Motivation

- Shows the quality of any algorithm.
- Interesting property of any problem.
- Interesting techniques to prove lower bounds.
  - No assumption about the used algorithms
  - Have to show a property for all algorithms and some inputs.
  - For all algorithms there is an input, such that the running time is at least....
  - Typically more complicated than upper bounds.
- Here we start with lower bounds for coloring cycles.
Motivation

Ideas

- Model distributed computers, connected in a cycle.
- No assumption about structure of the algorithm.
- Assume the running time is $t$ on a cycle of length $n$.
- Step one: Normalize the behavior of the algorithm.
- Step two: Extend the possible inputs for the algorithms, such that the algorithm works still correct.
- Step three: find some contradiction.
Step one: Normalize the behavior of the algorithm

After $t$ steps a node may know the identifiers of $2t + 1$ nodes. Let

$$W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n \land x_i = x_j \Rightarrow i = j\}$$

be the set of possible surroundings.

It is not necessary to color any node before step $t$:

- Each node may simulate the behavior of the $2t + 1$ nodes in the surrounding.
- Or each node sends also the history of colors.

Thus after $t$ rounds node $v$ has the topological information $\zeta(v)$:

$$\zeta(v) = (x_1, x_2, \ldots, x_s) \in W_{s,n} \text{ with } s = 2t + 1.$$

Any algorithm will use some deterministic strategy $\pi$ to find a coloring:

$$c(v) \leftarrow \Phi_\pi(\zeta(v)) \text{ with } \Phi_\pi : W_{s,n} \mapsto \{1, 2, \ldots, c_{\text{max}}\}.$$
Step two: Extend the possible inputs

- The set of nodes is $W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\}$.
- The set of edges is $E_{s,n}$. They contain any possible edge is any cycle:
  \[ E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]
- This graph $B_{s,n} = (W_{s,n}, E_{s,n})$ has $\binom{n}{s} s!$ nodes of degree $n - s$. Thus it has $(n - s) \binom{n}{s} s!$ edges.

**Theorem (Coloring $B_{s,n}$)**

*If an algorithm $\pi_t$ colors any cycle of length $n$ with $c$ colors in $t$ steps, then it will define a legal coloring of $B_{s,n}$.***
Step two: Extend the possible inputs

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \quad \text{and} \quad E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

**Theorem (Coloring \( B_{s,n} \))**

*If an algorithm \( \pi_t \) colors any cycle of length \( n \) with \( c \) colors in \( t \) steps, then it will define a legal coloring of \( B_{s,n} \).*

- Assume algorithm \( \pi_t \) colors cycle of length \( n \) correct, but not the \( B_{s,n} \).
- Thus there is an edge \( e = ((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \in E_{s,n} \) which is not colored correctly.
- Take this edge and extend it to a cycle of length \( n \) using the missing identifiers.
- This cycle with this order is not colored correctly.
- Contradiction.
Lower Bound for even length cycle

\[ W_{s,n} = \{(x_1, x_2, ..., x_s) \mid 1 \leq x_i \leq n\} \quad \text{and} \quad E_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

Theorem (Distributed Coloring \( C_{2n} \))

Any deterministic distributed algorithm uses \( n - 1 \) rounds to color a cycle of length \( 2n \) with 2 colors.

- Assume the algorithm runs in time \( t \leq n - 2 \).
- Then this algorithm will color the graph \( B_{2t+1,2n} \) with 2 colors.
- \( B_{2t+1,2n} \) is bipartite for \( t \leq n - 2 \).
- We will now construct the following cycle:

\[
(1, 2, 3, ..., 2t + 1) \rightarrow (2, 3, 4, ..., 2t + 2) \rightarrow \ldots \rightarrow (2t + 3, 1, 2, ..., 2t) \rightarrow (1, 2, 3, ..., 2t + 1)
\]
Lower Bound for even length cycle

\begin{align*}
W_{s,n} &= \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \quad \text{and} \quad E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_s+1)) \mid x_1 \neq x_{s+1}\}
\end{align*}

**Theorem (Parallel Coloring $C_{2n}$)**

*Any deterministic parallel algorithm uses $\log n$ rounds to color a cycle of length $2n$ with 2 colors.*

- Assume the algorithm runs in time $t \leq \log n$.
- The best way to collect information is doubling (see lower bound for broadcast/accumulation).
- Then we may use its strategy to construct a distributed version running in $t$ time.
- Contradiction.
Step four: find some contradiction

\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \text{ and } E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\} \]

- We want a lower bound for the 3-coloring of cycles.
- Step a) Show \(\chi(B_{2t+1,n}) \geq \log^{2t} n\).
- Step b) Show \(\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})\).
- Step c) Use the line-graph construction.
- Step d) Show property for coloring a line-graph.
- Step e) Put everything together.
Construction of $\tilde{B}_{s,n}$

$W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\}$ and $E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$

- Remember:
  - $W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n \land x_i = x_j \Rightarrow i = j\}$
  - $E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$
  - $B_{s,n} = (W_{s,n}, E_{s,n})$

Construct now:

- $\tilde{W}_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_1 < x_2 < \ldots < x_s \leq n\}$
- $\tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$
- $\tilde{B}_{s,n} = (\tilde{W}_{s,n}, \tilde{E}_{s,n})$

Thus $\tilde{B}_{s,n}$ is by definition a non-directed sub-graph of $B_{s,n}$.

Lemma

We have: $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$. 
**Line-Graphs**

$\tilde{W}_{s,n} = \{ (x_1, \ldots, x_s) | x_1 < x_2 < \ldots < x_s \}$, $\tilde{E}_{s,n} = \{ \{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1} \}$, $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$.

**Definition (Line-Graphs)**

Let $G = (V, E)$ be an directed graph. $DL(G) = (E, E')$ is called line-graph of $G$, iff

$$E' = \{ (e, e') | e, e' \in E \land e \cap e' \neq \emptyset \}.$$  

A graph $H$ is called directed line-graph, iff a graph $G$ exists, with $DL(G) = H$. 

![Diagram of line-graphs with vertices a, b, c and edges x, y linking them]
Line-Graphs

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) | x_1 < x_2 < \ldots < x_s\}, \tilde{E}_{s,n} = \{\{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1}\}, \chi(\tilde{B}_{s,n}) \leq \chi(\tilde{B}_{s,n}) \]

**Definition (Line-Graphs)**

Let \( G = (V, E) \) be an undirected graph. \( L(G) = (E, E') \) is called line-graph of \( G \), iff

\[
E' = \{\{e, e'\} | e, e' \in E \land e \cap e' \neq \emptyset\}.
\]

A graph \( H \) is called line-graph, iff a graph \( G \) exists, with \( L(G) = H \).
Example 1

\[ \tilde{W}_{s,n} = \{ (x_1, \ldots, x_s) | x_1 < x_2 < \ldots < x_s \}, \tilde{E}_{s,n} = \{ (x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1}) | x_1 \neq x_{s+1} \}, \chi(\tilde{B}_{s,n}) \leq \chi(\tilde{B}_{s,n}) \]
Example 2

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < x_2 < \ldots < x_s\}, \tilde{E}_{s,n} = \{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1}) \mid x_1 \neq x_{s+1}\}, \chi(\tilde{B}_{s,n}) \leq \chi(\tilde{W}_{s,n}) \leq \chi(\tilde{E}_{s,n}) \]
Example 3

\[ \tilde{W}_{s,n} = \{(x_1, ..., x_s) \mid x_1 < x_2 < ... < x_s\}, \tilde{E}_{s,n} = \{(x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1}) \mid x_1 \neq x_{s+1}\}, \chi(\tilde{B}_{s,n}) \leq \chi(\tilde{G}_{s,n}) \]
DeBruijn network of dimension $d$

$\tilde{W}_{s,n} = \{(x_1, \ldots, x_s) \mid x_1 < x_2 < \ldots < x_s\}$, $\tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$, $\chi(\tilde{B}_{s,n}) \leq \chi(\tilde{B}_{s,n})$

- DeBruijn network:
  
  \[
  DB(d) = (V_{DB(d)}, E_{DB(d)}^s \cup E_{DB(d)}^{se})
  \]

  \[
  V_{DB(d)} = \{0, 1\}^d
  \]

  \[
  E_{DB(d)}^s = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}
  \]

  \[
  E_{DB(d)}^{se} = \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \]

  Number of nodes: $2^d$  
  Degree: $2 + 2$

  Number of edges: $2^{d+1}$  
  Diameter: $d$

**Lemma**

We have: $DB(d + 1) = DL(DB(d))$ for $d \geq 1$. 
Line-Graph Properties of $\tilde{B}_{s,n}$

$\tilde{W}_{s,n} = \{(x_1, \ldots, x_s) | x_1 < x_2 < \ldots < x_s\}$, $\tilde{E}_{s,n} = \{\{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1}\}$, $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$

**Lemma**

1. $\tilde{B}_{1,n}$ is the complete directed graph of $n$ nodes.
2. We have $\tilde{B}_{s+1,n} = LG(\tilde{B}_{s,n})$ for $s \geq 1$.

**Proof:**

1. By definition: $\tilde{E}_{s,n} = \{\{(x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})\} | x_1 \neq x_{s+1}\}$.

2. By construction:

   - In $\tilde{B}_{s,n}$: $(x_1, x_2, \ldots x_s) \rightarrow (x_2, x_3, \ldots, x_{s+1})$ and $(x_2, x_3, \ldots, x_{s+1}) \rightarrow (x_3, x_4, \ldots, x_{s+2})$.
   - In $V(DL(\tilde{B}_{s+1,n}))$: $((x_1, x_2, \ldots, x_s), (x_2, x_3, \ldots, x_{s+1}))$ and $((x_2, x_3, \ldots, x_{s+1}), (x_3, x_4, \ldots, x_{s+2}))$.
   - In $V(DL(\tilde{B}_{s+1,n}))$: $(x_1, x_2, \ldots x_s, x_{s+1})$ and $(x_2, x_3, \ldots x_{s+1}, x_{s+2})$ (simplified).
   - In $E(DL(\tilde{B}_{s+1,n}))$: $((x_1, x_2, \ldots x_s, x_{s+1}), (x_2, x_3, \ldots x_{s+1}, x_{s+2}))$. 

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Motivation

Coloring Cycles

9:16 Results

Walter Unger 29.11.2016 20:13  WS2016/17
Bounds for Coloring Line-Graphs

\[ \tilde{W}_{s,n} = \{ (x_1, \ldots, x_s) \mid x_1 < x_2 < \ldots < x_s \}, \tilde{E}_{s,n} = \{ (x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1}) \mid x_1 \neq x_{s+1} \}, \chi(\tilde{B}_{s,n}) \leq \chi(H) \]

**Lemma**

Let \( H \) be any directed graph, then we have \( \chi(DL(H)) \geq \log(\chi(H)) \).

**Proof:**

- Let \( k = \chi(DL(H)) \), thus we can color the nodes from \( DL(H) \) with \( k \) colors.

- Thus we may color the edges from \( H \) with \( k \) colors: \( \chi'(H) \leq k \).

- For any edge \( e = (v, w) \) of \( H \) let \( c'(e) \) be the color of \( e \).

- Define now a coloring of the nodes \( v \) of \( H \):
  \[ c(v) = \bigcup_{v \in e} c'(e). \]

- This is a correct \( 2^k \) node-coloring of \( H \).

- Thus \( \chi(H) \leq 2^k = 2^{\chi(DL(H))} \).

- Thus \( \log(\chi(H)) \leq \chi(DL(H)) \).
Lemma

We have $\chi(\tilde{B}_s, n) \geq \log^{(s-1)} n$.

Proof:

- $\tilde{B}_{1,n}$ is the complete directed graph of $n$ nodes.
- $\chi(\tilde{B}_{1,n}) = n$.
- We have $\tilde{B}_{s+1,n} = LG(\tilde{B}_s, n)$ for $s \geq 1$.
- We have already: $\chi(DL(H)) \geq \log(\chi(H))$.
- Thus we get $\chi(\tilde{B}_{s+1,n}) \geq \log(\chi(\tilde{B}_s, n))$.
- Thus we get $\chi(\tilde{B}_s, n) \geq \log^{(s-1)}(\chi(\tilde{B}_1, n))$.
- Thus we get $\chi(\tilde{B}_s, n) \geq \log^{(s-1)}(n)$.
Theorem

Any deterministic distributed algorithm needs at least $1/2(\log^* n - 1)$ rounds to color a cycle of length $n$ with 3 colors.

Proof:

- We have already: $\chi(\tilde{B}_s, n) \geq \log^{(s-1)} n$, resp.
- We have already: $\chi(\tilde{B}_{2t+1}, n) \geq \log^{(2t)} n$.
- We also have: $\chi(\tilde{B}_{2t+1}, n) \leq 3$.
- Thus we get: $\log^{(2t)} n \leq 3$ and finally
- $2t \geq \log^* n - 1$. 