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Definition of a Broadcasts and Accumulation

**Definition of Broadcast:**

Given are $G = (V, E)$ and $v \in V$.
- $v$ has information $I(v)$ and
- no node from $V \setminus \{v\}$ knows $I(v)$.
- Each node of $V \setminus \{v\}$ has to receive information $I(v)$.

**Definition of Accumulation:**

Given are $G = (V, E)$ and $v \in V$.
- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\bigcup_{w \in V} I(w)$. 
Definition of a Gossip

Definition of Accumulation:

Given are $G = (V, E)$ and $v \in V$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\bigcup_{w \in V} I(w)$.

Definition (Gossip):

Given is $G = (V, E)$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Each node of $v \in V$ should receive the information $\bigcup_{w \in V} I(w)$.
Types of Communication

- Telegraph-Mode: Communication is directed.
  - Is also called one-way communication.

- Telephone-Mode: Information is exchanged.
  - Is also called two-way communication.
  - Communication only between neighbours.
  - Communication is done in rounds.
  - In each round the active edges are a matching.
  - Each round uses one time-unit.
In the broadcast-problem the information of one node is transfered to all others.

The accumulation-problem is a “inverse” broadcast.

A gossip distributes the sum of all informations to all nodes.

In each round the communication is done by a matching.

The communication on an edge may be one-way or two-way, depending on the mode.

The size of send date is ignored.
Definition

- By $\text{comm}(A)$ we denote the complexity (number of rounds) of a communication-algorithm.

- $r(G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the gossip-problem on } G\}$

- $r_2(G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the gossip-problem on } G\}$

- $b(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the broadcast-problem on } G \text{ and } v\}$

- $a(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the accumulations-problem on } G \text{ and } v\}$
First Results

- For each graph $G$ and $v \in V$ we have:
  - $a(v, G) = b(v, G)$
  - $a(G) = b(G)$
  - $\text{mina}(G) = \text{minb}(G)$

- Note: reverse broadcast is accumulation.

- There exists a graph $G$ with: $r(G) = 2 \cdot r_2(G)$.

- Note: 2-clique or cycle of length four.

- The following holds: $\text{minb}(G) \leq b(G) \leq r_2(G) \leq r(G) \leq 2 \cdot r_2(G)$.

- The inequalities result from the definitions.

- $\text{minb}(L(n)) = \lceil n/2 \rceil$

- Optimal broadcast on a line start in the center of the line.

- $b(L(n)) = n - 1$

- A message from the left has to traverse all edges.
First Results II

Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots, E_z$ be the corresponding one-way broadcast-algorithm.
- Let $B = F_1, F_2, \cdots, F_z$ be the corresponding one-way accumulation-algorithm.
- Then is $F_1, F_2, \cdots, F_z, E_1, E_2, \cdots, E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:
For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and
\[
b(G) = r(G)
\]

Proof (for \( n = 8 \)):

Both broadcasts together are a gossip-algorithm.
First Results IV

- $\text{rad}(G) \leq \text{minb}(G)$.
- $\text{rad}(G) \leq \text{diam}(G) \leq b(G)$.
- Let $G = (V, E)$ and $H = (V, F)$ with $F \subseteq E$. Then we have:
  - $b(G) \leq b(H)$.
  - $\text{minb}(G) \leq \text{minb}(H)$.
  - $r(G) \leq r(H)$.
  - $r_2(G) \leq r_2(H)$.
- $\text{minb}(G) \leq (\deg(G) - 1) \cdot \text{rad}(G) + 1$.
- $b(G) \leq (\deg(G) - 1) \cdot \text{diam}(G) + 1$.
- $b(G) \leq \deg(G) \cdot \text{rad}(G)$.
- $r(G) \leq 2(\deg(G) - 1) \cdot \text{rad}(G) + 2$.
- $r_2(G) \leq 2(\deg(G) - 1) \cdot \text{rad}(G) + 1$.
Lower Bound

**Lemma**

Let $G = (V, E)$ be a graph with $n$ nodes. Then we have:

- $b(G) \geq \min b(G) \geq \lceil \log n \rceil$

**Proof:**

- Let $A(t)$ be the number of informed nodes after $t$ rounds.
- $A(0) = 1$
- $A(t + 1) \leq 2 \cdot A(t)$
- $A(t) \leq 2^t$
- At the end $2^t \geq n$ must hold.
Optimal Broadcast-Tree

Each informed node has to send in each round the information to a non-informed node:

A tree $T_i$ is a broadcast-tree, iff

- the root of $T_i$ has $i$ successors $v_0, v_1, \ldots, v_{i-1}$ and
- $v_j$ is the root of a $T_j$. 
First Results

Lemma

We have:

- \( \min b(K(n)) = b(K(n)) = \lceil \log n \rceil \) and
- \( \min b(HQ(m)) = b(HQ(m)) = m. \)

Proof \((K(n))\):

\[
\text{for } t = 1 \text{ to } \lceil \log n \rceil \text{ do} \\
\quad \text{for all } i \in \{0, 1, \ldots, 2^{t-1} - 1\} \text{ do in parallel} \\
\quad \quad \text{if } i + 2^{t-1} \leq n \text{ then} \\
\quad \quad \quad i \text{ sends to } i + 2^{t-1}
\]

Proof \((HQ(m))\):

\[
\text{for } i = 1 \text{ to } m \text{ do} \\
\quad \text{for all } a_1, a_2, \ldots, a_{i-1} \in \{0, 1\} \text{ do in parallel} \\
\quad \quad a_1a_2\cdots a_{i-1}00\cdots0 \text{ sends to } a_1a_2\cdots a_{i-1}10\cdots0
\]
First Results II

Lemma
For all $k, m \geq 2$ we have: $\min b(T_k(m)) = k \cdot m$.

Idea of proof:

- $b(\varepsilon, T_k(m)) = k \cdot m$.
- $b(\varepsilon, T_k(m)) \leq b(\nu, T_k(m))$.
- Note that $\nu$ has to inform $\varepsilon$.
- and $\varepsilon$ has to inform the other successors.
Lemma

We have:

- \( b(\text{CCC}(k)) \leq 5k + O(1) \)
- \( b(\text{BF}(k)) \leq 4.5k + O(1) \)
- \( b(\text{SE}(k)) \leq 4k + O(1) \)
- \( b(\text{DB}(k)) \leq 3k + O(1) \)

Proof: Use the following statements:

- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1. \)
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G). \)
Theorem:

We have: \( \lceil 5k/2 \rceil - 2 \leq \min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5k/2 \rceil - 1. \)

- The following parts are proven:
  - \( \min b(\text{CCC}(k)) \geq \lceil 5k/2 \rceil - 2 \)
  - Algorithm for \( \lceil 5k/2 \rceil - 1 \) will be presented.
CCC, Proof $\min_b(\text{CCC}(k)) \geq \lceil 5 \cdot k/2 \rceil - 2$

- $\text{diam}(\text{CCC}(k)) = \lceil 5/2 \cdot k \rceil - 2$
- The statement holds for even $k$.
- Let $k$ be odd.
- Let $(0,00\cdots 0)$ be the origin of the message.
- The nodes $(\lfloor k/2 \rfloor, 11\cdots 1)$ and $(\lfloor k/2 \rfloor + 1, 11\cdots 1)$ are both in distance $(\lceil 5 \cdot k/2 \rceil - 2)$.
- Thus we need one round more then the diameter.
- The statement hold, because the CCC is node-symetric.
Algorithm BROADCAST-\(\text{CCC}_k\)
(0, 00...0) sends to (0, 10...0);
for \(i = 0\) to \(k - 1\) do begin
  for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel
    \((i - 1, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \((i, a_0 \ldots a_{i-1}00 \ldots 0)\);
  for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel
    \((i, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \((i, a_0 \ldots a_{i-1}10 \ldots 0)\);
end;
for all \(\alpha \in \{0, 1\}^k\) do in parallel
  Broadcast on cycle \(C_\alpha(k)\) starting from \((k - 1, \alpha)\);
Theorem:

We have: \( \min_b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5 \cdot k/2 \rceil - 2. \)

Idea of proof: Change the first phase and send in both directions.
Theorem:
We have: \( \min b(SE(k)) = b(SE(k)) = 2 \cdot k - 1 \)

Proof:
- The diameter provides the lower bound.
- Note \( SE(k) \) is not node-symmetric.
- We have to provide an algorithm for any node \( v \).
- Algorithm has to be without conflicts.
- And we do now show it here in detail.
**Theorem:**

We have: \( \lceil \frac{3m}{2} \rceil \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \)

- The diameter gives the lower bound.
- Algorithm will be provided in the following.
BF (Idea of proof)

- Distribute the information in two ways:
  - Prefer in the first strategy the cycle-edges.
  - Prefer in the second strategy the cross-edges.
- Split the butterfly into two isomorph parts.
- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

\[
\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
Splitting of $BF(m)$ in $F_0$ and $F_1$:
- $F_0$ has nodes: $\{(l, \alpha 0) \mid 0 \leq l \leq m-1, \alpha \in \{0,1\}^{m-1}\}$.
- $F_1$ has nodes: $\{(l, \alpha 1) \mid 0 \leq l \leq m-1, \alpha \in \{0,1\}^{m-1}\}$.
- $F_0$ and $F_1$ are isomorphic.

$\#_0(w)$ denotes the number of 0’en in $w$.
$\#_1(w)$ denotes the number of 1’en in $w$. 

$\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$
Consider $F_0$: from node $v_0 = (0, 00 \cdots 00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

Consider $F_1$: from node $v_1 = (m - 1, 00 \cdots 01)$ exists a unique path of length $m - 1$ to $w_1 = (0, \alpha 1)$ for $\alpha \in \{0, 1\}^{m-1}$.

First step of the algorithm $v_0$ informs $v_1$.

Then we use in $F_0$ and $F_1$ two different strategies.
BF (Proof III)

- **Aim:** Inform in \( \lceil 3m/2 \rceil \) steps the nodes \( w_0 = (m - 1, \alpha 0) \) and \( w_1 = (0, \alpha 1) \) for \( \alpha \in \{0, 1\}^{m-1} \).

- If a node \( w_0 = (m - 1, \alpha 0) \) gets informed, then it informs in the next step \( w_1 = (0, \alpha 1) \) (if necessary).

- If a node \( w_1 = (0, \alpha 1) \) gets informed, then it informs in the next step \( w_0 = (m - 1, \alpha 0) \) (if necessary).

\[ \lceil 3m/2 \rceil \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]
**BF (Proof IV)**

- In $F_0$ a informed node $(l, \alpha_0)$ sends first to $(l+1, \alpha_0)$ and then to $(l+1, \alpha(l)0)$. [$\alpha(l) = \alpha_1 \ldots \alpha_l \ldots$]
- In $F_1$ a informed node $(l, \alpha_1)$ sends first to $(l+1, \alpha(l)1)$ and then to $(l+1, \alpha_1)$.
- The time to inform from $v_0 = (0,00 \cdots 00)$ a node $w_0 = (m-1, \alpha_0)$ is: $1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha)$.
- The time to inform from $v_1 = (m-1,00 \cdots 01)$ a node $w_1 = (0, \alpha_1)$ is: $1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha)$.
**Case 1: m is odd:**

- **Case 1.1: \( \#_1(\alpha) < (m - 1)/2 \):**
  Node \( w_0 \) will be informed from \( v_0 \) at time
  \[
  m + \#_1(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor.
  \]
  After this \( w_0 \) sends to \( w_1 \).
  \( w_1 \) is informed at time \( \lfloor 3m/2 \rfloor \).

- **Case 1.2: \( \#_0(\alpha) < (m - 1)/2 \):**
  node \( w_1 \) will be informed from \( v_0 \) at time
  \[
  m + \#_0(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor.
  \]
  \( w_0 \) will be informed from \( w_1 \) at time \( \lfloor 3m/2 \rfloor \).

- **Case 1.3: \( \#_0(\alpha) = \#_1(\alpha) = (m - 1)/2 \):**
  \( w_0 \) is informed at time
  \[
  m + \#_1(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor.
  \]
  \( w_1 \) is informed at time \( m + \#_0(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor \).
BF (Proof V)

Case 2: $m$ is even:

- Case 2.1: $\#_1(\alpha) \leq (m - 2)/2$:
  node $w_0$ will be informed from $v_0$ at time $m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_1$ will be informed at time $\lfloor 3m/2 \rfloor$.

- Case 2.2: $\#_0(\alpha) \leq (m - 2)/2$:
  node $w_1$ will be informed from $v_0$ at time $m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_0$ will be informed at time $\lfloor 3m/2 \rfloor$.

In the last phase we distribute the information on the cycles.

- Running time is: $\lceil m/2 \rceil$ rounds.
- Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
Theorem:

We have: \( d \leq \min b(DB(d)) = b(DB(d)) \leq \lfloor 3/2 \cdot (d + 1) \rfloor. \)

Proof:

- Idea \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \bar{y}_1)\).
- The order is given by the parity.
- Let \(\alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2.\)
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \bar{\alpha})\).
- \((0011000)\) informs first \((0110000)\) and then \((0110001)\).
For $k \in \{0, 1\}$ consider the path $P_k$
from $(y_1, y_2, \ldots, y_d)$ to $(z_1, z_2, \ldots, z_{d-1}, z_d)$.

\begin{align*}
(y_1, \ldots, y_d), (y_2, \ldots, y_d, k), (y_3, \ldots, y_d, k, z_1), (y_4, \ldots, y_d, k, z_1, z_2), \ldots \hspace{1cm} \\
\ldots, (y_d, k, z_1, \ldots, z_{d-2}), (k, z_1, \ldots, z_{d-1}), (z_1, \ldots, z_{d-1}, z_d)
\end{align*}

Let $v_{0i} = (y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2})$ the i-th node on $P_0$.
Let $v_{1i} = (y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2})$ the i-th node on $P_1$.

We have different times (1 or 2) for sending:

- $(y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 0, z_1, \ldots, z_{i-2}, z_{i-1})$
- $(y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 1, z_1, \ldots, z_{i-2}, z_{i-1})$.

Thus the sum of running times is on $P_0$ and $P_1$: $3(d + 1)$.
Thus the running time for the broadcast is: $\left\lfloor \frac{3(d + 1)}{2} \right\rfloor$. 
Degree of the Nodes

Theorem:

Let $n \geq 5$ and $G = (V, E)$ be a graph with $n$ nodes:

- If $\Delta(G) = 3$ holds, we have: $b(G) \geq \min b(G) \geq 1.4404 \log(n) - 3$.
- If $\Delta(G) = 4$ holds, we have: $b(G) \geq \min b(G) \geq 1.1374 \log(n) - 2$.

Proof:

- Let $A$ be a broadcast-algorithm.
- Let $\text{Broad}_i^A(v_0)$ be the set of nodes, which are informed from $v_0$ by $A$ in $i$ rounds.
- Let $\text{Rec}_i^A(v_0) = \text{Broad}_i^A(v_0) \setminus \text{Broad}_{i-1}^A(v_0)$.
- Let $\text{Rec}_0^A(v_0) = \{v_0\}$.
- We have: $|\text{Broad}_i^A(v_0)| = \sum_{s=0}^{i} |\text{Rec}_s^A(v_0)|$. 


Building the Idea

We consider here only the case $\Delta(G) = 3$. The case $\Delta(G) = 4$ is similar.

- The initial node may send at most three times.
- The initial node sends only in rounds 1, 2, 3.
- Any other nodes will be informed at time $t$ via an edge $e$.
- No further node may be informed via $e$.
- Thus any other node may send at most two times.
- If a node $v$ is informed in round $t$ by $w$, then did $w$ receive the information at round $t-1$ or $t-2$.
- Thus the number of newly informed nodes in round $t > 3$, is at most the number of nodes which got informed in rounds $t-1$ and $t-2$. 
Proof

- Let $A(i) = |\text{Rec}_i^A(v_0)|$.
- $A(0) = 1$
- $A(1) = 1$
- $A(2) = 2$
- $A(3) = 4$
- $A(i) = A(i - 1) + A(i - 2)$ für $i \geq 4$.
- Show by induction: $A(i) \leq 1.61804^i$ for $i \geq 0$. 
Proof

- \( A(0) = 1 \leq 1 = 1.61804^0 \)
- \( A(1) = 1 \leq 1.61804 = 1.61804^1 \)
- \( A(2) = 2 \leq 2.61805 = 1.61804^2 \)
- \( A(3) = 4 \leq 4.23612 = 1.61804^3 \)

- Induction step \((i \geq 4):\)
  - We have: \( A(j) \leq 1.61804^j \) for any \( j \leq i - 1. \)
  - \( A(i) = A(i - 1) + A(i - 2) \leq 1.61804^{i-1} + 1.61804^{i-2} \leq 1.61804^i \)
  - Note for this: \( 1.61804 + 1 \leq 1.61804^2. \)

- Thus we have: \( n \leq |\text{Broadcast}_A(v_0)| = \sum_{i=0}^{t} |\text{Rec}_A(v_0)| \leq \sum_{i=0}^{t} A(i) \leq \sum_{i=0}^{t} 1.61804^i = \frac{1.61804^{t+1} - 1}{1.61804 - 1} \leq 3 \cdot 1.61804^t \)
- \( t \geq 1.4404 \cdot \log_2 n - 3. \)
- Proof of the second statement may be done in the same way.
More Results

Consequence:
\[ b(DB_k) \geq \min b(DB_k) \geq 1.1374 \cdot k - 2 \]

Theorem:
\[ b(BF_m) = \min b(BF_m) > 1.7396m \text{ for large enough } m. \]

Idea of Proof: Check the number of nodes in distance \( k \).

Theorem:
\[ b(DB_m) > 1.3042m \text{ for large enough } m. \]

Idea of Proof: Check the number of nodes in distance \( k \).
## Overview

| Graph  | $|V|$          | Diameter | Lower Bound  | Upper Bound  |
|--------|---------------|----------|--------------|--------------|
| $K_n$  | $n$           | 1        | $\lceil \log_2 n \rceil$ | $\lceil \log_2 n \rceil$ |
| $HQ_k$ | $2^k$         | $k$      | $k$          | $k$          |
| $CCC_k$| $k \cdot 2^k$| $\lfloor 5k/2 \rfloor - 2$ | $\lfloor 5k/2 \rfloor - 2$ | $\lfloor 5k/2 \rfloor - 2$ |
| $SE_k$ | $2^k$         | $2k - 1$ | $2k - 1$     | $2k - 1$     |
| $DB_k$ | $2^k$         | $k$      | $1.4404k$    | $3/2 (k + 1)$|
| $BF_k$ | $k \cdot 2^k$| $\lfloor 3k/2 \rfloor$ | $1.7609k$    | $2k - \frac{1}{2} \log \log k + c$ |
In edge-disjoint-path communication the information is passed on a set of edge-disjoint paths between the endpoint of each path. A sending or receiving node may not forward any information at the same time. A edge-disjoint communication algorithm for $G$ is a sequence of rounds $A_1, A_2, \ldots, A_k$, where each $A_i$ is a correct edge-disjoint-path communication.

- $edp-b(G) =$ maximal time to broadcast in edp-mode in $G$.
- $edp-a(G) =$ maximal time to accumulate in edp-mode in $G$.
- $edp-r_1(G) =$ minimal time to gossip in 1-way edp-mode in $G$.
- $edp-r_2(G) =$ minimal time to gossip in 2-way edp-mode in $G$. 
Idea \((edp-a(G))\)

**Definition**

A set of vertices \(K \subset V\) is called knowledge set, if the pieces of information residing in the vertices of \(K\) form the cumulative message.

**Definition**

Let \(T = (V, E)\) be some tree, we will refer to any vertex of degree \(> 2\) in \(T\) as a critical vertex, while all other vertices are called non-critical.

- Collect within two rounds all pieces of information in a subset \(K\) of non-critical vertices with \(|S| \leq \lfloor n/2 \rfloor\).
- In each of \(\lceil \log_2 n \rceil - 1\) communication rounds, reduce the size of a given knowledge set \(K\) by a factor of two.
**Lemma:**

Let $G = (V, E)$ a graph with $n$ nodes. Then we have:

$$r(G) \geq r_2(G) \geq \begin{cases} \lceil \log_2 n \rceil & n \text{ even,} \\ \lceil \log_2 n \rceil + 1 & n \text{ odd.} \end{cases}$$

Proof: Only the case, where $n$ is odd, has to be proven.

- **Show:** $r_2(G) \geq \lceil \log_2 n \rceil + 1$.

- Let $A$ be a communication-algorithm for the gossip-problem. $A$ has communication rounds (matchings) $E_1, E_2, \cdots, E_k$.

- **Show by induction:** After $i$ rounds has each node at most $2^i$ pieces of information.
  - $i = 0$: Each node has $2^0 = 1$ pieces of information.
  - $i - 1 \rightarrow i$: at most $2^{i-1} + 2^{i-1} = 2^i$ pieces of information may be collected by any node.

- In round $k$ is at least one node $v$ inactive.

- $v$ has after $k$ rounds at most $2^{k-1}$ pieces of information.
Lemma:

For any graph $G = (V, E)$ with $|V| = n$ we have:

- $r(G) \leq 2n - 2$, and
- $r_2(G) \leq 2n - 3$.

Proof: Follows from the following known statements:

- $\minb(G) \leq n - 1$ for any graph $G = (V, E)$ with $|V| = n$.
- $r(G) \leq 2 \cdot \minb(G)$
- $r_2(G) \leq 2 \cdot \minb(G) - 1$
Lemma:

We have:

- \( r(T_k(1)) = 2k \)
- \( r_2(T_k(1)) = 2k - 1 \)

Proof:

- Show: \( r(T_k(1)) \geq 2k \).
- \( r(T_k(1)) \) has one root and \( k \) leaves.
- The maximal matching is 1.
- In each round is only one leaf active.
- Each leaf has to send at least once.
- Each leaf has to receive at least once.
- Thus in total \( 2k \) rounds necessary.
- \( r_2(T_k(1)) \geq 2k - 1 \), is a simple exercise.
Theorem:

We have:

- \( r_2(L(n)) = n - 1 \) for any even number \( n \geq 2 \),
- \( r_2(L(n)) = n \) for any odd number \( n \geq 3 \),
- \( r(L(n)) = n \) for any even number \( n \geq 2 \) and
- \( r(L(n)) = n + 1 \) for any odd number \( n \geq 3 \).

Proof:

- All are more or less easy.
Gossip on arbitrary Trees

Lemma:
For any tree $T$ we have:
- $r(T) = 2 \cdot \min_b(T)$
- $r_2(T) = 2 \cdot \min_b(T) - 1$

Idea of the proof:
- We have already for any graph $G$: $r(G) \leq 2 \cdot \min_b(G)$.
- We have to show: $r(G) \geq 2 \cdot \min_b(G)$.
- Let $W = \bigcup_{v \in V} I(v)$ be the total information.
- Let $A$ be any communication algorithm on $T$.
- Let $t$ be the point in time, when some node knows $W$.
- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
Gossip on arbitrary Trees (Proof I)

Let $u \neq v$ be an other node which knows $W$ after $t$ steps.

Let $(u, y_1, y_2, \cdots, y_k, v)$ be the unique path connecting $u$ and $v$.

If $v$ sends to $y_k$ at time $t$, then $v$ did know $W$ at time $t - 1$.

So we have to consider the case: $y_k$ sends to $v$ at time $t$:

- In this case $y_k$ sends $v$ some missing information.
- $y_k$ knows at time $t - 1$ the full information, which has to be send from $y_k$ to $v$.
- The information, which has to be send from $v$ to $y_k$, is already send.
- Then the node $y_k$ know $W$ at time $t - 1$.

Contradiction, the node $u$ does not exist.

Thus we have: $t \geq \min b(T) = b(v, T)$. 

\[ u \quad y_1 \quad y_2 \quad y_3 \quad y_k \quad v \]
Gossip on arbitrary Trees (Proof II)

Consider the situation at node $v$ after round $t$.
Let w.l.o.g. $v$ be the root of $T$.
Let $v_1, v_2, \ldots, v_k$ be the successors of $v$.
Let $T_1, T_2, \ldots, T_k$ be the subtrees with roots $v_1, v_2, \ldots, v_k$.
In each subtree $T_i$ is some information $w_i$ missing.
Only the node $v$ knows $\bigcup_{j=1}^k w_j$.
Thus there are $b(v, T)$ steps to be done.
We finally have $r(T) \geq \min b(T) + b(v, T) \geq 2 \cdot \min b(T)$
Consider the two-way mode: by a similar way we may prove:

- At time $t$ only two neighbours nodes $u$ and $v$ know the total information. We get in the similar way the second statement.
Lemma:

For all \( m \geq 1 \) and \( k \geq 2 \) we have:

- \( r(T_k(m)) = 2 \min b(T_k(m)) = 2 \cdot k \cdot m \).
- \( r_2(T_k(m)) = 2 \min b(T_k(m)) - 1 = 2 \cdot k \cdot m - 1 \).
Gossip on Cycles

Theorem:

We have:
- \( r_2(C(k)) = k/2 \) for even \( k \).
- \( r_2(C(k)) = \lceil k/2 \rceil + 1 \) for odd \( k \).

Idea of the proof (\( k \) even): \([k \text{ odd: an easy exercise}]\)

- Let \( k \) be even.
- \( r_2(C(k)) \geq k/2 \) results by the diameter.
- \( r_2(C(k)) \leq k/2 \) is true by the following algorithm:
  1. \( \{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i + 1\}, \ldots, \{n - 2, n - 1\} \)
  2. \( \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\} \)
  3. \( \{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i + 1\}, \ldots, \{n - 2, n - 1\} \)
  4. \( \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\} \)
  5. \( \ldots \)

- Note: After \( i \) rounds knows each node \( 2 \cdot i \) Informationen.
1-Way Gossip on Cycles (Idea)

- Messages should traverse in both directions.
- Activate each \( f(n) \)-th node on the cycle.
- This will result in an additional \( \Theta(f(n)) \) steps.
- During the distribution we get \( \Theta\left(\frac{n}{2f(n)}\right) \) delays.
- Thus we will choose \( f(n) = \Theta(\sqrt{n}) \).
- By this idea we may get a lower and upper bound.
Gossip on Cycles (Idea)
Gossip on Cycles (Idea of the algorithm)

- Split the cycle in $\Theta(\sqrt{n})$ blocks $B_i$.
- Within block $B_i$ ($i \in \{1, 2, 3, \ldots, k\}$ with $k \in \Theta(\sqrt{n})$) do the following:
  - Phase 1:
    - The nodes $v_i [u_i]$ start a “wave” to the left [right].
    - The messages of $v_i$ and $u_i$ are delayed $\Theta(\sqrt{n})$ times by the other messages.
    - After $n/2 + \Theta(\sqrt{n})$ round know nodes $z_i$ the total information.
  - Phase 2:
    - Each node $z_i$ distribute the total information to $\Theta(\sqrt{n})$ nodes.
- Note: If $n$ is even, we have always a delay of one and the synchronization is easy.
Theorem:

We have:

- $r(C(n)) \leq n/2 + \sqrt{2n} - 1$ for even $n$.
- $r(C(n)) \leq \lceil n/2 \rceil + \lceil 2 \cdot \sqrt{\lceil n/2 \rceil} \rceil - 1$ for odd $n$.
- $r(C(n)) \geq n/2 + \sqrt{2n} - 1$ for even $n$.
- $r(C(n)) \geq \lceil n/2 \rceil + \lceil \sqrt{2n} - 1/2 \rceil - 1$ for odd $n$.

Proof: See literature.
Theorem:
For all $m \in \mathbb{N}$ we have: $r_2(HQ(m)) = m$

Proof:
- The lower bound is the diameter.
- Upper bound by the following algorithm:
  
  ```
  for $i = 1$ to $m$ do
    for all $a_1, a_2, \ldots, a_{m-1} \in \{0, 1\}$ do in parallel
    $a_1 a_2 \cdots a_{i-1} 0 a_i a_{i+1} \cdots a_{m-1}$ sends to
    $a_1 a_2 \cdots a_{i-1} 1 a_i a_{i+1} \cdots a_{m-1}$
  ```

Corollary:
For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
Consider one-way mode:
- Start with the first phase of the gossip-algorithm for cycles on all cycles.
- Then each $\Theta(\sqrt{n})$-th node on each cycle knows the total information of its cycles.
- In $\Theta(\sqrt{n})$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each $\Theta(\sqrt{n})$-th node of each cycle the total information.
- The final part is the second phase of the gossip-algorithm of cycles on all cycles.
- All nodes know now the total information.
Consider two-way mode:

- Start with the gossip algorithm for cycles on all cycles.
- Each node of the cycle knows now the total information of its cycle.
- In $\Theta(n/2)$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each node the total information.
Theorem:

Let $k \geq 3$, then we have:

- $r(\text{CCC}(k)) \leq r(\text{C}(k)) + 3k - 1 \leq \left\lceil \frac{7k}{2} \right\rceil + \left[ 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right] - 2.$
- $r(\text{BF}(k)) \leq r(\text{C}(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left[ 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right] - 1.$
- $r_2(\text{CCC}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.
- $r_2(\text{CCC}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$.
- $r_2(\text{BF}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.
- $r_2(\text{BF}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$. 
Gossip on Graphs with $2 \cdot m$ Nodes (0. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes (1. Idea)

Implication:

- For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
- For all $m \in \mathbb{N}$ we have: $r_2(K(m)) \leq \lceil \log m \rceil + 1$
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)

- Too many nodes where inactive for too long time.
- These nodes could not double their information.
- Idea: Try to double the information of any node.
- Detailed idea: In each step each node has an “interval” of information.
- To make the doubling easy split the nodes into two groups.
- Both groups should be the same size.
- In the first step pairs of node from each group share their information.
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes

Theorem:

For all $m \in \mathbb{N}$ we have: $r_2(K(2m)) = \lceil \log_2 m \rceil$

Proof: Split the nodes in groups $Q[i]$ and $R[i]$ ($0 \leq i \leq m - 1$).

- **algorithm:**
  
  for all $i \in \{0, \ldots, m - 1\}$ do in parallel
  
  Exchange the information between $Q[i]$ and $R[i]$

  for $t = 1$ to $\lceil \log_2 m \rceil$ do
  
  for all $i \in \{0, \ldots, m - 1\}$ do in parallel
  
  Exchange the information between $Q[i]$ and $R[(i + 2^{t-1}) \bmod m]$

- **Invariant:**
  
  - Let $\alpha[i]$ be the information of $Q[i]$ and $R[i]$ after their initial exchange.
  - After round $t$ know nodes $Q[i]$ and $R[(i + 2^{t-1}) \bmod m]$: $\cup_{0 \leq j \leq 2^t - 1} \alpha[(i + j) \bmod m]$

  The invariant is easy to be shown.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (a try)

- We need an extra round.
- A nice proof with this idea will become complicated.
- We will try to put some structure into the proof.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (Idea)

- How could this be an idea?
- We only have the edges of the first step.
- Idea: We could now choose a small even number of Nodes, which together have the total information.
- These nodes may perform the above gossip algorithm.
- In the last step we repeat the first round.
Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be all nodes.
- For all $i \in \{0, 1, \ldots, m - 1\}$ the node $v_{m+2+i}$ sends to $v_i$.
- The node $\{v_0, v_1, v_2, \ldots, v_m\}$ have now the total information.
- If $m+1$ is even, perform a gossip on the nodes $\{v_0, v_1, v_2, \ldots, v_m\}$.
- If $m+1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \ldots, v_{m+1}\}$.
- For all $i \in \{0, 1, \ldots, m - 1\}$ the nodes $v_i$ send to $v_{m+2+i}$.
- Correctness follows direct by the construction.

Running time for $m+1$ even:

$$r_2(K(m+1)) + 2 = \lceil \log_2(m+1) \rceil + 2 = \lceil \log_2 \left( \frac{n+1}{2} \right) \rceil + 2$$

$$= \lceil \log_2(n+1) \rceil + 1 = \lceil \log_2 n \rceil + 1$$

Running time for $m+1$ odd:

$$r_2(K(m+2)) + 2 = \lceil \log_2(m+2) \rceil + 2 = \lceil \log_2 \left( \frac{n+3}{2} \right) \rceil + 2$$

$$= \lceil \log_2(n+3) \rceil + 1 = \lceil \log_2 n \rceil + 1$$
1st Idea (Let the Knowledge grow)

We need more rounds.

A nice proof with this idea will become complicated.

We will try to put some structure into the proof.
2nd Idea (Let the Knowledge grow in a structured way)

We need an additional two rounds.

$v_x$ and $w_y$ alternate as sender and receiver.

The information grows in blocks (intervals) in the nodes.

With this idea we may do the proof.

Only the first two rounds are special.
2\textsuperscript{nd} Idea (Let the Knowledge grow in a structured way)

- After the first two rounds some node-pairs share their information.
- Consider this situation as the start:
  - All $v_x$ and $w_x$ have one information pair.
  - $v_i$ sends to $w_j$ and the $w_x$ have 2 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 3 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 5 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 8 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 13 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 21 information pairs.
- Thus the grow-rate and the algorithm is clearly visible.
algorithm

- Let $n = 2m$.

- Gossip-Algorithm:
  
  $t := 0$;
  
  for all $i \in \{0, \ldots, m - 1\}$ do in parallel $R[i]$ sends to $Q[i]$;
  
  for all $i \in \{0, \ldots, m - 1\}$ do in parallel $Q[i]$ sends to $R[i]$;
  
  while $\text{fib}(2t + 1) < m$ do begin
    
    $t := t + 1$;
    
    for all $i \in \{0, \ldots, m - 1\}$ do in parallel
    
    $R[(i + \text{fib}(2t - 1)) \mod m]$ sends to $Q[i]$;
    
    if $\text{fib}(2t) < m$ then
    
    for all $i \in \{0, \ldots, m - 1\}$ do in parallel
    
    $Q[(i + \text{fib}(2t)) \mod m]$ sends to $R[i]$
    
  end;

\[
\begin{align*}
\text{fib}(0) &= \text{fib}(1) = 1 \\
\text{fib}(i) &= \text{fib}(i - 1) + \text{fib}(i - 2)
\end{align*}
\]
One-Way-Gossip

**Theorem:**

Let $n = 2m$ and $k = \min\{x \mid \text{fib}(x) \geq m\}$. Then we have $r(K(n)) \leq k + 1$.

**Proof:**

- The algorithm stops, if $\text{fib}(2t + 1) \geq m$ or $\text{fib}(2t) \geq m$ holds.
- The number of rounds within the loop is $2t$ or $2(t - 1) + 1$.
- The total number of rounds is $(k - 1) + 2$.
- Correctness may be proven by the following invariant:

  Let $a[i]$ be the information, which share $R[i]$ and $Q[i]$ after two rounds.

  After $t$ loops we have:

  - $Q[i]$ knows $\bigcup_{0 \leq j \leq \text{fib}(2t + 1) - 1} \alpha[(i + j) \mod m]$
  - $R[i]$ knows $\bigcup_{0 \leq j \leq \text{fib}(2t + 2) - 1} \alpha[(i + j) \mod m]$

  The correctness is a direct result of this.
One-Way-Gossip

Theorem:
Let \( n = 2m - 1 \) and \( k = \min\{x \mid \text{fib}(x) \geq m\} \). Then we have \( r(K(n)) \leq k + 2 \).

Proof: Using the same idea as for the two-way mode.

Theorem:
Let \( n \) even. Then we have: \( r(K(n)) \geq 2 + \lceil \log_{\frac{1}{2}(1+\sqrt{5})} \frac{n}{2} \rceil \).

Proof: See literature (Idea is given the following).

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<td>Lower Bound</td>
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</tbody>
</table>
Idea for the lower Bound

- **Situation:**
  - Algorithm with “fibonacci growth”.
  - No idea to enlarge this growth.

- **Construction of a lower bound:**
  - Start with an arbitrary algorithm.
  - Use only the restriction of the algorithm.
  - Abstract.

- We will now try to do the abstraction.
- Try the get the core-problem.
- The core-problem ist:
  - “Fibonacci growth” could not be improved.
1. Abstraction

Definition:

The Network Counting Problem:
- Given a directed graph $G = (V, E)$.
- Each node stores a number.
- Initial just the number 1 is stored.
- The receiver add the number from the sender to his number after one communication.
- The objective is: all nodes should store a number larger then $|V|$.
- With $nc(G)$ we denote the minimal rounds to achieve this objective.

Lemma:

For any graph $G$ we have: $r(G) \geq nc(G)$.
2. Abstraction

- Let $G = (\{v_1, v_2, v_3, \cdots, v_n\}, E)$ be a directed Graph.
- Each node $v_i$ stores after $t$ rounds the number $z_i^t$.
- One situation of the network counting problem could be described by a vector:
  - Initial: $(1, 1, 1, \cdots, 1)^T$.
  - After $t$ rounds: $(z_1^t, z_2^t, z_3^t, z_n^t)^T$.
- One round of an algorithm for the network counting problem is given by a matrix $B$:
  - $A$ is a $n \times n$ matrix.
  - $a_{ij} = 1$ node $j$ sends to node $i$.
  - $A$ contains on the diagonal only ones.
  - $A$ has in each row at most two ones.
  - $A$ has in each column at most two ones.
  - If $a_{ij} = a_{kl} = 1$ ($i \neq j \neq k \neq l$), then we have $l \neq i \neq k$ and $l \neq j \neq k$.
  - Thus we get: $A \cdot (z_1^t, z_2^t, z_3^t, z_n^t)^T = (z_1^{t+1}, z_2^{t+1}, z_3^{t+1}, z_n^{t+1})^T$
2. Abstraction (Continuation)

- We consider now matrices of the above form.
- These are matrices $A$, for which there is a transformation $T$ with:

$$TAT^{-1} = \begin{pmatrix} B & 0 \\ \cdot & B \\ \cdot & \cdot & B \\ 0 & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & 1 \end{pmatrix}.$$ 

and $B = \begin{pmatrix} 11 \\ 01 \end{pmatrix}$.

- We will estimate the growth, which these matrices provide for the network counting problem.
Recollection (Norm, 3. Abstraction)

- Let $\|\cdot\|$ be the vector norm over $\mathbb{R}^n$. Then we have:
  - $\|x\| = 0 \Leftrightarrow x = 0^n$,
  - $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$,
  - $\|x + y\| \leq \|x\| + \|y\|$  
  - this holds for all $\alpha \in \mathbb{R}, x, y \in \mathbb{R}^n$

- The matrix norm for a vector norm $\|\cdot\|$ is defined by $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Then we have:
  - $\|A\| = 0 \Leftrightarrow A = 0$
  - $\|A + B\| \leq \|A\| + \|B\|$
  - $\|\alpha A\| = \alpha \cdot \|A\|$
  - $\|A \cdot B\| \leq \|A\| \cdot \|B\|$
  - $\|A \cdot x\| \leq \|A\| \cdot \|x\|$  
  - this holds for all $A, B \in \mathbb{R}^{n^2}, x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \alpha \geq 0$.

- Here we use: $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for $\text{einf} \ x = (x_1, .., x_n)$,

- Known: $\|A\| = \text{Spectral Norm}(A) = \sqrt{|\lambda_{\text{max}}(A^T \cdot A)|}$ with: $\lambda_{\text{max}}$ is the largest Eigenvalue.
2. Abstraction (Continuation)

- We compute the spectral norm:
- \(|A| = |TAT^{-1}| = |B|.
- \(B^T \cdot B = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \begin{pmatrix} 11 \\ 01 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix}.
- \(\Rightarrow (2 - \lambda)(1 - \lambda) - 1 = 0
- \(\Rightarrow \lambda^2 - 3\lambda + 1 = 0
- \(\Rightarrow \lambda_{\text{max}}(B^T B) = \frac{3}{2} + \sqrt{\frac{5}{4}}
- \(\Rightarrow \lambda_{\text{max}}(A^T A) = \sqrt{\lambda_{\text{max}}(A^T A)} = \frac{1}{2}(1 + \sqrt{5})
Theorem:
A algorithm, solving the network counting problem needs $2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) \frac{n}{2} \rceil$ rounds.

Proof:
- Let $A_j, 1 \leq j \leq r$ be matrices, which solve the problem in $r$ rounds.
- $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdots A_2 \cdot A_1 \cdot (1, 1, \cdots, 1)$.
- $||\alpha|| \leq (\prod_{i=1}^{r-2} ||A_i||) \cdot ||(1, \ldots, 1)|| \leq (\frac{1}{2}(1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}$
- Let $inf(i, t)$ be the number, which have the nodes $v_i$ after $t$ rounds.
- After round $t$ we have: $inf(i, t) \geq n$ for all $i \in \{1, 2, \cdots, n\}$.
- After round $t-1$ we have: $inf(i, t-1) \geq n$ for at least $n/2$ nodes.
- There could be some $i$ with: $inf(i, t-2) \geq n$.
- But if $\alpha_i < n$ and $inf(i, t-1) \geq n$, then there exists $j$ with: $\alpha_i + \alpha_j \geq n$. 
\[ \alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdots A_2 A_1 (1, 1, \cdots, 1) \]

- Let
  - \( c_1 \) be the number of cases with: \( \alpha_i \geq n \),
  - \( c_2 \) be the number of cases with: \( \alpha_i < n \) and \( \alpha_j \geq n \),
  - \( c_3 \) be the number of cases with: \( \alpha_i < n, \alpha_j < n \) and \( \alpha_i + \alpha_j \geq n \).

- Then we have: \( c_1 \geq c_2 \) and \( c_1 + c_2 + c_3 \geq n/2 \).
- Thus we also get: \( 2c_1 + c_3 \geq \frac{n}{2} \)

\[ \|\alpha\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \sqrt{c_1 n^2 + c_3 \cdot 2 \cdot \frac{n^2}{4}} \geq n \cdot \sqrt{\frac{1}{2} (2c_1 + c_3)} \geq \frac{n}{2} \sqrt{n}. \]

- We already have:
  \[ \|\alpha\| \leq \left( \prod_{i=1}^{r-2} \|A_i\| \right) \cdot \|(1, \ldots, 1)\| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}. \]

- And we get:
  \[ \frac{n}{2} \cdot \sqrt{n} \leq \|\alpha\| \leq \Phi^{r-2} \cdot \sqrt{n}, \]

- From which we conclude:
  \[ r \geq 2 + \left\lceil \log_{\frac{1}{2} (1 + \sqrt{5})} \frac{n}{2} \right\rceil \]
Quality of these Bounds

Lemma:

Let $n = 2m$ and let:

- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{\frac{1}{2}}(1 + \sqrt{5}) m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Let $\Phi = \frac{1}{2} (1 + \sqrt{5})$.
- Then we have: $\Phi^2 = \Phi + 1$.
- Furthermore we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Consider $n \in \mathbb{N}$ with: $n = 2 \cdot F(k)$ for some $k$.
- Then we have: $t_1 = k + 1$ and 
  
  $t_2 = 2 + \lceil \log_{\Phi} F(k) \rceil = 2 + k - 1 = k + 1$.
- From which we get: $t_1 = t_2$ for these $n$. 
Quality of these Bounds (Part 2)

**Lemma:**

Let $n = 2m$ and let:
- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) \cdot m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

**Proof:**

- Setze $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
- Then we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Let $n = 2 \cdot m$ arbitrary.
  - Let $i$ be defined by: $\Phi^{i-1} < m \leq \Phi^i$, then we have: $t_2 = 2 + i$.
  - Let $k$ be the smallest number with $F(k) \geq m$.
  - Note: $\Phi^{k-2} \leq F(k) \leq \Phi^{k-1}$.
  - Then we have: $i = k - 1$ oder $i = k - 2$.
  - From which we conclude: $t_1 = k + 1 \leq i + 3$. 
## Summary (Telefon-Mode)

| Graph  | $|V|$       | diam | Lower Bound                  | Upper Bound                  |
|--------|------------|------|------------------------------|------------------------------|
| $K_n$  | $n$        | 1    | $\lceil \log_2 n \rceil + \text{odd}(n)$ | $\lceil \log_2 n \rceil + \text{odd}(n)$ |
| $H_k$  | $2^k$      | $k$  | $n - \text{even}(n)$        | $n - \text{even}(n)$        |
| $P_n$  | $n$        | $n - 1$ | $\lceil \frac{n}{2} \rceil + \text{odd}(n)$ | $\lceil \frac{n}{2} \rceil + \text{odd}(n)$ |
| $C_n$  | $n$        | $\lceil \frac{n}{2} \rceil - 2$ | $\lceil \frac{5k}{2} \rceil - 2$ | $\lceil \frac{5k}{2} \rceil - 2$, $k$ even |
| CCC$_k$| $k \cdot 2^k$ | $\lceil \frac{5k}{2} \rceil + 1$, $k$ odd | $2k - 1$                      | $2k + 5$                      |
| SE$_k$ | $2^k$      | $2k - 1$ | $1.9770k$                    | $2.25 \cdot k + o(k)$        |
| BF$_k$ | $k \cdot 2^k$ | $\lceil \frac{3k}{2} \rceil$ | $1.5965k$                    | $2k + 5$                      |
| DB$_k$ | $2^k$      | $k$  |                              |                              |
### Summary (Telegraph-Mode)

| Graph  | $|V|$ | diam | Lower Bound | Upper Bound |
|--------|-----|------|-------------|-------------|
| $K_n$  | $n$ | 1    | $1.44 \log_2 n$ | $1.44 \log_2 n$ |
| $H_k$  | $2^k$ | $k$ | $1.44k$ | $1.88k$ |
| $P_n$  | $n$ | $n - 1$ | $n + \text{odd}(n)$ | $n + \text{odd}(n)$ |
| $C_n$  | $n$ even | $\left\lfloor \frac{n}{2} \right\rfloor$ | $\frac{n}{2} + \left\lceil \sqrt{2n} \right\rceil - 1$ | $\frac{n}{2} + \left\lceil \sqrt{2n} \right\rceil - 1$ |
|        | $n$ odd | $\left\lceil \frac{n}{2} \right\rceil$ | $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil - 1$ | $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil 2 \sqrt{\left\lfloor \frac{n}{2} \right\rfloor} \right\rceil - 1$ |
| $CCC_k$ | $k \cdot 2^k$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lfloor \frac{5k}{2} \right\rfloor - 2$ | $\left\lfloor \frac{7k}{2} \right\rfloor + \left\lceil 2 \sqrt{\left\lfloor \frac{k}{2} \right\rfloor} \right\rceil - 2$ |
| $SE_k$ | $2^k$ | $2k - 1$ | $2k - 1$ | $3k + 3$ |
| $BF_k$ | $k \cdot 2^k$ | $\left\lfloor \frac{3k}{2} \right\rfloor$ | $1.9770k$ | $\left\lfloor \frac{5k}{2} \right\rfloor + \left\lceil 2 \sqrt{\left\lfloor \frac{k}{2} \right\rfloor} \right\rceil - 1$ |
| $DB_k$ | $2^k$ | $k$ | $1.5965k$ | $3k + 3$ |
Lemma

\[ edp-r_1(G) \leq \min_{u \in V(G)} \{ edp-a_u(G) + edp-b_u(G) \} = 2 \cdot edp-b_{\min}(G) \]
\[ edp-r_2(G) \leq 2 \cdot edp-b_{\min}(G) - 1 \]

Lemma

For any graph \( G_n \) of \( n \) nodes, \( n \geq 2 \),

- \( \lceil \log_2 n \rceil \leq edp-r_2(G_n) \leq 2 \cdot \lceil \log_2 n \rceil + 1 \),
- \( \log_b(\lfloor n/2 \rfloor) + 2 \leq edp-r_1(G_n) \leq 2 \cdot \lceil \log_2 n \rceil + 2 \).
Results

Lemma

For each complete binary tree $T_h^2$ of depth $h \geq 3$ (and $n = 2^{h+1} - 1$ nodes),

- $2h + 3 = 2 \cdot \lceil \log_2 n \rceil + 1 \leq edp-r_1(C2T_h) \leq 2h + 4$,
- $2h + 2 = 2 \cdot \lceil \log_2 n \rceil \leq edp-r_2(C2T_h) \leq 2h + 3$.

Lemma

$edp-r_2(Gr_n^2) = 1.5 \cdot \log_2 n - \log_2 \log_2 n \pm O(1)$

$edp-r_2(Pl(n, h)) \geq 1.5 \log_2 n - \log_2 \log_2 n - 0.5 \log_2 h - 2$
Lemma

For $d \geq 3$

(i) $edp_r2(Gr^d_n) = (1 + 1/d) \cdot \log_2 n - \log_2 n \log_2 n \pm O(d)$,

(ii) $edp_r1(Gr^d_n) \leq (\log_b 2 + (2 - \log_b 2)/d) \cdot \log_2 n + O(d)$

$= (1.44... + 0.56.../d) \cdot \log_2 n + O(d)$.

Lemma

For every $X_k \in \{BF_k, CCC_k, Q_k\}$ of $n$ nodes and dimension $k$, $edp_r1(X_k) \leq r_1(K_n) + O(\log_2 \log_2 n)$.

Lemma

For every $Y_k \in \{BF_k, CCC_k\}$ of $n$ nodes and dimension $k$, $edp_r2(Y_k) \leq r_2(K_n) + O(\log_2 \log_2 n)$.
J. Hromkovič, et al.:
Legend

■ : Not of relevance
■ ■ : implicitly used basics
■ ■ ■ : idea of proof or algorithm
■ ■ ■ ■ : structure of proof or algorithm
■ ■ ■ ■ ■ : Full knowledge