v_i$ is $false$.
- $f(0, t) = 1$ if $v_0$ has the value $true$.
- Let $overflow(i)$ be the difference between the current input-flow and the output-flow.
  - $f((i, t)) = overflow(i)$ if $v_i$ is an and-node.
  - $f((i, s)) = overflow(i)$ if $v_i$ is an or-node.
- Note: the defined function $f$ is a flow.
Continuation of the Reduction (MAXFLOW)

**Lemma**

The defined flow is optimal.

- Use enlarging paths from $s$ to $t$:
  - An edge $e = (i, j)$ in the path is called forward-edge if $f(e) < c(e)$.
  - An edge $e = (j, i)$ in the path is called backward-edge if $f(e) > 0$.

- Known: Flow is maximal $\Leftrightarrow$ there is no enlarging path.

- Assume: there is an enlarging path.
  - A path starts at $s$ with a backward-edge.
  - A path ends at $t$ with a forward-edge.
Continuation of the Reduction (MAXFLOW)

- Thus we have three consecutive nodes \( j, i, k \) with:
  - \( j \neq t \).
  - \( k \neq s \).
  - \((j, i)\) is a backward-edge.
  - \((i, k)\) is a forward-edge.
  - \((i, j), (i, k)\) are edges in \( E' \).
  - \( f((i, j)) > 0 \) and \( f((i, k)) < c((i, k)) \).

- \( v_i \) may not be a input-node.
- \( v_i \) may not be an and-node, because from \( j \neq t \) and \( f((i, j)) > 0 \) we get that all outgoing edges are full.
- \( v_i \) may not be an or-node, because from \( k \neq s \) and \( f((i, k)) < c((i, k)) \) be get that all outgoing edges are without flow.
Legend

- : Not of relevance
- : implicitly used basics
- : idea of proof or algorithm
- : structure of proof or algorithm
- : Full knowledge