Theory of Parallel and Distributed Systems (WS2016/17)
Kapitel 1
First Algorithms for PRAM

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Lehrstuhl für Informatik 1

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1. There are limits to the computing power of a single Computer
2. Computers become cheaper
3. Specialized computers are expensive
4. There are tasks with large data
5. Many problems are very complex
   1. Weather and other Simulations
   2. Crash tests
   3. Military applications
   4. Large data: (SETI, ...)
   5. More similar problems
6. Thus there is the need for computers with more than one CPU
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Pipeline: (systolic array)

- There is a sequence of processors \((P_i)\) \(1 \leq i \leq n\).
- Processor \(P_1\) receives the input.
- Output of \(P_1\) will be passed as the input of \(P_2\).
- Output of \(P_i\) will be passed as the input of \(P_{i+1}\) \(1 \leq i < n\).
- Processor \(P_n\) delivers the final output.
- Processors may be different.
- Processors may run different programs.
- Intermediate outputs may be buffered.
- Pipelining is one important type of parallel system (in practice).
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- Idea: use more than one data stream.
- Data streams may interact each other.
- Each processor is the same.
- There is a global synchronization.
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Systolic Array with three data streams
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Vector Computer

- Vector of processes.
- Each processor has different data.
- But each processor executes the same program.
- Addition of two vectors:
  1. Read vector $A$
  2. Read vector $B$
  3. Add (each processor)
  4. Output the summation

- Single Instruction Multiple Data SIMD-Computer.
- Aim: Multiple Instruction Multiple Data MIMD-Computer.
- I.e. Fast processors with fast connections.
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Example: Transputer

- **Advantage:** very flexible, any fixed network of degree 4 possible.
- **Disadvantage:** long wires may be necessary, only a fixed network possible.
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![Diagram](image-url)
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- **Advantage:** “normal” CPUs.
- **Advantage:** fast links possible.
- **Advantage:** no special hardware.
- **Advantage:** variable network, may change during execution.
- **Advantage:** very large networks may be possible.
- **Disadvantage:** still a limited degree for the network.
- **Disadvantage:** large network are complicated.
- **Problem:** cooling large systems.
- **Problem:** fault tolerance.
- **Problem:** construct such a system.
- **Problem:** generate good data throughput with constant degree network.
- **Problem:** do the program structures fit the structure of the network.
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Look for good networks.

Trees, Grids, Pyramids, ...

HQ(n), CCC(n), BF(n), SE(n), DB(n), ...

Pancake Network and Burned Pancake Network.

Problem: Physical placement of the processors.

Problem: Length of wires.

Problem: Has the network a nice structure.

If the network becomes too large, we may use efficiency.

Solution: choose a mixed network structure.
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Parallel Computer V (Network)

1. CPU and memory are one logical unit:

```
CPU   RAM   CPU   RAM   CPU   RAM   CPU   RAM
```

```
Network
```

2. CPUs and memory are connected by a network:

```
CPU   CPU   CPU   CPU   CPU
```

```
RAM   RAM   RAM   RAM   RAM
```

```
Network
```

The difference is more on the practical side.
Parallel Computer V (Network)

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```
CPU | RAM | CPU | RAM | CPU | RAM | CPU | RAM
```

```
Network
```

2. CPUs and memory are connected by a network:

```
CPU  CPU  CPU  CPU  CPU
```

```
Network
```

```
RAM  RAM  RAM  RAM  RAM
```

The difference is more on the practical side.
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

   ![Diagram showing CPU and memory as one logical unit](network1.png)

2. CPUs and memory are connected by a network:

   ![Diagram showing CPUs and memory connected by a network](network2.png)

The difference is more on the practical side.
**Parallel Computer V (Network)**

1. **CPU and memory are one logical unit:**
   - CPU
   - RAM
   - CPU
   - RAM
   - CPU
   - RAM
   - CPU
   - RAM
   - CPU
   - RAM
   - Network

2. **CPUs and memory are connected by a network:**
   - CPU
   - CPU
   - CPU
   - CPU
   - CPU
   - RAM
   - RAM
   - RAM
   - RAM
   - RAM
   - Network

The difference is more on the practical side.
PRAM (theoretical model)

- Ignore/unify the costs for each computation step.
- Ignore/unify the costs for each communication step.
PRAM (theoretical model)

- Ignore/unify the costs for each computation step.
- Ignore/unify the costs for each communication step.
Definition RAM

- RAM: Random Access Machine
- CPU may access any memory cell
- Memory is unlimited
- Complexity measurements
  - uniform: each operation cost one unit
  - logarithmic: cost are measured according to the size of the numbers
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Motivation and History

PRAM Introduction

Efficiency

Selection

Merging

1:15 Definition

Walter Unger 6.12.2016 14:23
WS2016/17

Idea of PRAM

- Many processes
- Common program
- Program may select single processors
- Common memory
Definition PRAM

- Consists of processors $P_i$ with $1 \leq i \leq p$ (prozessor has id $i$).
- Consists of registers $R_j$ with $1 \leq j \leq m$.
- Each processor has some local registers.
- Each processor $P_i$ may access each register $R_j$.
- Each processor executes the same programm.
- The programm is synchronized, thus each processor executes the same instructions.
- A selection is possible by using the processor id.
- The input of length $n$ is written to registers $R_j$ with $1 \leq j \leq n$.
- The output is placed in some known registers.
- The registers contain words (numbers) in the uniform cost measurement.
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3. processor may do some local computation using local registers:
   
   \[ x := y \times 5. \]

For the access to the register we have the following variations:

- EREW Exclusive Read Exclusive Write
- CREW Concurrent Read Exclusive Write
- CRCW Concurrent Read Concurrent Write
- ERCW Exclusive Read Concurrent Write

Write conflicts may be solved using the following rules:

- Arbitrary: any processor gets access to the register.
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Computation of an “Or” (Idea)

$P_1$ $P_2$ $P_3$ $P_4$ $P_5$ $P_6$ $P_7$ $P_8$

$M_1$ $M_2$ $M_3$ $M_4$ $M_5$ $M_6$ $M_7$ $M_8$ $M_9$
Computation of an “Or” (Idea)

\[ P_1 \lor P_2 \lor P_3 \lor P_4 \lor P_5 \lor P_6 \lor P_7 \lor P_8 \]

\[ 0 \lor 1 \lor 0 \lor 0 \lor 1 \lor 0 \lor 0 \lor 1 \rightarrow 0 \]
Computation of an “Or” (Idea)

\[
P_1 \lor P_2 \lor P_3 \lor P_4 \lor P_5 \lor P_6 \lor P_7 \lor P_8
\]

\[
\begin{align*}
P_1 & : 0 \\
P_2 & : 1 \\
P_3 & : 0 \\
P_4 & : 0 \\
P_5 & : 1 \\
P_6 & : 0 \\
P_7 & : 0 \\
P_8 & : 1
\end{align*}
\]

\[
\begin{align*}
\text{Result} & : 0
\end{align*}
\]
Computation of an “Or” (Idea)

\[
\begin{align*}
x &= 0 \\
x &= 1 \\
x &= 0 \\
x &= 0 \\
x &= 1 \\
x &= 0 \\
x &= 0 \\
x &= 1 \\
\end{align*}
\]

\[
\begin{align*}
0 \lor 1 \lor 0 \lor 0 \lor 1 \lor 0 \lor 0 \lor 1 &\rightarrow 0
\end{align*}
\]
Computation of an “Or” (Idea)

\[
\begin{align*}
&x = 0 \quad x = 1 \\
&x = 0 \quad x = 0 \\
&x = 1 \\
&x = 0 \\
&x = 0 \\
&x = 1 \\
\end{align*}
\]

\[
\begin{align*}
&0 \lor 1 \\
&0 \lor 0 \\
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Computation of an “Or” (Idea)

\[ x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \]

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Computing an “Or”

- **Task:** Compute \( x = \bigvee_{i=1}^{n} x_i \).
- **Input:** \( x_i \) is in register \( R_i \) (1 \( \leq \) \( i \) \( \leq \) \( n \)).
- **Output:** computed in \( R_{n+1} \).
- **Model:** CRCW Arbitrary, Common oder Priority.
- **Program:** Or
  
  for all \( P_i \) where 1 \( \leq \) \( i \) \( \leq \) \( n \) do in parallel
  
  \[ R_i \rightarrow P_i(x) \]
  
  if \( x = true \) then \( P_i(x) \rightarrow R_{n+1} \)

- **Running time:** \( O(1) \) (exact 2 steps).
- **Number of processors:** \( n \).
- **Memory:** \( n + 1 \).
- **Possible models:** ERCW (Arbitrary, Common oder Priority).
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  - for all $P_i$ where $1 \leq i \leq n$ do in parallel
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  ```plaintext
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  - **for all** \( P_i \) where \( 1 \leq i \leq n \) **do in parallel**
    - \( R_i \rightarrow P_i(x) \)
    - **if** \( x = \text{true} \) **then** \( P_i(x) \rightarrow R_{n+1} \)
- **Running time:** \( O(1) \) (exact 2 steps).
- **Number of processors:** \( n \).
- **Memory:** \( n + 1 \).
- **Possible models:** ERCW (Arbitrary, Common oder Priority).
Computing an “Or”

- Task: Compute \( x = \bigvee_{i=1}^{n} x_i \).
- Input: \( x_i \) is in register \( R_i \) (1 \( \leq \) \( i \) \( \leq \) \( n \)).
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  for all \( P_i \) where 1 \( \leq \) \( i \) \( \leq \) \( n \) do in parallel
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Computing an “Or” (EREW)

- **Problem:**
  no writing of two processors
to the same register
at the same time.

- **Idea:** combine pairwise the results

- With this idea, computing the sum is also possible.

- Thus computing the “Or” is just a special case of computing a sum.
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Computing the Sum (Idea)
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12  6  34  5  7  23  4  11
Computing the Sum (Idea)

0 0 0 0

P₁  P₂  P₃  P₄

12  6  34  5  7  23  4  11
Computing the Sum (Idea)
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- $P_1$: 18, 12
- $P_2$: 39, 6
- $P_3$: 30, 34
- $P_4$: 15, 5

The numbers are divided among the processes to compute the sum.
Computing the Sum (Idea)

18
\[ \rightarrow \]
P_1

39
\[ \rightarrow \]
P_2

30
\[ \rightarrow \]
P_3

15
\[ \rightarrow \]
P_4

12 6 34 5 7 23 4 11
Computing the Sum (Idea)

\[ P_1 \rightarrow 18, \quad P_2 \rightarrow 39, \quad P_3 \rightarrow 30, \quad P_4 \rightarrow 15 \]
Computing the Sum (Idea)

57

45

30

15

P₁

P₂

P₃

P₄

12

6

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Computing the Sum (Idea)

\[
\begin{array}{cccc}
57 & 45 & 30 & 15 \\
\end{array}
\]

\[
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P_1 & P_2 & P_3 & P_4 \\
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Computing the sum (EREW)

- **Task:** compute $x = \sum_{i=1}^{n} x_i$ with $n = 2^k$.
- **Input:** $x_i$ is in register $R_i$ ($1 \leq i \leq n$).
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- **Modell:** EREW.
- **Programm:** Summe
  
  for all $P_i$ where $1 \leq i \leq n/2$ do in parallel
  
  $R_{2\cdot i-1} \rightarrow P_i(x)$

  for $j = 1$ to $k$ do
    if $(i - 1) \equiv 0 \pmod{2^{j-1}}$ then
      $R_{2\cdot i-1+2^{j-1}} \rightarrow P_i(y)$
    $x := x + y$
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- **Running time:** $O(k) = O(\log n)$ (precise $3 \cdot k + 1$ steps).
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Motivation and History
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Addition of Matrices

Let $A, B$ two $(n \times n)$-Matrices.

Sum $A + B$ is computable with $n^2$ processors in Zeit $O(1)$ on a EREW PRAM.

$R_1$ till $R_{n^2}$ contain $A$ (one row after the other).

$R_{1+n^2}$ bis $R_{2.n^2}$ contains $B$ (one row after the other).

Result in $R_{1+2.n^2}$ bis $R_{3.n^2}$.

Programm: MatSumme

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$x := x + y$

$P_i(x) \rightarrow R_{i+2.n^2}$

Running time: $O(1)$.

Number of processors: $O(n^2)$.

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\[ \text{Abbreviations: } \begin{array}{ll}
P & \text{Programm}
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- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$.

Programm: MatSumme

for all $P_i$ where $1 \leq i \leq n^2$ do in parallel

- $R_i \rightarrow P_i(x)$
- $R_{i+n^2} \rightarrow P_i(y)$
- $x := x + y$
- $P_i(x) \rightarrow R_{i+2\cdot n^2}$

Running time: $O(1)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$.

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

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Addition of Matrices

1. Let $A$, $B$ two $(n \times n)$-Matrices.
2. Sum $A + B$ is computable with $n^2$ processors in Zeit $O(1)$ on a EREW PRAM.
3. $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
4. $R_{1+n^2}$ bis $R_{2 \cdot n^2}$ contains $B$ (one row after the other).
5. Result in $R_{1+2 \cdot n^2}$ bis $R_{3 \cdot n^2}$.
6. Programm: MatSumme

   for all $P_i$ where $1 \leq i \leq n^2$ do in parallel
   
   $R_i \rightarrow P_i(x)$
   $R_{i+n^2} \rightarrow P_i(y)$
   $x := x + y$
   $P_i(x) \rightarrow R_{i+2 \cdot n^2}$

   Running time: $O(1)$.
7. Number of processors: $O(n^2)$.
8. Size of memory: $O(n^2)$.

Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$. 
Multiplication of Matrices

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2.n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2.n^2}$ bis $R_{3.n^2}$
- Register $A_{i,j} = R_{(i-1).n+j}$ ($1 \leq i, j \leq n$).
- Register $B_{i,j} = R_{(i-1).n+j+n^2}$ ($1 \leq i, j \leq n$).
- Register $C_{i,j} = R_{(i-1).n+j+2.n^2}$ ($1 \leq i, j \leq n$).
- processor $P_{i,j} = P_{(i-1).n+j}$ ($1 \leq i, j \leq n$).
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{\frac{n}{2}}$ contain $A$ (one row after the other).
- $R_{\frac{n}{2}}$ bis $R_n$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot\frac{n}{2}}$ bis $R_{3\cdot\frac{n}{2}}$
- Register $A_{i,j} = R_{(i-1)\cdot\frac{n}{2}+j}$ $(1 \leq i, j \leq n)$.
- Register $B_{i,j} = R_{(i-1)\cdot\frac{n}{2}+j+n^2}$ $(1 \leq i, j \leq n)$.
- Register $C_{i,j} = R_{(i-1)\cdot\frac{n}{2}+j+2\cdot\frac{n}{2}^2}$ $(1 \leq i, j \leq n)$.
- Processor $P_{i,j} = P_{(i-1)\cdot\frac{n}{2}+j}$ $(1 \leq i, j \leq n)$.
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
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- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ ($1 \leq i, j \leq n$).
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ ($1 \leq i, j \leq n$).
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- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.

Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$. 
Multiplication of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$
  - Register $A_{i,j} = R_{(i-1)\cdot n+j}$ \((1 \leq i, j \leq n)\).
  - Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ \((1 \leq i, j \leq n)\).
  - Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ \((1 \leq i, j \leq n)\).
  - Processor $P_{i,j} = P_{(i-1)\cdot n+j}$ \((1 \leq i, j \leq n)\).
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

Assume \( w.l.o.g \ n = 2^k \) for \( k \in \mathbb{N} \).

- Let \( A, B \) be two \( (n \times n) \)-Matrices.
- \( R_1 \) till \( R_{n^2} \) contain \( A \) (one row after the other).
- \( R_{1+n^2} \) bis \( R_{2 \cdot n^2} \) contains \( B \) (one row after the other).
- Result in \( R_{1+2 \cdot n^2} \) bis \( R_{3 \cdot n^2} \)
- Register \( A_{i,j} = R_{(i-1) \cdot n+j} \) (\( 1 \leq i,j \leq n \)).
- Register \( B_{i,j} = R_{(i-1) \cdot n+j+n^2} \) (\( 1 \leq i,j \leq n \)).
- Register \( C_{i,j} = R_{(i-1) \cdot n+j+2 \cdot n^2} \) (\( 1 \leq i,j \leq n \)).
- Processor \( P_{i,j} = P_{(i-1) \cdot n+j} \) (\( 1 \leq i,j \leq n \)).
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2 \cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2 \cdot n^2}$ bis $R_{3 \cdot n^2}$
- Register $A_{i,j} = R_{(i-1) \cdot n + j}$ ($1 \leq i, j \leq n$).
- Register $B_{i,j} = R_{(i-1) \cdot n + j + n^2}$ ($1 \leq i, j \leq n$).
- Register $C_{i,j} = R_{(i-1) \cdot n + j + 2 \cdot n^2}$ ($1 \leq i, j \leq n$).
- Processor $P_{i,j} = P_{(i-1) \cdot n + j}$ ($1 \leq i, j \leq n$).
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
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- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$
- Register $A_{i,j} = R_{(i-1)\cdot n+j} \ (1 \leq i, j \leq n)$.
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2} \ (1 \leq i, j \leq n)$.
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2} \ (1 \leq i, j \leq n)$.
- Processor $P_{i,j} = P_{(i-1)\cdot n+j} \ (1 \leq i, j \leq n)$.
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
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Multiplication of Matrices

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$
- Register $A_{i,j} = R_{(i-1)\cdot n+j} \ (1 \leq i, j \leq n)$.
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2} \ (1 \leq i, j \leq n)$.
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2} \ (1 \leq i, j \leq n)$.
- processor $P_{i,j} = P_{(i-1)\cdot n+j} \ (1 \leq i, j \leq n)$.
- Use the above notation to simplify the algorithm.
- Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$
- Register $A_{i,j} = R_{(i-1)\cdot n+j}$ $(1 \leq i, j \leq n)$.
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ $(1 \leq i, j \leq n)$.
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ $(1 \leq i, j \leq n)$.
- processor $P_{i,j} = P_{(i-1)\cdot n+j}$ $(1 \leq i, j \leq n)$.

Use the above notation to simplify the algorithm.

Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

Let $A, B$ be two $(n \times n)$-Matrices.

- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$

- Register $A_{i,j} = R_{(i-1)\cdot n+j}$ $(1 \leq i, j \leq n)$.
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ $(1 \leq i, j \leq n)$.
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ $(1 \leq i, j \leq n)$.
- Processor $P_{i,j} = P_{(i-1)\cdot n+j}$ $(1 \leq i, j \leq n)$.

Use the above notation to simplify the algorithm.

Each processor has to do some hidden local computation to implement the above expressions.
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

Programm: MatrProd 1
for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  $h = 0$
  for $l = 1$ to $n$ do
  \hspace{1cm} $A_{i,l} \rightarrow P_{i,j}(a)$
  \hspace{1cm} $B_{l,j} \rightarrow P_{i,j}(b)$
  \hspace{1cm} $h = h + a \cdot b$
  \hspace{1cm} $P_{i,j}(h) \rightarrow C_{i,j}$
- Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

\begin{align*}
A_{i,j} &= R_{(i-1) \cdot n + j} \\
B_{i,j} &= R_{(i-1) \cdot n + j + n^2} \\
C_{i,j} &= R_{(i-1) \cdot n + j + 2 \cdot n^2} \\
P_{i,j} &= P_{(i-1) \cdot n + j}
\end{align*}
Let $A, B$ be two $(n \times n)$-Matrices

Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

Programm: MatrProd 1
for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

$h = 0$
for $l = 1$ to $n$ do

$A_{i,l} \rightarrow P_{i,j}(a)$
$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$. 

Let $A_{i,j} = R(i-1) \cdot n + j$
$B_{i,j} = R(i-1) \cdot n + j + n^2$
$C_{i,j} = R(i-1) \cdot n + j + 2 \cdot n^2$
$P_{i,j} = P(i-1) \cdot n + j$
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices

- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

- Program: MatrProd 1
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    $h = 0$
    for $l = 1$ to $n$ do
      $A_{i,l} \rightarrow P_{i,j}(a)$
      $B_{l,j} \rightarrow P_{i,j}(b)$
      $h = h + a \cdot b$
      $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$. 
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

Programm: MatrProd 1

```plaintext
for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    $h = 0$
    for $l = 1$ to $n$ do
        $A_{i,l} \rightarrow P_{i,j}(a)$
        $B_{l,j} \rightarrow P_{i,j}(b)$
        $h = h + a \cdot b$
        $P_{i,j}(h) \rightarrow C_{i,j}$
```

- Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 
Multiplikation of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.
- Programm: MatrProd 1
  - for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    - $h = 0$
    - for $l = 1$ to $n$ do
      - $A_{i,l} \rightarrow P_{i,j}(a)$
      - $B_{l,j} \rightarrow P_{i,j}(b)$
      - $h = h + a \cdot b$
      - $P_{i,j}(h) \rightarrow C_{i,j}$
  - Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

\[ A_{i,j} = R(i-1) \cdot n + j \]
\[ B_{i,j} = R(i-1) \cdot n + j + n^2 \]
\[ C_{i,j} = R(i-1) \cdot n + j + 2 \cdot n^2 \]
\[ P_{i,j} = P(i-1) \cdot n + j \]
Multiplikation of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

Programm: MatrProd 1

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

$h = 0$

for $l = 1$ to $n$ do

$A_{i,l} \rightarrow P_{i,j}(a)$

$B_{i,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$. 

$A_{i,j} = R(i-1) \cdot n + j$

$B_{i,j} = R(i-1) \cdot n + j + n^2$

$C_{i,j} = R(i-1) \cdot n + j + 2 \cdot n^2$

$P_{i,j} = P(i-1) \cdot n + j$
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.
- Programm: MatrProd 1
  
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $h = 0$
  
  for $l = 1$ to $n$ do
    $A_{i,l} \rightarrow P_{i,j}(a)$
    $B_{l,j} \rightarrow P_{i,j}(b)$
    $h = h + a \cdot b$
    $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

\[
\begin{align*}
A_{i,j} &= R_{(i-1)\cdot n+j} \\
B_{i,j} &= R_{(i-1)\cdot n+j+n^2} \\
C_{i,j} &= R_{(i-1)\cdot n+j+2\cdot n^2} \\
P_{i,j} &= P_{(i-1)\cdot n+j}
\end{align*}
\]
Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices

- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

Programm: MatrProd 2
for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

$h = 0$

for $l = 1$ to $n$ do

$A_{i,l} \rightarrow P_{i,j}(a)$

$B_{i,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$. 
Let $A$, $B$ be two $(n \times n)$-Matrices

Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

Programm: MatrProd 2

for all $P_{i,j}$ where $1 \leq i,j \leq n$ do in parallel

$h = 0$

for $l = 1$ to $n$ do

$A_{i,l} \rightarrow P_{i,j}(a)$
$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$.
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Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

- Programm: MatrProd 2

  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $h = 0$

  for $l = 1$ to $n$ do
  
  $A_{i,l} \rightarrow P_{i,j}(a)$

  $B_{l,j} \rightarrow P_{i,j}(b)$

  $h = h + a \cdot b$

  $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$. 
Multiplikation of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.
- Programm: MatrProd 2
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  \[
  h = 0 \\
  \text{for } l = 1 \text{ to } n \text{ do} \\
  A_{i,l} \rightarrow P_{i,j}(a) \\
  B_{l,j} \rightarrow P_{i,j}(b) \\
  h = h + a \cdot b \\
  P_{i,j}(h) \rightarrow C_{i,j}
  \]

- Running time: $O(n)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

\[
\begin{align*}
A_{i,j} &= R_{(i-1)\cdot n+j} \\
B_{i,j} &= R_{(i-1)\cdot n+j+n^2} \\
C_{i,j} &= R_{(i-1)\cdot n+j+2\cdot n^2} \\
P_{i,j} &= P_{(i-1)\cdot n+j}
\end{align*}
\]
Multiplikation of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

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for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

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$A_{i,l} \rightarrow P_{i,j}(a)$

$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$. 
Multiplikation of Matrices

- Let $A, B$ be two $(n \times n)$-Matrices
- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

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$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$. 

\[
\begin{align*}
A_{i,j} &= R_{(i-1) \cdot n+j} \\
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C_{i,j} &= R_{(i-1) \cdot n+j+2 \cdot n^2} \\
P_{i,j} &= P_{(i-1) \cdot n+j}
\end{align*}
\]
Multiplikation von Matrizen

- Lasst $A$, $B$ zwei $(n \times n)$-Matrizen
- Produkt $A \cdot B$ ist computierbar mit $n^2$ Prozessoren in Zeit $O(n)$ auf einem EREW PRAM.

Programm: MatrProd 2

*for all* $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

1. $h = 0$
2. for $l = 1$ to $n$
   1. $A_{i,l} \rightarrow P_{i,j}(a)$
   2. $B_{l,j} \rightarrow P_{i,j}(b)$
   3. $h = h + a \cdot b$
   4. $P_{i,j}(h) \rightarrow C_{i,j}$

- Laufzeit: $O(n)$.
- Anzahl der Prozessoren: $O(n^2)$.
- Größe der Speicher: $O(n^2)$. 

$A_{i,j} = R(i-1) \cdot n + j$
$B_{i,j} = R(i-1) \cdot n + j + n^2$
$C_{i,j} = R(i-1) \cdot n + j + 2 \cdot n^2$
$P_{i,j} = P(i-1) \cdot n + j$
Problem:

- **Task:** Compute $s_i = \sum_{j=1}^{i} x_j$ for $1 \leq i \leq n$.
- **Input:** $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- **Output:** $s_i$ should be in register $R_i$ for $1 \leq i \leq n$. 


Compute the Prefixsum

Problem:

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for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel

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- **Running time:** \( O(k) = O(\log n) \) (precisely \( 3 \cdot k + 1 \) steps).
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- Task: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- Input: $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- Output: $m$ should be in register $R_{n+1}$.
- Possible with $n$ processors in time $O(\log n)$ using a EREW PRAM.
- Question: could it be done faster? (i.e. on a ERCW PRAM).
- A maximum is larger or equal than all other values.
- Idea: compare all pairs of numbers.
- The maximum will always win.
Motivation and History
PRAM Introduction
Efficiency
Selection
Merging

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| 26| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
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| 57| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
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| 26 | 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 59 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 67 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 52 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 61 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 27 | 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 49 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 67 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 23 | 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 56 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 34 | 12 | 14 | 56 | 23 | 67 | 49 | 27 | 61 | 52 | 67 | 59 | 26 | 41 | 33 | 22 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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Compute the Maximum (Idea)

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Compute the Maximum (Idea)

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Computing the Maximum

- **Task:** Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- **Input:** \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
- **Output:** \( m \) in register \( R_{n+1} \).
- **Model:** CRCW.

**Program:** Maximum

```plaintext
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
\( P_{i,1}(1) \rightarrow W_i \)

for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
\( R_i \rightarrow P_{i,j}(a) \)
\( R_j \rightarrow P_{i,j}(b) \)
if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
\( W_i \rightarrow P_{i,1}(h) \)
if \( h = 1 \) then
\( R_i \rightarrow P_{i,1}(h) \)
\( P_{i,1}(h) \rightarrow R_{n+1} \)
```
Computing the Maximum

- **Task**: Compute \( m = \max_{j=1}^{\frac{n}{2}} x_j \) with \( n = 2^k \).
- **Input**: \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
- **Output**: \( m \) in register \( R_{n+1} \).
- **Model**: CRCW.
- **Program**: Maximum
  
  for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  
  \( P_{i,1}(1) \rightarrow W_i \)

  for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
  
  \( R_i \rightarrow P_{i,j}(a) \)
  
  \( R_j \rightarrow P_{i,j}(b) \)

  if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)

  for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  
  \( W_i \rightarrow P_{i,1}(h) \)

  if \( h = 1 \) then
  
  \( R_i \rightarrow P_{i,1}(h) \)

  \( P_{i,1}(h) \rightarrow R_{n+1} \)
Computing the Maximum

- **Task:** Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
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**Program:**

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Computing the Maximum

- **Task**: Compute $m = \max_{j=1}^i x_j$ with $n = 2^k$.
- **Input**: $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
- **Output**: $m$ in register $R_{n+1}$.
- **Model**: CRCW.

**Program:**

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for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $P_{i,1}(1) \rightarrow W_i$
for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    $R_i \rightarrow P_{i,j}(a)$
    $R_j \rightarrow P_{i,j}(b)$
    if $a < b$ then $P_{i,j}(0) \rightarrow W_i$
for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $W_i \rightarrow P_{i,1}(h)$
    if $h = 1$ then
        $R_i \rightarrow P_{i,1}(h)$
        $P_{i,1}(h) \rightarrow R_{n+1}$
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Computing the Maximum

- Task: Compute $m = \max_{j=1}^{n} x_j$ with $n = 2^k$.
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- Model: CRCW.

Programm: Maximum

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel

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$R_i \rightarrow P_{i,j}(a)$

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for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel

$W_i \rightarrow P_{i,1}(h)$

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Computing the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
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Programm: Maximum

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel

$P_{i,1}(1) \rightarrow W_i$

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$P_{i,1}(h) \rightarrow R_{n+1}$
Computing the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- Input: \( x_j \) is in register \( R_j \) \((1 \leq x_j \leq n)\).
- Output: \( m \) in register \( R_{n+1} \).
- Model: CRCW.

Program: Maximum

```plaintext
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
\( P_{i,1}(1) \rightarrow W_i \)
for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
\( R_i \rightarrow P_{i,j}(a) \)
\( R_j \rightarrow P_{i,j}(b) \)
if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
\( W_i \rightarrow P_{i,1}(h) \)
if \( h = 1 \) then
\( R_i \rightarrow P_{i,1}(h) \)
\( P_{i,1}(h) \rightarrow R_{n+1} \)
```
Computing the Maximum

- Task: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- Input: $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
- Output: $m$ in register $R_{n+1}$.
- Model: CRCW.

Programm: Maximum

```plaintext
for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $P_{i,1}(1) \rightarrow W_i$

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    $R_i \rightarrow P_{i,j}(a)$
    $R_j \rightarrow P_{i,j}(b)$
    if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $W_i \rightarrow P_{i,1}(h)$
    if $h = 1$ then
        $R_i \rightarrow P_{i,1}(h)$
        $P_{i,1}(h) \rightarrow R_{n+1}$
```
Computing the Maximum

- Program: Maximum
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $P_{i,1}(1) \rightarrow W_i$
  
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $R_i \rightarrow P_{i,j}(a)$
  
  $R_j \rightarrow P_{i,j}(b)$
  
  if $a < b$ then $P_{i,j}(0) \rightarrow W_i$
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $W_i \rightarrow P_{i,1}(h)$
  
  if $h = 1$ then
  
  $R_i \rightarrow P_{i,1}(h)$
  
  $P_{i,1}(h) \rightarrow R_{n+1}$

- Running time: $O(1)$.

- Number of processors: $O(n^2)$.

- Memory: $O(n)$.
Computing the Maximum

- Program: Maximum
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $P_{i,1}(1) \rightarrow W_i$
  for all $P_{i,j}$ where $1 \leq i,j \leq n$ do in parallel
    $R_i \rightarrow P_{i,j}(a)$
    $R_j \rightarrow P_{i,j}(b)$
    if $a < b$ then $P_{i,j}(0) \rightarrow W_i$
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $W_i \rightarrow P_{i,1}(h)$
    if $h = 1$ then
      $R_i \rightarrow P_{i,1}(h)$
      $P_{i,1}(h) \rightarrow R_{n+1}$

- Running time: $O(1)$.
- Number of processors: $O(n^2)$.
- Memory: $O(n)$. 
Computing the Maximum

- Program: Maximum
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $P_{i,1}(1) \rightarrow W_i$

  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $R_i \rightarrow P_{i,j}(a)$
  
  $R_j \rightarrow P_{i,j}(b)$

  if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $W_i \rightarrow P_{i,1}(h)$

  if $h = 1$ then
  
  $R_i \rightarrow P_{i,1}(h)$
  
  $P_{i,1}(h) \rightarrow R_{n+1}$

- Running time: $O(1)$.

- Number of processors: $O(n^2)$.

- Memory: $O(n)$. 
Computing the Maximum

- **Programm:** Maximum

  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $P_{i,1}(1) \rightarrow W_i$

  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $R_i \rightarrow P_{i,j}(a)$
  
  $R_j \rightarrow P_{i,j}(b)$
  
  if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $W_i \rightarrow P_{i,1}(h)$
  
  if $h = 1$ then
  
  $R_i \rightarrow P_{i,1}(h)$
  
  $P_{i,1}(h) \rightarrow R_{n+1}$

- Running time: $O(1)$.

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Computing the Maximum

- **Programm: Maximum**
  
  for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  
  \[
  P_{i,1}(1) \rightarrow W_i
  \]

  for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
  
  \[
  R_i \rightarrow P_{i,j}(a) \\
  R_j \rightarrow P_{i,j}(b)
  \]

  if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)

  for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  
  \[
  W_i \rightarrow P_{i,1}(h)
  \]

  if \( h = 1 \) then
  
  \[
  R_i \rightarrow P_{i,1}(h) \\
  P_{i,1}(h) \rightarrow R_{n+1}
  \]

- Running time: \( O(1) \).
- Number of processors: \( O(n^2) \).
- Memory: \( O(n) \).
Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till $n$
- Input: Father of node $i$ is written in register $R_i$
- For the roots $i$ we have: in register $R_i$ is written $i$.

Program: Ranking

forall $P_i$ where $1 \leq i \leq n$ do in parallel
  forall $j = 1$ to $\lceil \log n \rceil$ do
    $R_i \rightarrow P_i(h)$
    $R_h \rightarrow P_i(h)$
    $P_i(h) \rightarrow R_i$

Running time: $O(\log n)$.

Number of processors: $O(n)$.

Memory: $O(n)$.

Model: CREW.
Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till \( n \)
- Input: Father of node \( i \) is written in register \( R_i \).
- For the roots \( i \) we have: in register \( R_i \) is written \( i \).
- Program: Ranking
  
  for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel
    
    for \( j = 1 \) to \( \lceil \log n \rceil \) do
      
      \( R_i \rightarrow P_i(h) \)
      \( R_h \rightarrow P_i(h) \)
      \( P_i(h) \rightarrow R_i \)

  Running time: \( O(\log n) \).

Number of processors: \( O(n) \).
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Model: CREW.
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```
for all $P_i$ where $1 \leq i \leq n$ do in parallel
  for $j = 1$ to $\lceil \log n \rceil$ do
    $R_i \rightarrow P_i(h)$
    $R_h \rightarrow P_i(h)$
    $P_i(h) \rightarrow R_i$
```

Running time: $O(\log n)$.

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Model: CREW.
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Program: Ranking

```plaintext
for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel
    for \( j = 1 \) to \( \lceil \log n \rceil \) do
        \( R_i \rightarrow P_i(h) \)
        \( R_h \rightarrow P_i(h) \)
        \( P_i(h) \rightarrow R_i \)
```

Running time: \( O(\log n) \).

Number of processors: \( O(n) \).

Memory: \( O(n) \).

Model: CREW.
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- Nodes are identified by numbers from 1 till $n$.
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Program: Ranking

for all $P_i$ where $1 \leq i \leq n$ do in parallel

for $j = 1$ to $\lceil \log n \rceil$ do

$R_i \rightarrow P_i(h)$

$R_h \rightarrow P_i(h)$

$P_i(h) \rightarrow R_i$

Running time: $O(\log n)$.

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Running time: $O(\log n)$.

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  for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel
  
  for \( j = 1 \) to \( \lceil \log n \rceil \) do
    
    \( R_i \rightarrow P_i(h) \)
    
    \( R_h \rightarrow P_i(h) \)
    
    \( P_i(h) \rightarrow R_i \)

  Running time: \( O(\log n) \).

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Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till $n$.
- Input: Father of node $i$ is written in register $R_i$.
- For the roots $i$ we have: in register $R_i$ is written $i$.
- Programm: Ranking
  \begin{verbatim}
  for all $P_i$ where $1 \leq i \leq n$ do in parallel
    for $j = 1$ to $\lceil \log n \rceil$ do
      $R_i \rightarrow P_i(h)$
      $R_h \rightarrow P_i(h)$
      $P_i(h) \rightarrow R_i$
  \end{verbatim}
  Running time: $O(\log n)$.
  Number of processors: $O(n)$.
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- Nodes are identified by numbers from 1 till $n$
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$R_i \rightarrow P_i(h)$
$R_h \rightarrow P_i(h)$
$P_i(h) \rightarrow R_i$

Running time: $O(\log n)$.

Number of processors: $O(n)$.

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Model: CREW.
## Short Summary

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Question: May we save some processors?
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**Question:** May we save some processors?  
May we do this saving in any situation?
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May we do this saving in any situation?
How do we estimate the efficiency of a parallel algorithm?
Cost Measurement

Let $A$ be any parallel algorithm, we denote:

- $T_A(n)$ the running time of $A$.
- $P_A(n)$ the number of processors used by $A$.
- $R_A(n)$ the number of registers used by $A$.
- $W_A(n)$ the number of accesses to registers done by $A$.
- $ST(n)$ the running time of the best [known] sequential algorithm.
- $Eff_A(n) := \frac{ST(n)}{P_A(n) \cdot T_A(n)}$ the efficiency of $A$.
- $AEff_A(n) := \frac{W_A(n)}{P_A(n) \cdot T_A(n)}$ the usage efficiency of $A$. 
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Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select($k, S$)

\begin{verbatim}
if $|S| \leq 50$ then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
$m := \text{Select}(\lceil |M|/2 \rceil, M)$
$S_1 := \{s \in S \mid s < m\}$
$S_2 := \{s \in S \mid s = m\}$
$S_3 := \{s \in S \mid s > m\}$
if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $\text{Select}(k - |S_1| - |S_2|, S_3)$
\end{verbatim}
**k-th Element**

- **Task**: Compute the k-th (k-smallest) element in an unsorted sequence $S = \{s_1, \ldots, s_n\}$.

- **Lower bound**: $n - 1$ comparisons

- **Start with a nice sequential algorithm**

- **Programm**: Select(k,S)

  
  ```
  if |S| ≤ 50 then return k-th number in S
  
  Split S in ⌈n/5⌉ sub-sequences $H_i$ of size ≤ 5
  
  Sort each $H_i$
  
  Let $M$ be the sequence of the middle elements in $H_i$
  
  $m := Select(\lceil |M|/2 \rceil, M)$
  
  $S_1 := \{s \in S \mid s < m\}$
  
  $S_2 := \{s \in S \mid s = m\}$
  
  $S_3 := \{s \in S \mid s > m\}$
  
  if $|S_1| ≥ k$ then return Select($k, S_1$)
  
  if $|S_1| + |S_2| ≥ k$ then return $m$
  
  return Select($k - |S_1| - |S_2|, S_3$)
  ```
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \cdots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select($k, S$)

if $|S| \leq 50$ then return $k$-th number in $S$

Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$

Sort each $H_i$

Let $M$ be the sequence of the middle elements in $H_i$

$m := \text{Select}(\lceil |M|/2 \rceil, M)$

$S_1 := \{s \in S \mid s < m\}$

$S_2 := \{s \in S \mid s = m\}$

$S_3 := \{s \in S \mid s > m\}$

if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$

if $|S_1| + |S_2| \geq k$ then return $m$

return $\text{Select}(k - |S_1| - |S_2|, S_3)$
k-th Element

- **Task**: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.
- **Lower bound**: $n - 1$ comparisons
- **Start with a nice sequential algorithm**

**Programm: Select(k,S)**

```plaintext
if |S| ≤ 50 then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
\[ m := \text{Select}(\lceil|M|/2\rceil, M) \]
\[ S_1 := \{ s \in S \mid s < m \} \]
\[ S_2 := \{ s \in S \mid s = m \} \]
\[ S_3 := \{ s \in S \mid s > m \} \]
if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $\text{Select}(k - |S_1| - |S_2|, S_3)$
```
Task: Compute the $k$-th ($k$-smallest) element in an unsorted sequence
$S = \{s_1, \cdots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select($k,S$)

if $|S| \leq 50$ then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
m := Select($\lceil |M|/2 \rceil$, $M$)
$S_1$ := \{s $\in$ $S$ $|$ s $<$ m\}
$S_2$ := \{s $\in$ $S$ $|$ s = m\}
$S_3$ := \{s $\in$ $S$ $|$ s $>$ m\}
if $|S_1| \geq k$ then return Select($k$, $S_1$)
if $|S_1| + |S_2| \geq k$ then return m
return Select($k - |S_1| - |S_2|$, $S_3$)
**k-th Element**

- **Task:** Compute the \( k \)-th \((k\text{-smallest})\) element in an unsorted sequence \( S = \{s_1, \ldots, s_n\} \).

- **Lower bound:** \( n - 1 \) comparisons

- **Start with a nice sequential algorithm**

- **Program:** \texttt{Select}(k, S)
  
  \begin{algorithmic}
  \begin{align*}
  &\text{if } |S| \leq 50 \text{ then return } k\text{-th number in } S \\
  &\text{Split } S \text{ in } \lceil n/5 \rceil \text{ sub-sequences } H_i \text{ of size } \leq 5 \\
  &\text{Sort each } H_i \\
  &\text{Let } M \text{ be the sequence of the middle elements in } H_i \\
  &\text{set } m := \texttt{Select}(\lceil |M|/2 \rceil, M) \\
  &\text{set } S_1 := \{s \in S \mid s < m\} \\
  &\text{set } S_2 := \{s \in S \mid s = m\} \\
  &\text{set } S_3 := \{s \in S \mid s > m\} \\
  &\text{if } |S_1| \geq k \text{ then return } \texttt{Select}(k, S_1) \\
  &\text{if } |S_1| + |S_2| \geq k \text{ then return } m \\
  &\text{return } \texttt{Select}(k - |S_1| - |S_2|, S_3)
  \end{align*}
  \end{algorithmic}
**k-th Element**

- **Task:** Compute the $k$-th ($k$-smallest) element in an unsorted sequence $S = \{s_1, \ldots, s_n\}$.
- **Lower bound:** $n - 1$ comparisons
- **Start with a nice sequential algorithm**

**Program:** `Select(k, S)`

- **if** $|S| \leq 50$ **then return** $k$-th number in $S$
- **Split** $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
- **Sort** each $H_i$
- Let $M$ be the sequence of the middle elements in $H_i$
  
  - $m := \text{Select}(\lceil |M|/2 \rceil, M)$
  - $S_1 := \{s \in S \mid s < m\}$
  - $S_2 := \{s \in S \mid s = m\}$
  - $S_3 := \{s \in S \mid s > m\}$
  - **if** $|S_1| \geq k$ **then return** $\text{Select}(k, S_1)$
  - **if** $|S_1| + |S_2| \geq k$ **then return** $m$
  - **return** $\text{Select}(k - |S_1| - |S_2|, S_3)$
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \cdots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select(k,S)

if $|S| \leq 50$ then return $k$-th number in $S$

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Sort each $H_i$

Let $M$ be the sequence of the middle elements in $H_i$

$m := \text{Select} (\lceil |M|/2 \rceil, M)$

$S_1 := \{s \in S \mid s < m\}$

$S_2 := \{s \in S \mid s = m\}$

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if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$

if $|S_1| + |S_2| \geq k$ then return $m$

return $\text{Select}(k - |S_1| - |S_2|, S_3)$
k-th Element

- Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.
- Lower bound: $n - 1$ comparisons
- Start with a nice sequential algorithm

Programm: \texttt{Select}(k, S)

\begin{itemize}
  \item \textbf{if} $|S| \leq 50$ \textbf{then return} $k$-th number in $S$
  \item Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
  \item Sort each $H_i$
  \item Let $M$ be the sequence of the middle elements in $H_i$
  \item $m := \texttt{Select}(\lceil |M|/2 \rceil, M)$
  \item $S_1 := \{s \in S \mid s < m\}$
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  \item $S_3 := \{s \in S \mid s > m\}$
  \item \textbf{if} $|S_1| \geq k$ \textbf{then return} $\texttt{Select}(k, S_1)$
  \item \textbf{if} $|S_1| + |S_2| \geq k$ \textbf{then return} $m$
  \item \textbf{return} $\texttt{Select}(k - |S_1| - |S_2|, S_3)$
\end{itemize}
**k-th Element**

- Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \cdots, s_n\}$.

- Lower bound: $n - 1$ comparisons

- Start with a nice sequential algorithm

**Programm: Select(k,S)**

1. If $|S| \leq 50$ then return $k$-th number in $S$
2. Split $S$ in $[n/5]$ sub-sequences $H_i$ of size $\leq 5$
3. Sort each $H_i$
4. Let $M$ be the sequence of the middle elements in $H_i$
5. $m := Select(\lceil |M|/2 \rceil, M)$
6. $S_1 := \{s \in S \mid s < m\}$
7. $S_2 := \{s \in S \mid s = m\}$
8. $S_3 := \{s \in S \mid s > m\}$
9. If $|S_1| \geq k$ then return $Select(k, S_1)$
10. If $|S_1| + |S_2| \geq k$ then return $m$
11. Return $Select(k - |S_1| - |S_2|, S_3)$
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \cdots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select$(k,S)$

- if $|S| \leq 50$ then return $k$-th number in $S$
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if $|S_1| \geq k$ then return Select$(k, S_1)$

if $|S_1| + |S_2| \geq k$ then return $m$

return Select$(k - |S_1| - |S_2|, S_3)$
**Example for the k-th Element (Slow Motion)**

**Input/Data:**

<table>
<thead>
<tr>
<th>48</th>
<th>46</th>
<th>55</th>
<th>74</th>
<th>4</th>
<th>92</th>
<th>8</th>
<th>82</th>
<th>49</th>
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</table>

**M:**

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**sorted M:**

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Example for the k-th Element (Slow Motion)

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<table>
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<tr>
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sorted M:

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</tbody>
</table>

\( M:\)

| 59 | 46 | 55 | 74 | 24 | 33 | 8  | 44 | 45 | 23 | 31 | 29 | 66 | 50 | 76 | 68 | 53 | 48 | 15 | 37 | 25 |

sorted \( M:\)
Example for the k-th Element (Slow Motion)

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M:

| 59 | 46 | 55 | 74 | 24 | 33 | 8 | 44 | 45 | 23 | 31 | 29 | 66 | 50 | 76 | 68 | 53 | 48 | 15 | 37 | 25 |

sorted M:
Example for the k-th Element (Slow Motion)

**Input/Data:**

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**M:**

| 59 | 46 | 55 | 74 | 24 | 33 | 8 | 44 | 45 | 23 | 31 | 29 | 66 | 50 | 76 | 68 | 53 | 48 | 15 | 37 | 25 |

**sorted M:**

| 8 | 15 | 23 | 24 | 25 | 29 | 31 | 33 | 37 | 44 | 45 | 46 | 48 | 50 | 53 | 55 | 59 | 66 | 68 | 74 | 76 |
Example for the k-th Element (Slow Motion)

Input/Data:

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</table>

M:

| 59 | 46 | 55 | 74 | 24 | 33 | 8  | 44 | 45 | 23 | 31 | 29 | 66 | 50 | 76 | 68 | 53 | 48 | 15 | 37 | 25 |

sorted M:

| 8  | 15 | 23 | 24 | 25 | 29 | 31 | 33 | 37 | 44 | 45 | 46 | 48 | 50 | 53 | 55 | 59 | 66 | 68 | 74 | 76 |
Example for the k-th Element

Input/Data:

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sorted M:

|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
Example for the k-th Element

Input/Data:

| 27 | 89 | 78 | 93 | 63 | 38 | 61 | 81 | 15 | 68 | 68 | 91 | 72 | 22 | 24 | 26 | 7 | 29 | 52 | 90 | 6 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|
| 2  | 95 | 53 | 43 | 50 | 54 | 90 | 36 | 12 | 59 | 75 | 97 | 6  | 34 | 75 | 13 | 92 | 54 | 43 | 8  | 64 |
| 29 | 55 | 4  | 14 | 96 | 20 | 76 | 51 | 13 | 47 | 39 | 33 | 90 | 28 | 85 | 87 | 41 | 69 | 58 | 83 | 45 |
| 94 | 32 | 33 | 81 | 56 | 2  | 27 | 32 | 95 | 52 | 15 | 57 | 63 | 9  | 9  | 75 | 77 | 28 | 91 | 55 | 64 |
| 97 | 81 | 59 | 5  | 96 | 30 | 91 | 57 | 79 | 6  | 83 | 46 | 23 | 64 | 34 | 54 | 45 | 36 | 37 | 80 | 87 |

M:

| 29 | 81 | 53 | 43 | 63 | 30 | 76 | 51 | 15 | 52 | 68 | 57 | 63 | 28 | 34 | 54 | 45 | 36 | 52 | 80 | 64 |

sorted M:
Example for the k-th Element

Input/Data:

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M:

| 29 | 81 | 53 | 43 | 63 | 30 | 76 | 51 | 15 | 52 | 68 | 57 | 63 | 28 | 34 | 54 | 45 | 36 | 52 | 80 | 64 |

sorted M:

| 15 | 28 | 29 | 30 | 34 | 36 | 43 | 45 | 51 | 52 | 52 | 53 | 54 | 57 | 63 | 63 | 64 | 68 | 76 | 80 | 81 |
Example for the k-th Element

Input/Data:

| 27 | 89 | 78 | 93 | 63 | 38 | 61 | 81 | 15 | 68 | 68 | 91 | 72 | 22 | 24 | 26 | 7 | 29 | 52 | 90 | 6 |
| 2  | 95 | 53 | 43 | 50 | 54 | 90 | 36 | 12 | 59 | 75 | 97 | 6  | 34 | 75 | 13 | 92 | 54 | 43 | 8  | 64 |
| 29 | 55 | 4  | 14 | 96 | 20 | 76 | 51 | 13 | 47 | 39 | 33 | 90 | 28 | 85 | 87 | 41 | 69 | 58 | 83 | 45 |
| 94 | 32 | 33 | 81 | 56 | 2  | 27 | 32 | 95 | 52 | 15 | 57 | 63 | 9  | 9  | 75 | 77 | 28 | 91 | 55 | 64 |
| 97 | 81 | 59 | 5  | 96 | 30 | 91 | 57 | 79 | 6  | 83 | 46 | 23 | 64 | 34 | 54 | 45 | 36 | 37 | 80 | 87 |

M:

| 29 | 81 | 53 | 43 | 63 | 30 | 76 | 51 | 15 | 52 | 68 | 57 | 63 | 28 | 34 | 54 | 45 | 36 | 52 | 80 | 64 |

sorted M:

| 15 | 28 | 29 | 30 | 34 | 36 | 43 | 45 | 51 | 52 | 52 | 53 | 54 | 57 | 63 | 63 | 64 | 68 | 76 | 80 | 81 |
Example for the k-th Element (Worst Case)

Input/Data:

| 87 | 84 | 75 | 72 | 86 | 93 | 75 | 88 | 74 | 89 | 62 | 81 | 56 | 50 | 63 | 73 | 86 | 55 | 62 | 66 | 79 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 79 | 84 | 53 | 70 | 87 | 75 | 78 | 80 | 65 | 65 | 94 | 66 | 63 | 58 | 51 | 67 | 70 | 75 | 85 | 56 | 94 |
| 44 | 14 | 39 | 32 | 44 | 43 | 35 | 30 | 11 | 29 | 21 | 55 | 80 | 63 | 66 | 70 | 68 | 72 | 84 | 81 | 83 |
| 14 | 4  | 7  | 44 | 32 | 6  | 8  | 0  | 29 | 43 | 33 | 51 | 78 | 68 | 91 | 59 | 77 | 94 | 54 | 93 | 69 |
| 12 | 12 | 23 | 19 | 30 | 16 | 27 | 15 | 35 | 22 | 18 | 81 | 66 | 77 | 81 | 70 | 78 | 84 | 59 | 80 | 65 |

M:

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sorted M:

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
Example for the k-th Element (Worst Case)

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<td>81</td>
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<td>81</td>
<td>70</td>
<td>78</td>
<td>84</td>
<td>59</td>
<td>80</td>
<td>65</td>
</tr>
</tbody>
</table>

M:

| 44 | 14 | 39 | 44 | 33 | 43 | 35 | 30 | 35 | 43 | 33 | 66 | 66 | 63 | 66 | 70 | 77 | 75 | 62 | 80 | 79 |

sorted M:
**Example for the k-th Element (Worst Case)**

**Input/Data:**

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 87| 84| 75| 72| 86| 93| 75| 88| 74| 89| 62| 81| 56| 50| 63| 73| 86| 55| 62| 66| 79|   |   |   |   |   |   |   |   |   |   |   |
| 79| 84| 53| 70| 87| 75| 78| 80| 65| 65| 94| 66| 63| 58| 51| 67| 70| 75| 85| 56| 94|   |   |   |   |   |   |   |   |   |   |   |
| 44| 14| 39| 4 | 33| 43| 35| 30| 11| 29| 21| 55| 80| 63| 66| 70| 68| 72| 84| 81| 83|   |   |   |   |   |   |   |   |   |   |   |
| 14| 4 | 7 | 44| 32| 6 | 8 | 0 | 29| 43| 33| 51| 78| 68| 91| 59| 77| 94| 54| 93| 69|   |   |   |   |   |   |   |   |   |   |   |
| 12| 12| 23| 19| 30| 16| 27| 15| 35| 22| 18| 81| 66| 77| 81| 70| 78| 84| 59| 80| 65|   |   |   |   |   |   |   |   |   |   |   |

**M:**

| 44 | 14 | 39 | 44 | 33 | 43 | 35 | 30 | 35 | 43 | 33 | 66 | 66 | 63 | 66 | 70 | 77 | 75 | 62 | 80 | 79 |

**sorted M:**

| 14 | 30 | 33 | 33 | 35 | 35 | 39 | 43 | 43 | 44 | 44 | 62 | 63 | 66 | 66 | 66 | 70 | 75 | 77 | 79 | 80 |
Example for the k-th Element (Worst Case)

**Input/Data:**

| 87 | 84 | 75 | 72 | 86 | 93 | 75 | 88 | 74 | 89 | 62 | 81 | 56 | 50 | 63 | 73 | 86 | 55 | 62 | 66 | 79 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 79 | 84 | 53 | 70 | 87 | 75 | 78 | 80 | 65 | 65 | 94 | 66 | 63 | 58 | 51 | 67 | 70 | 75 | 85 | 56 | 94 |
| 44 | 14 | 39 | 4  | 33 | 43 | 35 | 30 | 11 | 29 | 21 | 55 | 80 | 63 | 66 | 70 | 68 | 72 | 84 | 81 | 83 |
| 14 | 4  | 7  | 44 | 32 | 6  | 8  | 0  | 29 | 43 | 33 | 51 | 78 | 68 | 91 | 59 | 77 | 94 | 54 | 93 | 69 |
| 12 | 12 | 23 | 19 | 30 | 16 | 27 | 15 | 35 | 22 | 18 | 81 | 66 | 77 | 81 | 70 | 78 | 84 | 59 | 80 | 65 |

**M:**

| 44 | 14 | 39 | 44 | 33 | 43 | 35 | 30 | 35 | 43 | 33 | 66 | 66 | 63 | 66 | 70 | 77 | 75 | 62 | 80 | 79 |

**sorted M:**

| 14 | 30 | 33 | 33 | 35 | 35 | 39 | 43 | 43 | 44 | 44 | 62 | 63 | 66 | 66 | 66 | 70 | 75 | 77 | 79 | 80 |
Running Time

- For some constants $c, d$ we get:
  - $T(n) \leq d \cdot n$ for $n \leq 50$
  - $T(n) \leq c \cdot n + T(n/5) + T(3n/4)$

if $|S| \leq 50$ then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
$m := \text{Select}(\lceil |M|/2 \rceil, M)$
$S_1 := \{s \in S \mid s < m\}$
$S_2 := \{s \in S \mid s = m\}$
$S_3 := \{s \in S \mid s > m\}$
if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $\text{Select}(k - |S_1| - |S_2|, S_3)$
Running Time

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  - $T(n) \leq d \cdot n$ for $n \leq 50$
  - $T(n) \leq c \cdot n + T(n/5) + T(3n/4)$

```plaintext
if $|S| \leq 50$ then return $k$-th number in $S$
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```
Running Time

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\[ m := \text{Select} \left( \lceil |M|/2 \rceil, M \right) \]
\[ S_1 := \{ s \in S \mid s < m \} \]
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Running Time

- Claim: \( T(n) \leq 20 \cdot r \cdot n \) with \( r = \max(d, c) \).
- Proof:
  - \( n = 50 \):
    \[
    T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}
    \]
  - \( n \geq 50 \):
    \[
    T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)
    \]
- Running time \( T(n) \) is in \( O(n) \).
Running Time

- **Claim:** $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.

- **Proof:**
  - $n = 50$:
    
    $$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$
  
  - $n > 50$:
    
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$

- Running time $T(n)$ is in $O(n)$. 
Running Time

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  - $n = 50$:
  
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  - $n > 50$:
  
  $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$

  $$T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n$$

- Running time $T(n)$ is in $O(n)$. 

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Proof:

- \( n = 50 \):

\[
T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}
\]

- \( n > 50 \):

\[
T(n) \leq c \cdot n + T\left( \frac{d \cdot n}{5} \right) + T\left( \frac{3 \cdot d \cdot n}{4} \right)
\]

\[
T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n
\]

Running time \( T(n) \) is in \( O(n) \).
Running Time

- **Claim:** $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.

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  - $n = 50$:
    \[
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    \]
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    \[
    T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)
    \]
    \[
    T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n
    \]

- Running time $T(n)$ is in $O(n)$. 
Running Time

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  - $n > 50$:
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$
  - $T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n$
- Running time $T(n)$ is in $O(n)$. 
Parallel k-Select

- **Input** $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
- Each $P_i$ knows $n, P(n)$.
- Each $P_i$ works on $\lceil n^x \rceil$ elements.
- We will now create a parallel version of the program Select(k,S).
- We will get a parallel recursive program.

1. Easy solution for small $S$.
2. Split $S$ into small sub-sequences for the processors.
3. Compute parallel the median of the sub-sequences.
4. Compute parallel and recursive the median of medians.
5. Compute the splitting into the three sub-sequences.
6. Do the final recursion.
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- **Each** $P_i$ **knows** $n, P(n)$.
- **Each** $P_i$ **works on** $\lceil n^x \rceil$ **elements**.
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1. **Easy solution for small** $S$.
2. **Split** $S$ **into small sub-sequences for the processors**.
3. **Compute parallel the median of the sub-sequences**.
4. **Compute parallel and recursive the median of medians**.
5. **Compute the splitting into the three sub-sequences**.
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Parallel k-Select

- Input \( S = \{s_1, \cdots, s_n\} \).
- Processors \( P_1, P_2, \cdots P_{\lceil n^{1-x} \rceil} \), thus \( P(n) = \lceil n^{1-x} \rceil \).
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Parallel k-Select

- Input $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots P_{[n^{1-x}]}$, thus $P(n) = [n^{1-x}]$.
- Each $P_i$ knows $n, P(n)$.
- Each $P_i$ works on $[n^x]$ elements.

We will now create a parallel version of the program Select($k$, $S$).
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Parallel k-Select

- Input $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots, P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
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- Each \( P_i \) knows \( n, P(n) \).
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Parallel $k$-Select

- Input $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots, P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
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Example for the k-th Element

Input/Data:

```
50 71 30 75 85 11 5 10 49 17 60 26 33 62 50 56 83 61 80 26 63 88 4 74 40 74 82 54 85 85 57 89 26 2 4
67 28 13 78 0 38 38 31 59 14 43 92 71 85 13 18 19 35 76 1 52 21 46 87 59 41 43 92 84 28 27 85 84 42 57
25 96 50 47 4 53 15 47 80 53 30 42 16 62 81 46 55 27 43 75 16 79 14 82 4 56 38 68 50 46 60 29 38 76 72
0 95 55 3 80 60 35 88 4 29 26 1 29 59 91 0 81 78 39 52 41 43 66 90 40 33 65 51 40 35 26 41 81 35 57
10 46 61 77 10 59 18 94 41 17 51 84 54 38 35 83 50 30 50 26 54 43 16 74 91 81 85 27 75 90 86 44 47 54 57
67 30 73 22 67 5 20 76 34 66 22 85 84 64 4 88 88 52 87 84 66 8 1 55 45 2 31 12 75 21 84 69 36 20 50
62 19 2 54 54 72 45 7 1 95 24 68 65 39 64 71 26 73 59 1 11 65 17 48 55 61 84 65 3 11 97 27 78 49 65
78 60 1 68 60 83 95 48 67 13 52 71 18 72 8 28 52 17 81 79 86 46 92 26 74 6 69 41 91 77 54 91 78 56 62 84 9
11 54 43 81 54 11 18 62 65 84 82 83 33 63 70 66 21 10 84 66 20 17 20 11 96 54 20 7 48 21 5 89 40 42 71
92 22 47 5 91 25 93 41 80 73 66 48 24 88 29 13 69 60 51 91 92 12 23 9 8 36 60 21 82 51 31 3 64 24 23
50 93 50 84 66 38 27 84 68 87 95 44 75 21 0 54 58 57 40 6 48 69 61 54 43 90 65 64 14 30 97 51 2 39 94
53 24 57 93 31 2 31 73 79 0 39 32 10 2 57 9 69 9 1 59 89 12 26 85 28 57 65 52 67 8 88 94 67 78 7
88 88 37 34 93 65 81 32 85 53 18 90 3 38 6 62 77 89 37 30 91 86 10 79 37 41 1 59 29 87 71 95 75 85 54
49 59 66 0 11 40 43 39 5 7 20 88 60 13 79 38 80 81 27 73 55 26 28 39 70 49 88 27 15 44 23 16 0 62 7
25 5 43 34 8 40 9 50 64 96 56 43 73 65 17 76 48 20 73 61 62 55 15 11 13 73 39 39 80 31 56 13 2 19
```

M:

```
P_1  P_2  P_3  P_4  P_5  P_6  P_7  P_8  P_9  P_{10}  P_{11}  P_{12}  P_{13}  P_{14}  P_{15}  P_{16}  P_{17}  P_{18}  P_{19}  P_{20}  P_{21}  P_{22}  P_{23}  P_{24}  P_{25}  P_{26}  P_{27}  P_{28}  P_{29}  P_{30}  P_{31}  P_{32}  P_{33}  P_{34}  P_{35}
```

sorted M:

```
```

```
## Motivation and History

### Example for the k-th Element

**Input/Data:**

```
50 71 30 75 85 11 5 10 49 17 60 26 33 62 50 56 83 61 80 26 63 88 4 74 40 74 70 82 54 85 85 57 89 26 2 4
67 28 13 78 0 38 38 31 59 14 43 92 71 85 13 18 19 35 76 1 52 21 46 87 59 41 43 92 84 28 27 85 84 42 57
25 96 50 47 4 53 15 47 80 53 30 42 16 62 81 46 55 27 43 75 16 79 14 82 4 56 38 68 50 46 60 29 38 76 72
0 95 55 3 80 50 35 88 4 29 26 1 29 59 91 0 81 78 39 52 41 43 66 90 40 33 65 51 40 35 26 41 81 35 57
10 46 61 77 10 59 18 94 41 17 51 84 54 38 35 83 50 50 26 54 43 16 74 91 81 85 27 75 90 86 44 47 54 57
67 30 73 22 67 5 20 76 34 66 22 85 84 54 4 88 88 52 87 84 66 8 1 55 45 2 31 12 75 21 84 69 36 20 50
62 19 2 54 54 72 45 7 1 95 24 68 65 39 64 71 26 73 59 1 11 65 17 48 55 61 84 65 3 11 97 27 78 49 65
78 60 1 68 60 83 95 48 67 13 52 71 87 28 52 17 81 79 86 46 92 26 74 6 69 41 91 77 54 91 78 56 62 84 9
11 54 43 81 54 11 18 62 65 84 82 83 33 63 70 66 21 10 84 66 20 17 20 11 96 54 20 7 48 21 5 89 40 42 71
92 22 47 5 91 25 93 41 66 76 44 28 48 22 99 23 68 51 91 92 12 23 9 8 36 60 21 82 51 31 3 64 24 23
50 93 50 84 66 38 27 84 68 87 95 75 21 0 54 58 57 40 6 48 69 61 54 43 90 65 64 14 30 97 51 2 39 94
53 24 57 93 31 2 31 73 79 0 39 32 10 2 57 9 69 9 1 59 89 12 26 85 28 57 65 52 67 8 88 94 67 78 7
88 88 37 34 93 65 81 32 85 53 18 90 3 38 6 62 77 89 37 30 91 86 10 79 37 41 1 59 29 87 71 95 75 85 54
49 59 66 0 11 40 43 39 5 7 20 88 60 13 79 38 80 81 27 73 55 26 28 39 70 49 88 27 15 44 23 16 0 62 7
25 5 43 34 8 40 9 50 64 96 56 43 73 65 17 76 48 20 73 61 62 55 15 11 13 73 39 39 38 80 31 56 13 2 19
```

**P:**

```
P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8 P_9 P_10 P_11 P_12 P_13 P_14 P_15 P_16 P_17 P_18 P_19 P_20 P_21 P_22 P_23 P_24 P_25 P_26 P_27 P_28 P_29 P_30 P_31 P_32 P_33 P_34 P_35
```

**M:**

```
50 54 47 54 40 31 48 64 53 43 68 54 59 50 54 69 57 51 52 55 43 20 55 43 54 65 52 50 44 60 56 47 42 54
```

sorted M:

```
50 54 47 54 40 31 48 64 53 43 68 54 59 50 54 69 57 51 52 55 43 20 55 43 54 65 52 50 44 60 56 47 42 54
```
**Example for the k-th Element**

**Input/Data:**

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**M:**

| 50 | 54 | 47 | 54 | 40 | 31 | 48 | 64 | 53 | 43 | 68 | 54 | 59 | 50 | 54 | 69 | 57 | 51 | 52 | 55 | 43 | 20 | 55 | 43 | 54 | 65 | 52 | 50 | 44 | 60 | 56 | 47 | 42 | 54|

sorted M:

| 20 | 31 | 40 | 42 | 43 | 43 | 44 | 47 | 48 | 50 | 50 | 50 | 51 | 52 | 52 | 53 | 54 | 54 | 54 | 54 | 54 | 55 | 55 | 56 | 57 | 59 | 60 | 64 | 65 | 68 | 69 |
### Example for the k-th Element

**Input/Data:**

```
50 71 30 75 85 11 5 10 49 17 60 26 33 62 50 56 83 61 80 26 63 88 4 74 40 74 82 54 85 85 57 89 26 2 4
67 28 13 78 0 38 38 31 59 14 43 92 71 85 13 18 19 35 76 1 52 21 46 87 59 41 43 92 84 28 27 85 84 42 57
25 96 50 47 4 53 15 47 80 53 30 42 16 62 81 46 55 27 43 75 16 79 14 82 4 56 38 68 50 46 60 29 38 76 72
0 95 55 3 80 60 35 88 4 29 26 1 29 59 91 0 81 78 39 52 41 43 66 90 40 33 65 51 40 35 26 41 81 35 57
10 46 61 77 10 59 18 94 41 17 51 84 54 38 35 83 50 30 50 26 54 43 16 74 91 81 85 27 75 90 86 44 47 54 57
67 30 73 22 67 5 20 76 34 66 22 85 84 64 4 88 88 52 87 84 66 8 1 55 45 2 31 12 75 21 84 69 36 20 50
62 19 2 54 54 72 45 7 1 95 24 68 65 39 64 71 26 73 59 1 11 65 17 48 55 61 84 65 3 11 97 27 78 49 65
78 60 1 68 60 83 95 48 67 13 52 71 87 28 52 17 81 79 86 46 92 26 74 6 69 41 91 77 54 91 78 56 62 84 9
11 54 43 81 54 11 18 62 65 84 82 83 33 63 70 66 21 10 84 66 20 17 20 11 96 54 20 7 48 21 5 89 40 42 71
92 22 47 5 91 25 93 41 80 73 66 48 24 88 29 13 69 60 51 91 92 12 23 9 8 36 60 21 82 51 31 3 64 24 23
50 93 50 84 66 38 27 84 68 87 95 44 75 21 0 54 58 57 40 6 48 69 61 54 43 90 65 64 14 30 97 51 2 39 94
53 24 57 93 31 2 31 73 79 0 39 32 10 2 57 9 69 9 1 59 89 12 26 85 28 57 65 52 67 8 88 94 67 78 7
88 88 37 34 93 65 81 32 85 53 18 90 3 38 6 62 77 89 37 30 91 86 10 79 37 41 1 59 29 87 71 95 75 85 54
49 59 66 0 11 40 43 39 5 7 20 88 60 13 79 38 80 81 27 73 55 26 28 39 70 49 88 27 15 44 23 16 0 62 7
25 5 43 34 8 40 9 50 64 96 56 43 73 65 17 76 48 20 73 61 62 55 15 11 13 73 39 39 38 80 31 56 13 2 19
```

**M:**

```
50 54 47 54 54 40 31 48 64 53 43 68 54 59 50 54 69 57 51 52 55 43 20 55 43 54 65 52 50 44 60 56 47 42 54
```

**sorted M:**

```
20 31 40 42 43 43 43 44 47 47 48 50 50 50 51 52 52 53 54 54 54 54 54 54 54 55 55 56 57 59 60 64 65 68 69
```
Parallel k-Select

Programm: ParSelect(k,S)
1:
   if \(|S| \leq k_1\) then \(P_1\) returns \(Select(k, S)\).
2:
   \(S\) is split into \(\lceil |S|^{1-x} \rceil\) sub-sequences \(S_i\) with \(|S_i| \leq \lceil n^x \rceil\).
   \(P_i\) stores the start-address of \(S_i\).
3:
   for all \(P_i\) where \(1 \leq i \leq \lceil n^{1-x} \rceil\) do in parallel
      \(m_i := Select(\lceil |S_i|/2 \rceil, S_i)\)
      \(P_i(m_1) \rightarrow R_i\).
   Assume in the following that \(M\) is the sequence of these values.
4:
   \(m := ParSelect(\lceil |M|/2 \rceil, M)\).
5: More to come!
Parallel k-Select

Programm: ParSelect(k,S)
1:  
\textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \ \textbf{return}s \ Select(k, S).
2:  
\textbf{S is split into} \ \lceil |S|^{1-x} \rceil \ \textbf{sub-sequences} \ S_i \ \textbf{with} \ |S_i| \leq \lceil n^x \rceil
\textbf{P}_i \ \textbf{stores the start-address of} \ S_i.
3:  
\textbf{for all} \ P_i \ \textbf{where} \ 1 \leq i \leq \lceil n^{1-x} \rceil \ \textbf{do in parallel}
\quad m_i := \ Select(\lceil |S_i|/2 \rceil, S_i)
\quad P_i(m_1) \rightarrow R_i.
\textbf{Assume in the following that} \ M \ \textbf{is the sequence of these values.}
4:  
\quad m := \ ParSelect(\lceil |M|/2 \rceil, M).
5:  
\textbf{More to come!}
Parallel k-Select

Programm: ParSelect(k,S)
1:  
   \textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \ \textbf{returns} \ Select(k,S).
2:  
   S \ \text{is split into} \ \lceil |S|^{1-x} \rceil \ \text{sub-sequences} \ S_i \ \text{with} \ |S_i| \leq \lceil n^x \rceil 
   \ \text{P}_i \ \text{stores the start-address of} \ S_i.
3:  
   \textbf{for all} \ P_i \ \text{where} \ 1 \leq i \leq \lceil n^{1-x} \rceil \ \textbf{do in parallel}
   \quad m_i := Select(\lceil |S_i|/2 \rceil, S_i)
   \quad P_i(m_1) \rightarrow R_i.
   \quad \text{Assume in the following that} \ M \ \text{is the sequence of these values.}
4:  
   m := ParSelect(\lceil |M|/2 \rceil, M).
5:  
   \text{More to come!}
Programm: ParSelect(k,S)
1: 
   if $|S| \leq k_1$ then $P_1$ returns $Select(k, S)$.
2: 
   $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   $P_i$ stores the start-address of $S_i$.
3: 
   for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
   
   $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
   
   $P_i(m_1) \rightarrow R_i$.
   Assume in the following that $M$ is the sequence of these values.
4: 
   $m := ParSelect(\lceil |M|/2 \rceil, M)$.
5: 
   More to come!
Parallel k-Select

Programm: ParSelect(k,S)
1:
   if $|S| \leq k_1$ then $P_1$ returns $Select(k, S)$.
2:
   $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   $P_i$ stores the start-address of $S_i$.
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   for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
      $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
      $P_i(m_1) \rightarrow R_i$.
   Assume in the following that $M$ is the sequence of these values.
4:
   $m := ParSelect(\lceil |M|/2 \rceil, M)$.
5: More to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:
Distribute $m$ via broadcast to all $P_i$.

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{s \in S_i \mid s < m\}$
$E_i := \{s \in S_i \mid s = m\}$
$G_i := \{s \in S_i \mid s > m\}$

5.2:

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$

5.3:

Even more to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:
Distribute $m$ via broadcast to all $P_i$.

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{ s \in S_i \mid s < m \}$

$E_i := \{ s \in S_i \mid s = m \}$

$G_i := \{ s \in S_i \mid s > m \}$

5.2:
Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$

5.3:
Even more to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:
Distribute $m$ via broadcast to all $P_i$.

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$\begin{align*}
L_i &:= \{ s \in S_i \mid s < m \} \\
E_i &:= \{ s \in S_i \mid s = m \} \\
G_i &:= \{ s \in S_i \mid s > m \}
\end{align*}$

5.2:
Compute with Parallel Prefix:

$\begin{align*}
l_i &:= \sum_{j=1}^{i} |L_j| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \\
e_i &:= \sum_{j=1}^{i} |E_j| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \\
g_i &:= \sum_{j=1}^{i} |G_j| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil. \\
Let: l_0 = e_0 = g_0 = 0
\end{align*}$

5.3:
Even more to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5+6

5.3:
Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

\( P_i \) writes \( L_i \) in \( R_{l_{i-1}+1}, \ldots, R_{l_i} \).
\( P_i \) writes \( E_i \) in \( R_{e_{j-1}+1}, \ldots, R_{e_i} \).
\( P_i \) writes \( G_i \) in \( R_{g_{i-1}+1}, \ldots, R_{g_i} \).

6:
if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Parallel k-Select

Programm: ParSelect(k,S) Steps 5+6

5.3:
Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \) and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

\( P_i \) writes \( L_i \) in \( R_{l_i-1+1}, \ldots, R_{l_i} \).
\( P_i \) writes \( E_i \) in \( R_{e_i-1+1}, \ldots, R_{e_i} \).
\( P_i \) writes \( G_i \) in \( R_{g_i-1+1}, \ldots, R_{g_i} \).

6:

if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: 
   \textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \ \textbf{returns} \ Select(k, S).

2: 
   S \ \text{is split into} \ \lceil |S|^{1-x} \rceil \ \text{sub-sequences} \ S_i \ \text{with} \ |S_i| \leq \lceil n^x \rceil \ 
   P_i \ \text{stores the start-address of} \ S_i.

3: 
   \textbf{for all} \ P_i \ \text{where} \ 1 \leq i \leq \lceil n^{1-x} \rceil \ \textbf{do in parallel}
   m_i := \text{Select}(\lceil |S_i|/2 \rceil, S_i)
   P_i(m_1) \rightarrow R_i.
   \text{Assume in the following that} \ M \ \text{is the sequence of these values}

4: 
   m := \text{ParSelect}(\lceil |M|/2 \rceil, M).
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: \( O(1) \)
   \[
   \text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } Select(k, S).
   \]

2:
   \[
   S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil
   \]
   \[
   P_i \text{ stores the start-address of } S_i.
   \]

3:
   \[
   \text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel}
   \]
   \[
   m_i := Select(\lceil |S_i|/2 \rceil, S_i)
   \]
   \[
   P_i(m_1) \rightarrow R_i.
   \]
   \[
   \text{Assume in the following that } M \text{ is the sequence of these values}
   \]

4:
   \[
   m := ParSelect(\lceil |M|/2 \rceil, M).
   \]
Parallel $k$-Select (Running Time)

**Programm: ParSelect($k$, $S$)**

1: $O(1)$
   - if $|S| \leq k_1$ then $P_1$ returns $Select(k, S)$.

2: $O(\log_2(|S|^{1-x}))$ thus we have $O(\log n)$
   - $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   - $P_i$ stores the start-address of $S_i$.

3:
   - for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
     - $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
     - $P_i(m_1) \rightarrow R_i$.
     - Assume in the following that $M$ is the sequence of these values

4:
   - $m := ParSelect(\lceil |M|/2 \rceil, M)$. 
Parallel k-Select (Running Time)

Programm: ParSelect(k, S)

1: $O(1)$
   \[\text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } Select(k, S).\]

2: $O(\log_2(|S|^{1-x}))$ thus we have $O(\log n)$
   \[S \text{ is split into } \left\lceil |S|^{1-x} \right\rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \left\lceil n^x \right\rceil\]
   \[P_i \text{ stores the start-address of } S_i.\]

3: $O(n^x)$
   \[\text{for all } P_i \text{ where } 1 \leq i \leq \left\lceil n^{1-x} \right\rceil \text{ do in parallel}\]
   \[m_i := Select(\left\lceil |S_i|/2 \right\rceil, S_i)\]
   \[P_i(m_1) \rightarrow R_i.\]
   \[\text{Assume in the following that } M \text{ is the sequence of these values}\]

4: \[m := ParSelect(\left\lceil |M|/2 \right\rceil, M).\]
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: $O(1)$
   \textbf{if} $|S| \leq k_1$ \textbf{then} $P_1$ \textbf{returns} $Select(k,S)$.

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   $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   $P_i$ stores the start-address of $S_i$.

3: $O(n^x)$
   \textbf{for all} $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ \textbf{do in parallel}
   \hspace{1cm} $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
   \hspace{1cm} $P_i(m_1) \rightarrow R_i$.
   Assume in the following that $M$ is the sequence of these values

4: $T_{ParSelect}(n^{1-x})$
   $m := ParSelect(\lceil |M|/2 \rceil, M)$. 
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a:
Distribute $m$ via broadcast to all $P_i$.

5.1b:
for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
$L_i := \{ s \in S_i \mid s < m \}$
$E_i := \{ s \in S_i \mid s = m \}$
$G_i := \{ s \in S_i \mid s > m \}$

5.2:
Compute with Parallel Prefix:
$l_i := \sum_{j=1}^{i} |L_j|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$e_i := \sum_{j=1}^{i} |E_j|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$g_i := \sum_{j=1}^{i} |G_j|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
Let: $l_0 = e_0 = g_0 = 0$
Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$

Distribute $m$ via broadcast to all $P_i$.

5.1b:

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{s \in S_i \mid s < m\}$

$E_i := \{s \in S_i \mid s = m\}$

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5.2:

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$
Distribute $m$ via broadcast to all $P_i$.

5.1b: $O(|S_i|)$ thus we have $O(n^x)$
for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{ s \in S_i \mid s < m \}$
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Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$

Distribute $m$ via broadcast to all $P_i$.

5.1b: $O(|S_i|)$ thus we have $O(n^x)$

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{s \in S_i \mid s < m\}$

$E_i := \{s \in S_i \mid s = m\}$

$G_i := \{s \in S_i \mid s > m\}$

5.2: $O(\log_2(n^{1-x}))$

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5+6

5.3:
Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

\( P_i \) writes \( L_i \) in \( R_{l_i−1+1}, \ldots, R_{l_i} \).
\( P_i \) writes \( E_i \) in \( R_{e_i−1+1}, \ldots, R_{e_i} \).
\( P_i \) writes \( G_i \) in \( R_{g_i−1+1}, \ldots, R_{g_i} \).

6:

if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Parallel k-Select (Running Time)

Programm: ParSelect(k, S) Steps 5+6

5.3: $O(n^x)$

Compute $L = \{s \in S \mid s < m\}$, $E = \{s \in S \mid s = m\}$
and $G = \{s \in S \mid s > m\}$ as follows:
for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

- $P_i$ writes $L_i$ in $R_{l_{i-1}+1}, \ldots, R_{l_i}$.
- $P_i$ writes $E_i$ in $R_{e_{i-1}+1}, \ldots, R_{e_i}$.
- $P_i$ writes $G_i$ in $R_{g_{i-1}+1}, \ldots, R_{g_i}$.

6:

- if $|L| \geq k$ then return $\text{ParSelect}(k, L)$
- if $|L| + |E| \geq k$ then return $m$
- return $\text{Select}(k - |L| - |E|, G)$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5+6

5.3: $O(n^x)$

Compute $L = \{s \in S \mid s < m\}$, $E = \{s \in S \mid s = m\}$ and $G = \{s \in S \mid s > m\}$ as follows:

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

- $P_i$ writes $L_i$ in $R_{l_i-1+1}, \ldots, R_{l_i}$.
- $P_i$ writes $E_i$ in $R_{e_i-1+1}, \ldots, R_{e_i}$.
- $P_i$ writes $G_i$ in $R_{g_i-1+1}, \ldots, R_{g_i}$.

6: $T_{ParSelect}(3 \cdot n/4)$

if $|L| \geq k$ then return $ParSelect(k, L)$
if $|L| + |E| \geq k$ then return $m$
return $Select(k - |L| - |E|, G)$
Parallel k-Select (Running Time)

Adding all up we get:

\[ T_{\text{ParSelect}}(n) = c_1 \log n + c_2 \cdot n^x + T_{\text{ParSelect}}(n^{1-x}) + T_{\text{ParSelect}}(3/4 \cdot n). \]

\[ T_{\text{ParSelect}}(n) = O(n^x) \text{ with } P_{\text{ParSelect}}(n) = O(n^{1-x}). \]

\[ \text{Eff}_{\text{ParSelect}}(n) = \frac{O(n)}{O(n^x) \cdot O(n^{1-x})} = O(1) \]
Sequential Merging

- **Input:**
  \[ A = (a_1, a_2, \cdots, a_r) \text{ and } B = (b_1, b_2, \cdots, b_s) \] two sorted sequences

- **Output:**
  \[ C = (c_1, c_2, \cdots, c_n) \] sorted sequence of \( A \) and \( B \) with \( n = r + s \).

- **Programm:** Merge
  
  \[ i := 1; j := 1; n := r + s \]
  
  for \( k := 1 \) to \( n \) do
    
    if \( a_i < b_j \)
      
      then \( c_k := a_i; i := i + 1; \)
      
      else \( c_k := b_j; j := j + 1; \)

- Algorithm does not care about special cases.

- Running time: at most \( r + s \) comparisons, i.e. \( O(n) \).

- Lower bound on the number of comparisons is \( r + s \), i.e. \( \Omega(n) \).
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Input:
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Programm: Merge
\[ i := 1; j := 1; n := r + s \]
for \( k := 1 \) to \( n \) do
  if \( a_i < b_j \)
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- Programm: Merge
  \[
i := 1; j := 1; n := r + s
  \]
  \[
  \text{for } k := 1 \text{ to } n \text{ do}
  \]
  \[
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  \]
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- **Programm: Merge**
  
  \[
  \begin{align*}
  i &:= 1; j := 1; n := r + s \\
  \text{for } k := 1 \text{ to } n \text{ do} \\
  &\quad \text{if } a_i < b_j \\
  &\quad \quad \text{then } c_k := a_i; i := i + 1; \\
  &\quad \quad \text{else } c_k := b_j; j := j + 1;
  \end{align*}
  \]

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Idea for Parallel Merging (CREW)

- The border lines may not intersect each other.
- Thus we may separate the two sequences into disjoint blocks.
- Let $A_i$ the $i$ block of size $\lceil r/p \rceil$.
- Let $\hat{B}_i$ block in $B$ which should be merged with $A_i$.
- Thus we may uses a PRAM easily (in this case).
The border lines may not intersect each other.

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- Thus we may uses a PRAM easily (in this case).
Idea for Parallel Merging (CREW)

Let $A_i$ [resp. $B_i$] the $i$ block of size $\lceil r/p \rceil$ [resp. $\lceil s/p \rceil$].
Let $\hat{B}_i$ [resp. $A_i$] block in $B$ [resp. $A$] which should be merged with $A_i$ [resp. $B_i$].
$P_i$ cares about $A_i$ and $\hat{B}_i$ if $|\hat{B}_i| \leq \lceil r/p \rceil$.
Let $C$ be those where one $P_j$ takes already care of.
$P_i$ cares about $A_i \setminus C$ and $\hat{B}_i \setminus C$. 
Idea for Parallel Merging (CREW)

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$P_i$ cares about $A_i$ and $\hat{B}_i$ if $|\hat{B}_i| \leq \lceil r/p \rceil$.

Let $C$ be those where one $P_j$ takes already care of.

$P_i$ cares about $A_i \setminus C$ and $\hat{B}_i \setminus C$. 
Let \( A_i \) [resp. \( B_i \)] the \( i \) block of size \( \lceil r/p \rceil \) [resp. \( \lceil s/p \rceil \)].

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- Let $C$ be those where one $P_j$ takes already care of.
- $P_i$ cares about $A_i \setminus C$ and $\hat{B}_i \setminus C$. 
Parallel Merging (CREW)

1. Use $P(n)$ processors.
2. Each processor $P_i$ computes for $A [B]$ its part of size $r/P(n) [s/P(n)]$.
3. Each processor $P_i$ computes the part from $B [A]$ which should be merged with its $A$-block [$B$-block].
4. Each processor computes its $A$ or $B$ block, where only he is responsible for.
5. This block has size $O(n/P(n))$.
6. Each processor merges its block into the resulting sequence.
7. Time: $O(\log n + n/P(n))$.
8. Efficiency

$$\frac{n}{O(P(n)) \cdot O(\log n + n/P(n))}.$$

9. Efficiency is 1 for $P(n) \leq n/\log n$. 
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Idea for Merging (EREW)

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1. Use $P(n)$ processors.
2. Compute the median $m$ of the sequences $A$ and $B$.
3. Split the sequences $A$ and $B$ in two sub-sequences each of the “same” size ($-1 \leq |A| - |B| \leq 1$).
4. Continue recursively, till all sub-sequences are smaller than $n/P(n)$.
5. Do the merging in the same way as before.

Remaining problem: Find the median of two sequences.
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Example for the Median for two Sorted Sequences

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- Sequences A and B are sorted.
- Compute median $a$ of A and median $b$ of B.
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Motivation and History

PRAM Introduction

Efficiency

Selection

Merging

1:62 Parallel Merging

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- Explain the motivation behind parallel systems.
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Legende

- : Nicht relevant
- : Grundlagen, die implizit genutzt werden
- : Idee des Beweises oder des Vorgehens
- : Struktur des Beweises oder des Vorgehens
- : Vollständiges Wissen