Theory of Parallel and Distributed Systems (WS2016/17)
Kapitel 1
First Algorithms for PRAM

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Lehrstuhl für Informatik 1

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1. There are limits to the computing power of a single Computer
2. Computers become cheaper
3. Specialized computers are expensive
4. There are tasks with large data
5. Many problems are very complex
   1. Weather and other Simulations
   2. Crash tests
   3. Military applications
   4. Large data: (SETI, ...)
   5. More similar problems
6. Thus there is the need for computers with more than one CPU
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- Processor \(P_1\) receives the input.
- Output of \(P_1\) will be passed as the input of \(P_2\).
- Output of \(P_i\) will be passed as the input of \(P_{i+1}\) \(1 \leq i < n\).
- Processor \(P_n\) delivers the final output.
- Processors may be different.
- Processors may run different programs.
- Intermediate outputs may be buffered.
- Pipelining is one important type of parallel system (in practice).
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Systolic Arrays

- Idea: use more than one data stream.
- Data streams may interact each other.
- Each processor is the same.
- There is a global synchronization.
- Processors may run simple programs.
- Advantage: really fast (for special applications).
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Vector Computer

- **Vector of processes.**
- Each processor has different data.
- But each processor executes the same program.
- Addition of two vectors:
  1. Read vector $A$
  2. Read vector $B$
  3. Add (each processor)
  4. Output the summ

- **Single Instruction Multiple Data** (SIMD) Computer.
- **Aim:** **Multiple Instruction Multiple Data** (MIMD) Computer.
- I.e. Fast processors with fast connections.
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- Advantage: very flexible, any fixed network of degree 4 possible.
- Disadvantage: long wires may be necessary, only a fixed network possible.
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- Memory
- Link
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- **Advantage:** “normal” CPUs.
- Advantage: fast links possible.
- Advantage: no special hardware.
- Advantage: variable network, may change during execution.
- Advantage: very large networks may be possible.
- Disadvantage: still a limited degree for the network.
- Disadvantage: large network are complicated.
- Problem: cooling large systems.
- Problem: fault tolerance.
- Problem: construct such a system.
- Problem: generate good data throughput with constant degree network.
- Problem: do the program structures fit the structure of the network.
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Parallel Computer II (Goodput)

- Look for good networks.
- Trees, Grids, Pyramids, ...
- $HQ(n)$, $CCC(n)$, $BF(n)$, $SE(n)$, $DB(n)$, ...
- Pancake Network and Burned Pancake Network.
- Problem: Physical placement of the processors.
- Problem: Length of wires.
- Problem: Has the network a nice structure.
- If the network becomes too large, we may use efficiency.
- Solution: choose a mixed network structure.
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Solution: choose a mixed network structure.
Parallel Computer III (Network)
Parallel Computer IV (Network)
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

```
CPU  RAM  CPU  RAM  CPU  RAM  CPU  RAM
```

2. CPUs and memory are connected by a network:

```
CPU  CPU  CPU  CPU  CPU
```

```
RAM  RAM  RAM  RAM  RAM
```

The difference is more on the practical side.
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

   CPU  RAM  CPU  RAM  CPU  RAM  CPU  RAM  CPU  RAM

   Network

2. CPUs and memory are connected by a network:

   CPU  CPU  CPU  CPU  CPU

   Network

   RAM  RAM  RAM  RAM  RAM

The difference is more on the practical side.
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

   ![Diagram showing CPU and RAM connected as one logical unit]

2. CPUs and memory are connected by a network:

   ![Diagram showing separate CPUs and RAM connected by a network]

The difference is more on the practical side.
Parallel Computer V (Network)

1. CPU and memory are one logical unit:

```
CPU    RAM
CPU    RAM
CPU    RAM
CPU    RAM
CPU    RAM

Network
```

2. CPUs and memory are connected by a network:

```
CPU
CPU
CPU
CPU
CPU

Network

RAM
RAM
RAM
RAM
RAM
```

The difference is more on the practical side.
PRAM (theoretical model)

- Ignore/unify the costs for each computation step.
- Ignore/unify the costs for each communication step.
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- Ignore/unify the costs for each communication step.
Definition RAM

- **RAM:** Random Access Machine
- CPU may access any memory cell
- Memory is unlimited
- Complexity measurements
  - uniform: each operation cost one unit
  - logarithmic: cost are measured according to the size of the numbers
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Idea of PRAM

- Many processes
- Common program
- Program may select single processors
- Common memory
Definition PRAM

- Consists of processors $P_i$ with $1 \leq i \leq p$ (prozessor has id $i$).
- Consists of registers $R_j$ with $1 \leq j \leq m$.
- Each processor has some local registers.
- Each processor $P_i$ may access each register $R_j$.
- Each processor executes the same programm.
- The programm is synchronized, thus each processor executes the same instructions.
- A selection is possible by using the processor id.
- The input of length $n$ is written to registers $R_j$ with $1 \leq j \leq n$.
- The output is placed in some known registers.
- The registers contain words (numbers) in the uniform cost measurement.
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3. processor may do some local computation using local registers:
   
   $x := y \times 5$.

For the access to the register we have the following variations:

- EREW (Exclusive Read Exclusive Write)
- CREW (Concurrent Read Exclusive Write)
- CRCW (Concurrent Read Concurrent Write)
- ERCW (Exclusive Read Concurrent Write)

Write conflicts may be solved using the following rules:

- Arbitrary: any processor gets access to the register.
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Computation of an “Or” (Idea)
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\[
P_1 \lor P_2 \lor P_3 \lor P_4 \lor P_5 \lor P_6 \lor P_7 \lor P_8
\]

\[
0 \lor 1 \lor 0 \lor 0 \lor 1 \lor 0 \lor 0 \lor 1 \rightarrow 0
\]
Computation of an “Or” (Idea)

\[
\begin{array}{cccccccc}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 \\
0 & v & 1 & v & 0 & v & 0 & v \\
 & v & 0 & v & 1 & v & 0 & v \\
 & 0 & v & 1 & 0 & 1 & 0 & 0 \\
\end{array}
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Computation of an “Or” (Idea)

\[ x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \quad x = 0 \quad x = 0 \quad x = 1 \]

\[
\begin{array}{cccccccc}
0 & \lor & 1 & \lor & 0 & \lor & 0 & \lor & 1 & \lor & 0 & \lor & 0 & \lor & 1 & \rightarrow & 0
\end{array}
\]
Computation of an “Or” (Idea)

\[
x = 0 
\lor 
\lor 
\lor 
\lor 
\lor 
\lor 
\lor 
\rightarrow 
\]

\[
0 
\lor 
1 
\lor 
0 
\lor 
0 
\lor 
1 
\lor 
0 
\lor 
0 
\lor 
1 
\rightarrow 
0
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Computing an “Or”

- **Task**: Compute \( x = \bigvee_{i=1}^{n} x_i \).
- **Input**: \( x_i \) is in register \( R_i \) \((1 \leq i \leq n)\).
- **Output**: computed in \( R_{n+1} \).
- **Model**: CRCW Arbitrary, Common oder Priority.
- **Programm**: Or
  
  for all \( P_i \) where \( 1 \leq i \leq n \) do in parallel
  
  \[ R_i \rightarrow P_i(x) \]

  if \( x = \text{true} \) then \( P_i(x) \rightarrow R_{n+1} \)

- **Running time**: \( O(1) \) (exact 2 steps).
- **Number of processors**: \( n \).
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Programm: Or

\[
\text{for all } P_i \text{ where } 1 \leq i \leq n \text{ do in parallel} \\
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- Output computed in $R_{n+1}$.
- Program: Or
  
  for all $P_i$ where $1 \leq i \leq n$ do in parallel
  
  $R_i \rightarrow P_i(x)$
  
  if $x = true$ then $P_i(x) \rightarrow R_{n+1}$

- Running time: $O(1)$ (exact 2 steps).
- Number of processors: $n$.
- Memory: $n + 1$.
- Possible models: ERCW (Arbitrary, Common oder Priority).
Computing an “Or”

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Programm: Or

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Computing an “Or” (EREW)

- Problem:
  no writing of two processors
to the same register
at the same time.

- Idea: combine pairwise the results

- With this idea, computing the sum is also possible.

- Thus computing the “Or” is just a special case of computing a sum.
Computing an “Or” (EREW)

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Computing the Sum (Idea)
Computing the Sum (Idea)
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Computing the Sum (Idea)
Computing the Sum (Idea)
Computing the Sum (Idea)

\[ P_1 \quad P_2 \quad P_3 \quad P_4 \]

\[ 12 \quad 6 \quad 34 \quad 5 \quad 7 \quad 23 \quad 4 \quad 11 \]
Computing the Sum (Idea)

\[ \begin{align*}
0 & \quad 0 & \quad 0 & \quad 0 \\
P_1 & \quad P_2 & \quad P_3 & \quad P_4 \\
12 & \quad 6 & \quad 34 & \quad 5 \quad 7 \quad 23 \quad 4 \quad 11
\end{align*} \]
Computing the Sum (Idea)
Computing the Sum (Idea)

18 -> 12
39 -> 6
30 -> 34
15 -> 5

P1 -> P2
P1 -> P3
P1 -> P4
P2 -> P3
P2 -> P4
P3 -> P4

12 + 6 + 34 + 5 + 7 + 23 + 4 + 11
Computing the Sum (Idea)

\[ \begin{align*}
18 & \quad 39 & \quad 30 & \quad 15 \\
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\end{align*} \]
Computing the Sum (Idea)
Computing the Sum (Idea)
Computing the Sum (Idea)

P1: 57
P2: 45
P3: 30
P4: 15

12 6 34 5 7 23 4 11
Computing the Sum (Idea)

57  45  30  15

\( P_1 \)  \( P_2 \)  \( P_3 \)  \( P_4 \)

12  6  34  5  7  23  4  11
Computing the Sum (Idea)

\[ 57 \rightarrow 45 \rightarrow 30 \rightarrow 15 \]

\[ P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \]

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Computing the Sum (Idea)

\[
\begin{align*}
57 & \quad 45 & \quad 30 & \quad 15 \\
\downarrow & & \downarrow & \\
P_1 & \quad P_2 & \quad P_3 & \quad P_4 \\
\end{align*}
\]

\[
\begin{align*}
12 & \quad 6 & \quad 34 & \quad 5 & \quad 7 & \quad 23 & \quad 4 & \quad 11 \\
\end{align*}
\]
Computing the Sum (Idea)

103

45

30

15

P₁

P₂

P₃

P₄

12

6

34

5

7

23

4

11
Computing the sum (EREW)

- Task: compute $x = \sum_{i=1}^{n} x_i$ with $n = 2^k$.
- Input: $x_i$ is in register $R_i$ ($1 \leq i \leq n$).
- Output: should be in $R_1$ (input may be overwritten).
- Modell: EREW.
- Programm: Summe
  
  for all $P_i$ where $1 \leq i \leq n/2$ do in parallel
  
  $R_{2,i-1} \rightarrow P_i(x)$
  
  for $j = 1$ to $k$ do
  
  if $(i - 1) \equiv 0 \pmod{2^{j-1}}$ then
  
  $R_{2,i-1+2^{j-1}} \rightarrow P_i(y)$
  
  $x := x + y$
  
  $P_i(x) \rightarrow R_{2,i-1}$

- Running time: $O(k) = O(\log n)$ (precise $3 \cdot k + 1$ steps).
- Number of processors: $n/2$.
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  ```
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  \( R_{2 \cdot i - 1 + 2^{j-1}} \rightarrow P_i(y) \)
  
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  if \( (i - 1) \equiv 0 \pmod{2^{j-1}} \) then
  
  \[ R_{2,i-1+2j-1} \rightarrow P_i(y) \]
  
  \( x := x + y \)
  
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Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

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Motivation and History  
PRAM Introduction  
Efficiency  
Selection  
Merging  
Walter Unger 30.1.2017 11:52  
WS2016/17  
Z  
Computing the sum (EREW)  

1:23 Sum  9/9
Addition of Matrices

- Let $A$, $B$ two $(n \times n)$-Matrices.
- Sum $A + B$ is computable with $n^2$ processors in Zeit $O(1)$ on a EREW PRAM.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$.
- Program: MatSumme
  for all $P_i$ where $1 \leq i \leq n^2$ do in parallel
    $R_i \rightarrow P_i(x)$
    $R_{i+n^2} \rightarrow P_i(y)$
    $x := x + y$
    $P_i(x) \rightarrow R_{i+2\cdot n^2}$
- Running time: $O(1)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$.

Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$. 
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Programm: MatSumme

for all $P_i$ where $1 \leq i \leq n^2$ do in parallel

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Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

Let $n$ be an integer, then $n = 2^k$ for $k \in \mathbb{N}$.
Addition of Matrices

- Let \( A, B \) two \((n \times n)\)-Matrices.
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- Running time: \( O(1) \).
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Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

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Programm: MatSumme
for all $P_i$ where $1 \leq i \leq n^2$ do in parallel

- $R_i \rightarrow P_i(x)$
- $R_{i+n^2} \rightarrow P_i(y)$
- $x := x + y$
- $P_i(x) \rightarrow R_{i+2\cdot n^2}$

- Running time: $O(1)$.
- Number of processors: $O(n^2)$.
- Size of memory: $O(n^2)$. 

Assume w.l.o.g $n = 2^k$ for $k \in \mathbb{N}$. 

Let $A, B$ two $(n \times n)$-Matrices. 

Sum $A + B$ is computable with $n^2$ processors in Zeit $O(1)$ on a EREW PRAM. 

$R_1$ till $R_{n^2}$ contain $A$ (one row after the other).

$R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).

Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$.

Programm: MatSumme
for all $P_i$ where $1 \leq i \leq n^2$ do in parallel

- $R_i \rightarrow P_i(x)$
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Running time: $O(1)$.

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Size of memory: $O(n^2)$.
Multiplication of Matrices

Assume w.l.o.g. $n = 2^k$ for $k \in \mathbb{N}$.

- Let $A, B$ be two $(n \times n)$-Matrices.
- $R_1$ till $R_{n^2}$ contain $A$ (one row after the other).
- $R_{1+n^2}$ bis $R_{2\cdot n^2}$ contains $B$ (one row after the other).
- Result in $R_{1+2\cdot n^2}$ bis $R_{3\cdot n^2}$
- Register $A_{i,j} = R_{(i-1)\cdot n+j}$ ($1 \leq i, j \leq n$).
- Register $B_{i,j} = R_{(i-1)\cdot n+j+n^2}$ ($1 \leq i, j \leq n$).
- Register $C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2}$ ($1 \leq i, j \leq n$).
- Processor $P_{i,j} = P_{(i-1)\cdot n+j}$ ($1 \leq i, j \leq n$).
- Use the above notation to simplify the algorithm.
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Let $A, B$ be two $(n \times n)$-Matrices.

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- Register $C_{i,j} = R_{(i-1).n+j+2.n^2}$ ($1 \leq i, j \leq n$).
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- Register $C_{i,j} = R_{(i-1) \cdot n + j + 2 \cdot n^2}$ ($1 \leq i, j \leq n$).
- processor $P_{i,j} = P_{(i-1) \cdot n + j}$ ($1 \leq i, j \leq n$).
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Processor $P_{i,j} = P_{(i-1)\cdot n+j} \ (1 \leq i, j \leq n)$.

Use the above notation to simplify the algorithm.

Each processor has to do some hidden local computation to implement the above expressions.
Multiplication of Matrices

- Let \( A, B \) be two \((n \times n)\)-Matrices.
- \( R_1 \) till \( R_{n^2} \) contain \( A \) (one row after the other).
- \( R_{1+n^2} \) bis \( R_{2\cdot n^2} \) contains \( B \) (one row after the other).
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- Register \( C_{i,j} = R_{(i-1)\cdot n+j+2\cdot n^2} \) \((1 \leq i, j \leq n)\).
- processor \( P_{i,j} = P_{(i-1)\cdot n+j} \) \((1 \leq i, j \leq n)\).

Use the above notation to simplify the algorithm.

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Assume w.l.o.g \( n = 2^k \) for \( k \in \mathbb{N} \).
Let $A, B$ be two $(n \times n)$-Matrices.

$R_1$ till $R_{n^2}$ contain $A$ (one row after the other).

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processor $P_{i,j} = P((i-1)\cdot n+j)\ (1 \leq i, j \leq n)$.

Use the above notation to simplify the algorithm.

Each processor has to do some hidden local computation to implement the above expressions.
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Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices

- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

- Programm: MatrProd 1
  
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $h = 0$
  
  for $l = 1$ to $n$
  
  $A_{i,l} \rightarrow P_{i,j}(a)$
  
  $B_{i,j} \rightarrow P_{i,j}(b)$
  
  $h = h + a \cdot b$
  
  $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$.
Let $A$, $B$ be two $(n \times n)$-Matrices.

Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

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$B_{l,j} \rightarrow P_{i,j}(b)$

$h = h + a \cdot b$

$P_{i,j}(h) \rightarrow C_{i,j}$

Running time: $O(n)$.

Number of processors: $O(n^2)$.

Size of memory: $O(n^2)$. 

\[
A_{i,j} = R(i-1) \cdot n + j \\
B_{i,j} = R(i-1) \cdot n + j + n^2 \\
C_{i,j} = R(i-1) \cdot n + j + 2 \cdot n^2 \\
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\end{align*}
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Multiplikation of Matrices

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- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a CREW PRAM.

Programm: MatrProd 1

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

1. $h = 0$
2. for $l = 1$ to $n$ do
   1. $A_{i,l} \rightarrow P_{i,j}(a)$
   2. $B_{l,j} \rightarrow P_{i,j}(b)$
   3. $h = h + a \cdot b$
   4. $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.
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Multiplikation of Matrices

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Multiplikation of Matrices

- Let $A$, $B$ be two $(n \times n)$-Matrices

- Product $A \cdot B$ is computable with $n^2$ processors in time $O(n)$ on a EREW PRAM.

- Programm: MatrProd 2
  
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $h = 0$
  
  for $l = 1$ to $n$ do
  
  $A_{i,l} \rightarrow P_{i,j}(a)$
  
  $B_{l,j} \rightarrow P_{i,j}(b)$
  
  $h = h + a \cdot b$
  
  $P_{i,j}(h) \rightarrow C_{i,j}$

- Running time: $O(n)$.

- Number of processors: $O(n^2)$.

- Size of memory: $O(n^2)$. 

\[
A_{i,j} = R_{(i-1)\cdot n+j} \\
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Multiplikation of Matrices

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Compute the Prefixsum

Problem:

- **Task**: Compute \( s_i = \sum_{j=1}^{i} x_j \) for \( 1 \leq i \leq n \).
- **Input**: \( x_j \) is in register \( R_j \) (\( 1 \leq j \leq n \)).
- **Output**: \( s_i \) should be in register \( R_i \) for \( 1 \leq i \leq n \).
Problem:

- Task: Compute \( s_i = \sum_{j=1}^{i} x_j \) for \( 1 \leq i \leq n \).
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Compute the Prefixsum

Problem:

- Task: Compute $s_i = \sum_{j=1}^{i} x_j$ for $1 \leq i \leq n$.
- Input: $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- Output: $s_i$ should be in register $R_i$ for $1 \leq i \leq n$. 
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)

\[
P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7 \quad P_8
\]

\[
x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8
\]
Computing Prefixsum (Idea)

\[ P_1 \rightarrow x_1 \]
\[ P_2 \rightarrow x_2 \]
\[ P_3 \rightarrow x_3 \]
\[ P_4 \rightarrow x_4 \]
\[ P_5 \rightarrow x_5 \]
\[ P_6 \rightarrow x_6 \]
\[ P_7 \rightarrow x_7 \]
\[ P_8 \rightarrow x_8 \]
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)

\[
\begin{align*}
X_{1..1} & \rightarrow X_{1..2} & \rightarrow X_{2..3} & \rightarrow X_{3..4} & \rightarrow X_{4..5} & \rightarrow X_{5..6} & \rightarrow X_{6..7} & \rightarrow X_{7..8} \\
X_{1..1} & \rightarrow X_{1..2} & \rightarrow X_{2..3} & \rightarrow X_{3..4} & \rightarrow X_{4..5} & \rightarrow X_{5..6} & \rightarrow X_{6..7} & \rightarrow X_{7..8}
\end{align*}
\]
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)

\[
x_1..1 \quad x_1..2 \quad x_1..3 \quad x_1..4 \quad x_2..5 \quad x_3..6 \quad x_4..7 \quad x_5..8
\]
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)
Computing Prefixsum (Idea)

Done!
Computing the Prefixsum

- **Task:** Compute $s_i = \sum_{j=1}^{i} x_j$ for $1 \leq i \leq n$.

- **Input:** $x_j$ is in register $R_j$ ($1 \leq j \leq n$).

- **Output:** $s_i$ should be in register $R_i$ for $1 \leq i \leq n$.

- **Model:** EREW

- **Programm:** Summe

  for all $P_i$ where $1 \leq i \leq n$ do in parallel
  
  $R_i \rightarrow P_i(x)$

  for $j = 1$ to $k$ do
    
    if $i > 2^{j-1}$ then
      
      $R_{i-2^{j-1}} \rightarrow P_i(y)$

      $x := x + y$

      $P_i(x) \rightarrow R_i$

- **Running time:** $O(k) = O(\log n)$ (precisely $3 \cdot k + 1$ steps).

- **Number of processors:** $n$.

- **Size of memory:** $n$. 
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- **Task:** Compute \( s_i = \sum_{j=1}^{i} x_j \) for \( 1 \leq i \leq n \).
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```

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- Running time: \( O(k) = O(\log n) \) (precisely \( 3 \cdot k + 1 \) steps).
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for all $P_i$ where $1 \leq i \leq n$ do in parallel

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if $i > 2^{j-1}$ then

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- **Running time**: $O(k) = O(\log n)$ (precisely $3 \cdot k + 1$ steps).
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  for all $P_i$ where $1 \leq i \leq n$ do in parallel
  
  $R_i \rightarrow P_i(x)$
  
  for $j = 1$ to $k$
  
  if $i > 2^j - 1$ then
  
  $R_{i-2^j-1} \rightarrow P_i(y)$
  
  $x := x + y$
  
  $P_i(x) \rightarrow R_i$
  
  Running time: $O(k) = O(\log n)$ (precisely $3 \cdot k + 1$ steps).
  
  - **Number of processors:** $n$.
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- Running time: $O(k) = O(\log n)$ (precisely $3 \cdot k + 1$ steps).
- Number of processors: $n$.
- Size of memory: $n$. 
Compute the Maximum

- **Task**: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- **Input**: $x_j$ is in register $R_j$ ($1 \leq j \leq n$).
- **Output**: $m$ should be in register $R_{n+1}$.
- Possible with $n$ processors in time $O(\log n)$ using a EREW PRAM.
- **Question**: could it be done faster? (i.e. on a ERCW PRAM).
- A maximum is larger or equal than all other values.
- **Idea**: compare all pairs of numbers.
- The maximum will always win.
Compute the Maximum

- **Task:** Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
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## Compute the Maximum (Idea)

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Compute the Maximum (Idea)

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| 22| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 33| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 41| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 26| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 59| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 57| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 52| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 61| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 27| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 49| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 67| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 23| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 56| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 14| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 12| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 34| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

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1:32 Maximum 3/5
### Compute the Maximum (Idea)

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| 41| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 26| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 59| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 57| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 52| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 61| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 27| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 49| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 67| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 23| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 56| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 14| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 34| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 34|   | 12| 14| 56| 23| 67| 49| 27| 61| 52| 57| 59| 26| 41| 33| 22|
### Compute the Maximum (Idea)

|   | 22 | 33 | 41 | 26 | 59 | 57 | 52 | 61 | 27 | 49 | 67 | 23 | 56 | 14 | 12 | 34 | 34 | 12 | 14 | 56 | 23 | 67 | 49 | 27 | 61 | 52 | 57 | 59 | 26 | 41 | 33 | 22 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
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| 0 | 1  | 1  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 1  | 0  | 1  | 1  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 0 | 1  | 1  | 0  | 1  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 0 | 1  | 1  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 0 | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 1  | 1  | 0  | 1  | 1  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 0 | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 0  | 0  | 1  | 1  | 0  | 1  | 0  | 0  | 1  | 1  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
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| 0 | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
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| 26| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 59| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 67| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 52| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 27| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
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| 12| 0 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 34| 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
Compute the Maximum (Idea)

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Walter Unger 30.1.2017 11:52  WS2016/17
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Computing the Maximum

- **Task:** Compute $m = \max_{j=1}^i x_j$ with $n = 2^k$.
- **Input:** $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
- **Output:** $m$ in register $R_{n+1}$.
- **Model:** CRCW.

**Program:**

```plaintext
for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $P_{i,1}(1) \rightarrow W_i$

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    $R_i \rightarrow P_{i,j}(a)$
    $R_j \rightarrow P_{i,j}(b)$
    if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
    $W_i \rightarrow P_{i,1}(h)$
    if $h = 1$ then
        $R_i \rightarrow P_{i,1}(h)$
        $P_{i,1}(h) \rightarrow R_{n+1}$
```
Computing the Maximum

- Task: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- Input: $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
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Programm: Maximum

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel

$P_{i,1}(1) \rightarrow W_i$

for all $P_{i,j}$ where $1 \leq i,j \leq n$ do in parallel

$R_i \rightarrow P_{i,j}(a)$
$R_j \rightarrow P_{i,j}(b)$

if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel

$W_i \rightarrow P_{i,1}(h)$

if $h = 1$ then

$R_i \rightarrow P_{i,1}(h)$
$P_{i,1}(h) \rightarrow R_{n+1}$
Computing the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- Input: \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
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- Model: CRCW.

Programm: Maximum

\[
\text{for all } P_{i,1} \text{ where } 1 \leq i \leq n \text{ do in parallel} \\
\quad P_{i,1}(1) \rightarrow W_i \\
\text{for all } P_{i,j} \text{ where } 1 \leq i, j \leq n \text{ do in parallel} \\
\quad R_i \rightarrow P_{i,j}(a) \\
\quad R_j \rightarrow P_{i,j}(b) \\
\quad \text{if } a < b \text{ then } P_{i,j}(0) \rightarrow W_i \\
\text{for all } P_{i,1} \text{ where } 1 \leq i \leq n \text{ do in parallel} \\
\quad W_i \rightarrow P_{i,1}(h) \\
\quad \text{if } h = 1 \text{ then} \\
\quad \quad R_i \rightarrow P_{i,1}(h) \\
\quad \quad P_{i,1}(h) \rightarrow R_{n+1}
\]
Computing the Maximum

- **Task**: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- **Input**: \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
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- **Model**: CRCW.

**Programm: Maximum**

```plaintext
for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  \( P_{i,1}(1) \to W_i \)

for all \( P_{i,j} \) where \( 1 \leq i,j \leq n \) do in parallel
  \( R_i \to P_{i,j}(a) \)
  \( R_j \to P_{i,j}(b) \)
  if \( a < b \) then \( P_{i,j}(0) \to W_i \)

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
  \( W_i \to P_{i,1}(h) \)
  if \( h = 1 \) then
    \( R_i \to P_{i,1}(h) \)
    \( P_{i,1}(h) \to R_{n+1} \)
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Computing the Maximum

- Task: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- Input: $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
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- Model: CRCW.
- Programm: Maximum
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $P_{i,1}(1) \rightarrow W_i$

  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  $R_i \rightarrow P_{i,j}(a)$
  
  $R_j \rightarrow P_{i,j}(b)$

  if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $W_i \rightarrow P_{i,1}(h)$

  if $h = 1$ then
  
  $R_i \rightarrow P_{i,1}(h)$
  
  $P_{i,1}(h) \rightarrow R_{n+1}$
Computing the Maximum

- Task: Compute \( m = \max_{j=1}^{i} x_j \) with \( n = 2^k \).
- Input: \( x_j \) is in register \( R_j \) (\( 1 \leq x_j \leq n \)).
- Output: \( m \) in register \( R_{n+1} \).
- Model: CRCW.

Programm: Maximum

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel

\[ P_{i,1}(1) \rightarrow W_i \]

for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel

\[ R_i \rightarrow P_{i,j}(a) \]
\[ R_j \rightarrow P_{i,j}(b) \]
if \( a < b \) then \( P_{i,j}(0) \rightarrow W_i \)

for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel

\[ W_i \rightarrow P_{i,1}(h) \]
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Programm: Maximum

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P_{i,1}(1) \rightarrow W_i
\]
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\text{for all } P_{i,j} \text{ where } 1 \leq i, j \leq n \text{ do in parallel}
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R_i \rightarrow P_{i,1}(h)
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P_{i,1}(h) \rightarrow R_{n+1}
\]
Computing the Maximum

- Task: Compute $m = \max_{j=1}^{i} x_j$ with $n = 2^k$.
- Input: $x_j$ is in register $R_j$ ($1 \leq x_j \leq n$).
- Output: $m$ in register $R_{n+1}$.
- Model: CRCW.

Programm: Maximum

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
$P_{i,1}(1) \rightarrow W_i$

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
$R_i \rightarrow P_{i,j}(a)$
$R_j \rightarrow P_{i,j}(b)$
if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
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Programm: Maximum

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for all \( P_{i,1} \) where \( 1 \leq i \leq n \) do in parallel
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for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel
\( R_i \rightarrow P_{i,j}(a) \)
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```
Computing the Maximum

- Program: Maximum
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
  $P_{i,1}(1) \rightarrow W_i$

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  $R_i \rightarrow P_{i,j}(a)$
  
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  if $a < b$ then $P_{i,j}(0) \rightarrow W_i$

  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
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  $P_{i,1}(h) \rightarrow R_{n+1}$

- Running time: $O(1)$.

- Number of processors: $O(n^2)$.

- Memory: $O(n)$. 
Computing the Maximum

- **Programm: Maximum**
  
  for all $P_{i,1}$ where $1 \leq i \leq n$ do in parallel
  
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  $R_i \rightarrow P_{i,j}(a)$
  
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- Programm: Maximum
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    $P_{i,1}(1) \rightarrow W_i$
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- Running time: $O(1)$.
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- Memory: $O(n)$. 
Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till \( n \)
- Input: Father of node \( i \) is written in register \( R_i \).
- For the roots \( i \) we have: in register \( R_i \) is written \( i \).

Program: Ranking

\[
\text{for all } P_i \text{ where } 1 \leq i \leq n \text{ do in parallel}
\]

\[
\text{for } j = 1 \text{ to } \lceil \log n \rceil \text{ do}
\]

\[
R_i \rightarrow P_i(h)
\]

\[
R_h \rightarrow P_i(h)
\]

\[
P_i(h) \rightarrow R_i
\]

Running time: \( O(\log n) \).

Number of processors: \( O(n) \).

Memory: \( O(n) \).

Model: CREW.
Identify the Roots of a Forest

- Nodes are identified by numbers from 1 till \( n \)
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- Program: Ranking

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\text{for all } P_i \text{ where } 1 \leq i \leq n \text{ do in parallel}
\]

\[
\text{for } j = 1 \text{ to } \lceil \log n \rceil \text{ do}
\]

\[
R_i \to P_i(h)
\]

\[
R_h \to P_i(h)
\]

\[
P_i(h) \to R_i
\]

Running time: \( O(\log n) \).

Number of processors: \( O(n) \).

Memory: \( O(n) \).

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- Nodes are identified by numbers from 1 till $n$
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for all $P_i$ where $1 \leq i \leq n$ do in parallel
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    $R_i \rightarrow P_i(h)$
    $R_h \rightarrow P_i(h)$
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for all $P_i$ where $1 \leq i \leq n$ do in parallel

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$R_i \rightarrow P_i(h)$

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**Programm: Ranking**

```plaintext
for all $P_i$ where $1 \leq i \leq n$ do in parallel
  for $j = 1$ to ⌈log $n$⌉ do
    $R_i \rightarrow P_i(h)$
    $R_h \rightarrow P_i(h)$
    $P_i(h) \rightarrow R_i$
```

Running time: $O(\log n)$.

Number of processors: $O(n)$.

Memory: $O(n)$.

Model: CREW.
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- Program: Ranking
  
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  Running time: $O(\log n)$.

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\]

Running time: \( O(\log n) \).

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Question: May we save some processors?
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Question: May we save some processors?
## Motivation and History

### PRAM Introduction

### Efficiency

Selection

Merging

### 1:37 Situation 7/11

Walter Unger 30.1.2017 11:52 WS2016/17

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May we do this saving in any situation?
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**Question:** May we save some processors?  
May we do this saving in any situation?  
How do we estimate the efficiency of a parallel algorithm?
Cost Measurement

Let $A$ be any parallel algorithm, we denote:

- $T_A(n)$ the running time of $A$.
- $P_A(n)$ the number of processors used by $A$.
- $R_A(n)$ the number of registers used by $A$.
- $W_A(n)$ the number of accesses to registers done by $A$.
- $ST(n)$ the running time of the best [known] sequential algorithm.
- $Eff_A(n) := \frac{ST(n)}{P_A(n) \cdot T_A(n)}$ the efficiency of $A$.
- $AEff_A(n) := \frac{W_A(n)}{P_A(n) \cdot T_A(n)}$ the usage efficiency of $A$. 
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<td>1</td>
<td>ERCW</td>
</tr>
<tr>
<td>Or</td>
<td>O(n/log n)</td>
<td>O(log n)</td>
<td>O(n)</td>
<td>1</td>
<td>EREW</td>
</tr>
<tr>
<td>Maximum Sum</td>
<td>O(n^2/t)</td>
<td>O(t)</td>
<td>O(n)</td>
<td>1</td>
<td>CRCW</td>
</tr>
<tr>
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<td>O(log n)</td>
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<tr>
<td>Prefixsum</td>
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<td>O(log n)</td>
<td>O(n)</td>
<td>1</td>
<td>EREW</td>
</tr>
<tr>
<td>Mat.sum</td>
<td>O(n^2/log n)</td>
<td>O(log n)</td>
<td>O(n^2)</td>
<td>1</td>
<td>EREW</td>
</tr>
<tr>
<td>Mat.prod.</td>
<td>O(n^2/log n)</td>
<td>O(n log n)</td>
<td>O(n^2.276)</td>
<td>O(n^-0.734)</td>
<td>EREW</td>
</tr>
<tr>
<td>Mat.prod.</td>
<td>O(n^3/log n)</td>
<td>O(log n)</td>
<td>O(n^2.276)</td>
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<td>Mat.prod.</td>
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<td>O(n^2.276)</td>
<td>O(n^-0.734)</td>
<td>EREW</td>
</tr>
</tbody>
</table>
k-th Element

- **Task**: Compute the k-th (k-smallest) element in an unsorted sequence $S = \{s_1, \ldots, s_n\}$.
- **Lower bound**: $n - 1$ comparisons
- **Start with a nice sequential algorithm**

**Programm**: Select(k,S)

1. If $|S| \leq 50$ then return k-th number in S
2. Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
3. Sort each $H_i$
4. Let $M$ be the sequence of the middle elements in $H_i$
5. $m := Select(\lceil |M|/2 \rceil, M)$
6. $S_1 := \{s \in S \mid s < m\}$
7. $S_2 := \{s \in S \mid s = m\}$
8. $S_3 := \{s \in S \mid s > m\}$
9. If $|S_1| \geq k$ then return Select(k, $S_1$)
10. If $|S_1| + |S_2| \geq k$ then return $m$
11. Return $Select(k - |S_1| - |S_2|, S_3)$
k-th Element

- Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

- Lower bound: $n – 1$ comparisons

- Start with a nice sequential algorithm

Programm: Select(k,S)

\begin{verbatim}
if |S| ≤ 50 then return k-th number in S
Split S in ⌈n/5⌉ sub-sequences $H_i$ of size ≤ 5
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
\[ m := \text{Select}(\lceil |M|/2 \rceil, M) \]
\[ S_1 := \{s \in S | s < m\} \]
\[ S_2 := \{s \in S | s = m\} \]
\[ S_3 := \{s \in S | s > m\} \]
if $|S_1| \geq k$ then return Select($k, S_1$)
if $|S_1| + |S_2| \geq k$ then return $m$
return Select($k – |S_1| – |S_2|, S_3$)
\end{verbatim}
Task: Compute the $k$-th ($k$-smallest) element in an unsorted sequence
$S = \{s_1, \ldots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: $\text{Select}(k,S)$

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if $|S| \leq 50$ then return $k$-th number in $S$
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\end{verbatim}
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: Select($k, S$)

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k-th Element

- Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.
- Lower bound: $n - 1$ comparisons
- Start with a nice sequential algorithm

**Programm: Select(k,S)**

```plaintext
if |S| \leq 50 then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size \leq 5
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
$m := Select(\lceil |M|/2 \rceil, M)$
$S_1 := \{s \in S \mid s < m\}$
$S_2 := \{s \in S \mid s = m\}$
$S_3 := \{s \in S \mid s > m\}$
if $|S_1| \geq k$ then return $Select(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $Select(k - |S_1| - |S_2|, S_3)$
```
**Task:** Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

**Lower bound:** $n - 1$ comparisons

**Start with a nice sequential algorithm**

**Program:** \texttt{Select(k,S)}

\begin{verbatim}
if $|S| \leq 50$ then return $k$-th number in $S$

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Sort each $H_i$

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- return $Select(k - |S_1| - |S_2|, S_3)$
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

Lower bound: $n - 1$ comparisons

Start with a nice sequential algorithm

Programm: \texttt{Select}(k,S)
\begin{enumerate}
  \item if $|S| \leq 50$ then return $k$-th number in $S$
  \item Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
  \item Sort each $H_i$
  \item Let $M$ be the sequence of the middle elements in $H_i$
  \item $m := \texttt{Select}(\lceil |M|/2 \rceil, M)$
  \item $S_1 := \{s \in S \mid s < m\}$
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  \item $S_3 := \{s \in S \mid s > m\}$
  \item if $|S_1| \geq k$ then return $\texttt{Select}(k, S_1)$
  \item if $|S_1| + |S_2| \geq k$ then return $m$
  \item return $\texttt{Select}(k - |S_1| - |S_2|, S_3)$
\end{enumerate}
Task: Compute the \(k\)-th (\(k\)-smallest) element in a unsorted sequence 
\(S = \{s_1, \ldots, s_n\}\).

Lower bound: \(n - 1\) comparisons

Start with a nice sequential algorithm

Programm: \(\text{Select}(k,S)\)

\textbf{if} \(|S| \leq 50\) \textbf{then return} \(k\)-th number in \(S\)

Split \(S\) in \(\lceil n/5 \rceil\) sub-sequences \(H_i\) of size \(\leq 5\)

Sort each \(H_i\)

Let \(M\) be the sequence of the middle elements in \(H_i\)

\(m := \text{Select}(\lceil |M|/2 \rceil, M)\)

\(S_1 := \{s \in S \mid s < m\}\)

\(S_2 := \{s \in S \mid s = m\}\)

\(S_3 := \{s \in S \mid s > m\}\)

\textbf{if} \(|S_1| \geq k\) \textbf{then return} \(\text{Select}(k, S_1)\)

\textbf{if} \(|S_1| + |S_2| \geq k\) \textbf{then return} \(m\)

\textbf{return} \(\text{Select}(k - |S_1| - |S_2|, S_3)\)
Task: Compute the $k$-th ($k$-smallest) element in a unsorted sequence $S = \{s_1, \ldots, s_n\}$.

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return $\text{Select}(k - |S_1| - |S_2|, S_3)$
Example for the k-th Element (Slow Motion)

**Input/Data:**

| 45 | 81 | 75 | 38 | 33 | 58 | 84 | 10 | 47 | 55 | 89 | 61 | 73 | 48 | 74 | 64 | 18 | 60 | 9 | 35 | 23 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 27 | 74 | 34 | 8  | 12 | 56 | 34 | 87 | 8  | 76 | 90 | 81 | 37 | 16 | 89 | 15 | 94 | 15 | 78 | 68 | 96 |
| 26 | 40 | 29 | 72 | 72 | 85 | 30 | 39 | 64 | 26 | 62 | 62 | 84 | 49 | 6  | 97 | 64 | 21 | 46 | 89 | 45 |
| 7  | 52 | 16 | 11 | 13 | 51 | 67 | 95 | 26 | 3  | 63 | 22 | 12 | 2  | 69 | 42 | 40 | 49 | 18 | 77 | 31 |
| 36 | 57 | 13 | 55 | 61 | 79 | 25 | 19 | 82 | 86 | 46 | 31 | 11 | 33 | 61 | 82 | 96 | 54 | 1  | 53 | 65 |

**M:**

```

```

**sorted M:**

```

```
Example for the k-th Element (Slow Motion)

Input/Data:

```
45  81  75  72  72  85  84  95  82  86  90  81  84  49  89  97  96  60  78  89  96
36  74  34  55  61  79  67  87  64  76  89  62  73  48  74  82  94  54  46  77  65
27  57  29  38  33  58  34  39  47  55  63  61  37  33  69  64  64  49  18  68  45
26  52  16  11  13  56  30  19  26  26  62  31  12  16  61  42  40  21  9  53  31
 7  40  13  8  12  51  25  10  8  3  46  22  11  2  6  15  18  15  1  35  23
```

M:

```
[ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]
```

sorted M:

```
[ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]  [ ]
```
Example for the k-th Element (Slow Motion)

Input/Data:

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<th>72</th>
<th>81</th>
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M:

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<td>25</td>
<td>10</td>
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<td>3</td>
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</table>

sorted M:
Example for the k-th Element (Slow Motion)

**Input/Data:**

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<tr>
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<th>45</th>
<th>81</th>
<th>75</th>
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<td>96</td>
<td>54</td>
<td>1</td>
<td>53</td>
<td>65</td>
<td></td>
</tr>
</tbody>
</table>

**M:**

|   | 27 | 57 | 29 | 38 | 33 | 58 | 34 | 39 | 47 | 55 | 63 | 61 | 37 | 33 | 69 | 64 | 64 | 49 | 18 | 68 | 45 |

**sorted M:**

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

**M:**

|   | 27 | 57 | 29 | 38 | 33 | 58 | 34 | 39 | 47 | 55 | 63 | 61 | 37 | 33 | 69 | 64 | 64 | 49 | 18 | 68 | 45 |

**sorted M:**

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
Example for the k-th Element (Slow Motion)

Input/Data:

| 45 | 81 | 75 | 38 | 33 | 58 | 84 | 10 | 47 | 55 | 89 | 61 | 73 | 48 | 74 | 64 | 18 | 60 | 9 | 35 | 23 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
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| 26 | 40 | 29 | 72 | 72 | 85 | 30 | 39 | 64 | 26 | 62 | 62 | 84 | 49 | 6 | 97 | 64 | 21 | 46 | 89 | 45 |
| 7 | 52 | 16 | 11 | 13 | 51 | 67 | 95 | 26 | 3 | 63 | 22 | 12 | 2 | 69 | 42 | 40 | 49 | 18 | 77 | 31 |
| 36 | 57 | 13 | 55 | 61 | 79 | 25 | 19 | 82 | 86 | 46 | 31 | 11 | 33 | 61 | 82 | 96 | 54 | 1 | 53 | 65 |

M:

| 27 | 57 | 29 | 38 | 33 | 58 | 34 | 39 | 47 | 55 | 63 | 61 | 37 | 33 | 69 | 64 | 64 | 49 | 18 | 68 | 45 |

sorted M:

| 18 | 27 | 29 | 33 | 33 | 34 | 37 | 38 | 39 | 45 | 47 | 49 | 55 | 57 | 58 | 61 | 63 | 64 | 64 | 68 | 69 |
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Input/Data:

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M:

|  27  |  57  |  29  |  38  |  33  |  58  |  34  |  39  |  47  |  55  |  63  |  61  |  37  |  33  |  69  |  64  |  49  |  18  |  68  |  45  |

sorted M:

|  18  |  27  |  29  |  33  |  33  |  34  |  37  |  38  |  39  |  45  |  47  |  49  |  55  |  57  |  58  |  61  |  63  |  64  |  64  |  68  |  69  |
Example for the k-th Element

Input/Data:

\[
\begin{array}{cccccccccccccccc}
86 & 60 & 7 & 89 & 38 & 72 & 81 & 35 & 73 & 43 & 87 & 50 & 85 & 60 & 86 & 33 & 1 & 17 & 29 & 78 & 13 \\
7 & 58 & 52 & 70 & 48 & 82 & 57 & 64 & 63 & 32 & 21 & 55 & 37 & 71 & 95 & 23 & 5 & 51 & 85 & 16 & 8 \\
74 & 95 & 45 & 45 & 46 & 24 & 21 & 63 & 22 & 73 & 58 & 28 & 96 & 37 & 70 & 65 & 64 & 93 & 34 & 58 & 46 \\
50 & 12 & 22 & 11 & 19 & 97 & 52 & 84 & 47 & 27 & 93 & 25 & 45 & 28 & 60 & 9 & 81 & 71 & 1 & 75 & 87 \\
\end{array}
\]

M:

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

sorted M:

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]
**Example for the k-th Element**

**Input/Data:**

<table>
<thead>
<tr>
<th>96</th>
<th>47</th>
<th>13</th>
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</tbody>
</table>

**M:**

| 74 | 58 | 22 | 62 | 38 | 72 | 52 | 63 | 47 | 43 | 58 | 47 | 74 | 41 | 70 | 23 | 29 | 51 | 29 | 58 | 29 |

**sorted M:**

|         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
Example for the k-th Element

Input/Data:

<table>
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<tr>
<th>96</th>
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</tbody>
</table>

M:

| 74 | 58 | 22 | 62 | 38 | 72 | 52 | 63 | 47 | 43 | 58 | 47 | 74 | 41 | 70 | 23 | 29 | 51 | 29 | 58 | 29 |

sorted M:

| 22 | 23 | 29 | 29 | 29 | 38 | 41 | 43 | 47 | 47 | 51 | 52 | 58 | 58 | 58 | 62 | 63 | 70 | 72 | 74 | 74 |
Example for the k-th Element

Input/Data:

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</table>

M:

| 74 | 58 | 22 | 62 | 38 | 72 | 52 | 63 | 47 | 43 | 58 | 47 | 74 | 41 | 70 | 23 | 29 | 51 | 29 | 58 | 29 |

sorted M:

| 22 | 23 | 29 | 29 | 29 | 38 | 41 | 43 | 47 | 47 | 51 | 52 | 58 | 58 | 58 | 62 | 63 | 70 | 72 | 74 | 74 |
Example for the k-th Element (Worst Case)

Input/Data:

|  57 |  73 |  83 |  65 |  50 |  79 |  67 |  91 |  71 |  90 |  55 |  76 |  93 |  77 |  51 |  92 |  66 |  69 |  80 |  55 |  71 |
|  82 |  61 |  55 |  87 |  89 |  66 |  94 |  77 |  65 |  53 |  63 |  60 |  91 |  65 |  54 |  90 |  87 |  60 |  90 |  56 |  51 |
|  2  |  22 |  16 |  32 |  20 |  8  |  28 |  6  |  9  |  3  |  24 |  86 |  52 |  70 |  70 |  66 |  83 |  94 |  80 |  64 |  67 |
| 33  |  41 |  1  |  28 |  33 |  36 |  22 |  3  |  6  |  0  | 14  |  86 |  63 |  57 |  94 |  79 |  88 |  51 |  70 |  50 |  73 |
|  6  |  6  |  7  |  5  | 17  |  43 |  28 |  7  | 32  | 38  |  7  | 82  |  87 |  85 |  74 |  71 | 87  |  72 |  74 |  51 |  94 |

M:

|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |

sorted M:

|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
Example for the k-th Element (Worst Case)

Input/Data:

| 57 | 73 | 83 | 65 | 50 | 79 | 67 | 91 | 71 | 90 | 55 | 76 | 93 | 77 | 51 | 92 | 66 | 69 | 80 | 55 | 71 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 82 | 61 | 55 | 87 | 89 | 66 | 94 | 77 | 65 | 53 | 63 | 60 | 91 | 65 | 54 | 90 | 87 | 60 | 90 | 56 | 51 |
| 2  | 22 | 16 | 32 | 20 | 8  | 28 | 6  | 9  | 3  | 24 | 86 | 52 | 70 | 70 | 66 | 83 | 94 | 80 | 64 | 67 |
| 33 | 41 | 1  | 28 | 33 | 36 | 22 | 3  | 6  | 0  | 14 | 86 | 63 | 57 | 94 | 79 | 88 | 51 | 70 | 50 | 73 |
| 6  | 6  | 7  | 5  | 17 | 43 | 28 | 7  | 32 | 38 | 7  | 82 | 87 | 85 | 74 | 71 | 87 | 72 | 74 | 51 | 94 |

M:

| 33 | 41 | 16 | 32 | 33 | 43 | 28 | 7 | 32 | 38 | 24 | 82 | 87 | 70 | 70 | 79 | 87 | 69 | 80 | 55 | 71 |

sorted M:
Example for the k-th Element (Worst Case)

Input/Data:

\[
\begin{array}{cccccccccccccccccccc}
82 & 61 & 55 & 87 & 89 & 66 & 94 & 77 & 65 & 53 & 63 & 60 & 91 & 65 & 54 & 90 & 87 & 60 & 90 & 56 & 51 \\
2 & 22 & 16 & 32 & 20 & 8 & 28 & 6 & 9 & 3 & 24 & 86 & 52 & 70 & 70 & 66 & 83 & 94 & 80 & 64 & 67 \\
33 & 41 & 1 & 28 & 33 & 36 & 22 & 3 & 6 & 0 & 14 & 86 & 63 & 57 & 94 & 79 & 88 & 51 & 70 & 50 & 73 \\
6 & 6 & 7 & 5 & 17 & 43 & 28 & 7 & 32 & 38 & 7 & 82 & 87 & 85 & 74 & 71 & 87 & 72 & 74 & 51 & 94 \\
\end{array}
\]

\[M:\]

\[
\begin{array}{cccccccccccccccccccc}
33 & 41 & 16 & 32 & 33 & 43 & 28 & 7 & 32 & 38 & 24 & 82 & 87 & 70 & 70 & 79 & 87 & 69 & 80 & 55 & 71 \\
\end{array}
\]

\[\text{sorted } M:\]

\[
\begin{array}{cccccccccccccccccccc}
7 & 16 & 24 & 28 & 32 & 32 & 33 & 33 & 38 & 41 & 43 & 55 & 69 & 70 & 70 & 71 & 79 & 80 & 82 & 87 & 87 \\
\end{array}
\]
Example for the k-th Element (Worst Case)

Input/Data:

```
57  73  83  65  50  79  67  91  71  90  55  76  93  77  51  92  66  69  80  55  71
82  61  55  87  89  66  94  77  65  53  63  60  91  65  54  90  87  60  90  56  51
  2  22  16  32  20   8  28   6   9   3  24  86  52  70  70  66  83  94  80  64  67
33  41  1  28  33  36  22   3  6  14  86  63  57  94  79  88  51  70  50  73
  6   6   7   5  17  43  28   7  32  38   7  82  87  85  74  71  87  72  74  51  94
```

M:

```
33  41  16  32  33  43  28  7  32  38  24  82  87  70  70  79  87  69  80  55  71
```

sorted M:

```
 7  16  24  28  32  32  33  33  38  41  43  55  69  70  70  71  79  80  82  87  87
```
Running Time

- For some constants $c, d$ we get:
  - $T(n) \leq d \cdot n$ for $n \leq 50$
  - $T(n) \leq c \cdot n + T(n/5) + T(3n/4)$

if $|S| \leq 50$ then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
Sort each $H_i$
Let $M$ be the sequence of the middle elements in $H_i$
$m := \text{Select}(\lceil |M|/2 \rceil, M)$
$S_1 := \{s \in S | s < m\}$
$S_2 := \{s \in S | s = m\}$
$S_3 := \{s \in S | s > m\}$
if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $\text{Select}(k - |S_1| - |S_2|, S_3)$
Running Time

- For some constants \( c, d \) we get:
- \( T(n) \leq d \cdot n \) for \( n \leq 50 \)
- \( T(n) \leq c \cdot n + T(n/5) + T(3n/4) \)

```plaintext
if |S| \leq 50 then return k-th number in S
Split S in \( \lceil n/5 \rceil \) sub-sequences \( H_i \) of size \( \leq 5 \)
Sort each \( H_i \)
Let \( M \) be the sequence of the middle elements in \( H_i \)
\( m := \text{Select}(\lceil |M|/2 \rceil, M) \)
\( S_1 := \{ s \in S \mid s < m \} \)
\( S_2 := \{ s \in S \mid s = m \} \)
\( S_3 := \{ s \in S \mid s > m \} \)
if \( |S_1| \geq k \) then return \( \text{Select}(k, S_1) \)
if \( |S_1| + |S_2| \geq k \) then return \( m \)
return \( \text{Select}(k - |S_1| - |S_2|, S_3) \)
```
Running Time

- For some constants $c, d$ we get:
  - $T(n) \leq d \cdot n$ for $n \leq 50$
  - $T(n) \leq c \cdot n + T(n/5) + T(3n/4)$

if $|S| \leq 50$ then return $k$-th number in $S$
Split $S$ in $\lceil n/5 \rceil$ sub-sequences $H_i$ of size $\leq 5$
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if $|S_1| \geq k$ then return $\text{Select}(k, S_1)$
if $|S_1| + |S_2| \geq k$ then return $m$
return $\text{Select}(k - |S_1| - |S_2|, S_3)$
Running Time

- **Claim**: $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.

- **Proof**:
  
  - $n = 50$:
    
    $$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$
  
  - $n > 50$:
    
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$

- Running time $T(n)$ is in $O(n)$.  

Running Time

- **Claim:** $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.

- **Proof:**
  - $n = 50$:
    $$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$
  - $n > 50$:
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$

- Running time $T(n)$ is in $O(n)$. 
Running Time

- Claim: $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.
- Proof:
  - $n = 50$:
    $$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$
  - $n > 50$:
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$
    $$T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n$$
- Running time $T(n)$ is in $O(n)$. 

Running Time

- Claim: $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.
- Proof:
  - $n = 50$:
    $$T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}$$
  - $n \geq 50$:
    $$T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)$$
    $$T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n$$
- Running time $T(n)$ is in $O(n)$. 
Running Time

- Claim: $T(n) \leq 20 \cdot r \cdot n$ with $r = \max(d, c)$.
- Proof:
  - $n = 50$:
    \[
    T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}
    \]
  - $n > 50$:
    \[
    T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)
    \]
    \[
    T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n
    \]
- Running time $T(n)$ is in $O(n)$. 
Running Time

Claim: \( T(n) \leq 20 \cdot r \cdot n \) with \( r = \max(d, c) \).

Proof:

\( n = 50: \)

\[
T(n) \leq c \cdot n + \frac{d \cdot n}{5} + \frac{3 \cdot d \cdot n}{4}
\]

\( n > 50: \)

\[
T(n) \leq c \cdot n + T\left(\frac{d \cdot n}{5}\right) + T\left(\frac{3 \cdot d \cdot n}{4}\right)
\]

\[
T(n) \leq c \cdot n + 4 \cdot r \cdot n + 15 \cdot r \cdot n
\]

Running time \( T(n) \) is in \( O(n) \).
Parallel k-Select

- **Input** $S = \{s_1, \ldots, s_n\}$.
- Processes $P_1, P_2, \ldots, P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
- Each $P_i$ knows $n, P(n)$.
- Each $P_i$ works on $\lceil n^x \rceil$ elements.
- We will now create a parallel version of the program Select($k, S$).
- We will get a parallel recursive program.

1. Easy solution for small $S$.
2. Split $S$ into small sub-sequences for the processors.
3. Compute parallel the median of the sub-sequences.
4. Compute parallel and recursive the median of medians.
5. Compute the splitting into the three sub-sequences.
6. Do the final recursion.
Parallel k-Select

- **Input** $S = \{s_1, \cdots, s_n\}$.
- **Processors** $P_1, P_2, \cdots P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
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Parallel k-Select

- Input $S = \{s_1, \cdots, s_n\}$.
- Processors $P_1, P_2, \cdots P_{\lceil n^{1/x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
- Each $P_i$ knows $n, P(n)$.
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   2. Split $S$ into small sub-sequences for the processors.
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- Each \( P_i \) knows \( n, P(n) \).
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- We will now create a parallel version of the program Select(k,S).
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1. **Easy solution for small \( S \).**
2. **Split \( S \) into small sub-sequences for the processors.**
3. **Compute parallel the median of the sub-sequences.**
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Parallel k-Select

- Input $S = \{s_1, \ldots, s_n\}$.
- Processors $P_1, P_2, \ldots, P_{\lceil n^{1-x} \rceil}$, thus $P(n) = \lceil n^{1-x} \rceil$.
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Example for the k-th Element

**Input/Data:**

| 30  | 67 | 94 | 47 | 28 | 59 | 92 | 93 | 51 | 24 | 67 | 4 | 41 | 30 | 41 | 23 | 70 | 97 | 3 | 76 | 57 | 85 | 26 | 20 | 67 | 73 | 1 | 71 | 55 | 17 | 59 | 50 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 44  | 16 | 85 | 37 | 64 | 55 | 80 | 49 | 82 | 49 | 69 | 6 | 8 | 12 | 73 | 60 | 9 | 95 | 18 | 68 | 38 | 80 | 36 | 43 | 93 | 92 | 61 | 13 | 76 | 85 | 8 | 35 | 47 | 3 | 37 |
| 42  | 24 | 52 | 12 | 94 | 54 | 2 | 63 | 6 | 3 | 56 | 95 | 29 | 89 | 3 | 64 | 83 | 19 | 85 | 92 | 70 | 67 | 56 | 87 | 87 | 10 | 31 | 72 | 61 | 92 | 59 | 54 | 15 | 69 | 22 |
| 29  | 97 | 47 | 90 | 92 | 66 | 82 | 62 | 79 | 56 | 48 | 14 | 97 | 66 | 46 | 62 | 10 | 52 | 41 | 88 | 89 | 82 | 30 | 63 | 66 | 13 | 27 | 68 | 18 | 10 | 1 | 58 | 56 | 75 | 81 |
| 81  | 25 | 84 | 51 | 48 | 92 | 73 | 90 | 43 | 35 | 62 | 21 | 8 | 20 | 27 | 79 | 78 | 75 | 91 | 61 | 52 | 19 | 70 | 19 | 92 | 57 | 59 | 1 | 66 | 28 | 75 | 22 | 7 | 45 | 5 |
| 82  | 83 | 83 | 83 | 47 | 10 | 74 | 77 | 48 | 46 | 14 | 51 | 77 | 12 | 63 | 24 | 81 | 11 | 26 | 26 | 27 | 45 | 15 | 67 | 97 | 73 | 91 | 65 | 96 | 37 | 55 | 92 | 18 | 15 | 16 |
| 22  | 91 | 74 | 75 | 13 | 54 | 27 | 19 | 78 | 15 | 16 | 50 | 2 | 41 | 79 | 46 | 35 | 0 | 21 | 1 | 7 | 77 | 13 | 35 | 1 | 23 | 70 | 89 | 84 | 82 | 91 | 2 | 77 | 12 | 23 |
| 93  | 1 | 78 | 97 | 88 | 32 | 6 | 20 | 33 | 34 | 45 | 64 | 49 | 51 | 7 | 52 | 21 | 44 | 27 | 70 | 78 | 55 | 22 | 66 | 52 | 8 | 72 | 28 | 39 | 76 | 51 | 68 | 3 | 80 | 65 |
| 30  | 0 | 92 | 52 | 40 | 42 | 19 | 65 | 78 | 45 | 95 | 73 | 53 | 96 | 28 | 3 | 61 | 70 | 40 | 68 | 29 | 31 | 56 | 68 | 95 | 50 | 87 | 85 | 11 | 61 | 50 | 72 | 11 | 46 | 36 |
| 29  | 77 | 61 | 53 | 72 | 33 | 87 | 62 | 78 | 55 | 95 | 19 | 4 | 62 | 52 | 7 | 77 | 34 | 43 | 63 | 90 | 10 | 97 | 21 | 42 | 5 | 71 | 58 | 36 | 65 | 59 | 87 | 91 | 18 |
| 59  | 75 | 52 | 96 | 45 | 10 | 89 | 26 | 32 | 26 | 31 | 78 | 65 | 0 | 29 | 70 | 78 | 65 | 11 | 83 | 68 | 54 | 86 | 56 | 3 | 1 | 43 | 1 | 75 | 36 | 62 | 8 | 54 | 5 | 83 | 91 |
| 1   | 91 | 2 | 46 | 91 | 62 | 2 | 54 | 46 | 96 | 89 | 51 | 16 | 76 | 84 | 56 | 91 | 0 | 54 | 69 | 20 | 6 | 2 | 8 | 72 | 62 | 3 | 49 | 60 | 75 | 23 | 23 | 46 | 93 | 12 |
| 53  | 8 | 21 | 22 | 32 | 93 | 11 | 56 | 44 | 1 | 54 | 63 | 77 | 49 | 96 | 24 | 96 | 35 | 41 | 92 | 49 | 80 | 64 | 45 | 62 | 66 | 8 | 28 | 21 | 85 | 94 | 91 | 18 | 38 | 23 |
| 63  | 48 | 63 | 37 | 56 | 96 | 74 | 40 | 15 | 71 | 48 | 25 | 47 | 71 | 94 | 35 | 50 | 8 | 24 | 31 | 66 | 34 | 90 | 93 | 39 | 96 | 69 | 62 | 29 | 61 | 87 | 4 | 0 | 97 | 28 |
| 57  | 74 | 81 | 50 | 39 | 27 | 34 | 93 | 73 | 27 | 45 | 33 | 54 | 84 | 91 | 31 | 59 | 16 | 6 | 91 | 29 | 9 | 63 | 2 | 61 | 6 | 23 | 86 | 6 | 88 | 40 | 45 | 53 | 30 | 20 |


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Example for the k-th Element

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<thead>
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<th>Input/Data:</th>
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<tr>
<td>30 67 94 47 28 59 92 93 51 24 67 4 41 30 41 23 70 97 3 76 57 85 26 20 67 73 68 69 41 1 71 55 17 59 50</td>
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<td>44 16 85 37 64 55 80 49 82 49 69 6 8 12 73 60 9 95 18 68 38 80 36 43 93 92 61 13 76 85 8 35 47 3 37</td>
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<tr>
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<tr>
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<td>82 83 38 33 47 10 74 77 48 46 14 51 77 12 63 24 81 11 26 26 27 45 15 67 97 73 91 65 96 37 55 92 18 15 16</td>
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<td>22 91 74 75 13 54 27 19 78 15 16 50 2 41 79 46 35 0 21 1 7 77 13 35 1 23 70 89 84 82 91 2 77 12 23</td>
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<td>30 0 92 52 40 42 19 65 78 45 95 73 53 96 28 3 61 70 40 68 29 31 56 68 95 50 87 85 11 61 50 72 11 46 36</td>
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<td>29 77 61 53 72 33 87 97 28 55 95 48 19 4 62 52 7 77 34 43 63 90 10 97 21 42 5 71 58 36 65 59 87 91 18</td>
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<td>59 75 52 96 45 10 89 26 32 26 31 78 65 0 29 70 78 65 11 83 68 54 86 3 1 43 1 75 36 62 8 54 5 83 91</td>
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<td>53 8 21 22 32 93 11 56 44 1 54 63 77 49 96 24 96 35 41 92 49 80 64 45 62 66 8 28 21 85 94 91 18 38 23</td>
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<tr>
<td>63 48 63 37 56 96 74 40 15 71 48 25 47 71 94 35 50 8 24 31 66 34 90 93 39 96 69 62 29 61 87 4 0 97 28</td>
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<tr>
<td>57 74 81 50 39 27 34 93 73 27 45 33 54 84 91 31 59 16 6 91 29 9 63 2 61 6 23 86 6 88 40 45 53 30 20</td>
</tr>
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\[ P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7 \quad P_8 \quad P_9 \quad P_{10} \quad P_{11} \quad P_{12} \quad P_{13} \quad P_{14} \quad P_{15} \quad P_{16} \quad P_{17} \quad P_{18} \quad P_{19} \quad P_{20} \quad P_{21} \quad P_{22} \quad P_{23} \quad P_{24} \quad P_{25} \quad P_{26} \quad P_{27} \quad P_{28} \quad P_{29} \quad P_{30} \quad P_{31} \quad P_{32} \quad P_{33} \quad P_{34} \quad P_{35} \]

\[ M: \]

\[ 44 67 74 51 48 54 73 62 46 35 54 50 47 49 62 52 61 44 27 69 52 55 36 45 66 50 59 68 41 62 55 54 18 59 23 \]

\[ \text{sorted } M: \]
Example for the k-th Element

Input/Data:

| 30 | 67 | 94 | 47 | 28 | 59 | 92 | 93 | 51 | 24 | 67 | 4 | 41 | 30 | 41 | 23 | 70 | 97 | 3 | 76 | 57 | 85 | 26 | 20 | 67 | 73 | 68 | 69 | 41 | 1 | 71 | 55 | 17 | 59 | 50 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 44 | 16 | 85 | 37 | 64 | 55 | 80 | 49 | 82 | 49 | 69 | 6 | 8 | 12 | 73 | 60 | 9 | 95 | 18 | 68 | 38 | 80 | 36 | 43 | 93 | 92 | 61 | 13 | 76 | 85 | 8 | 35 | 47 | 3 | 37 |
| 42 | 24 | 52 | 12 | 94 | 54 | 2 | 63 | 6 | 3 | 56 | 95 | 29 | 89 | 3 | 64 | 83 | 19 | 85 | 92 | 70 | 67 | 56 | 87 | 87 | 10 | 31 | 72 | 61 | 92 | 59 | 54 | 15 | 69 | 22 |
| 29 | 97 | 47 | 90 | 92 | 66 | 82 | 62 | 79 | 56 | 48 | 14 | 97 | 66 | 46 | 62 | 10 | 52 | 41 | 88 | 89 | 82 | 30 | 63 | 66 | 13 | 27 | 68 | 18 | 10 | 1 | 58 | 56 | 75 | 81 |
| 81 | 25 | 84 | 51 | 48 | 92 | 73 | 90 | 43 | 35 | 62 | 21 | 8 | 20 | 27 | 79 | 78 | 75 | 91 | 61 | 52 | 19 | 70 | 19 | 92 | 57 | 59 | 1 | 66 | 28 | 75 | 22 | 7 | 45 | 5 |
| 82 | 83 | 83 | 83 | 47 | 10 | 74 | 77 | 48 | 46 | 14 | 51 | 77 | 12 | 63 | 24 | 81 | 11 | 26 | 26 | 27 | 45 | 15 | 67 | 97 | 73 | 91 | 65 | 96 | 37 | 55 | 92 | 18 | 15 | 16 |
| 22 | 91 | 74 | 75 | 13 | 54 | 27 | 19 | 78 | 15 | 16 | 50 | 2 | 41 | 79 | 46 | 35 | 0 | 21 | 1 | 7 | 77 | 13 | 35 | 1 | 23 | 70 | 89 | 84 | 82 | 91 | 2 | 77 | 12 | 23 |
| 93 | 1 | 78 | 97 | 88 | 32 | 6 | 20 | 33 | 34 | 45 | 64 | 49 | 51 | 7 | 52 | 21 | 44 | 27 | 70 | 78 | 55 | 22 | 66 | 52 | 8 | 72 | 28 | 39 | 76 | 51 | 68 | 3 | 80 | 65 |
| 30 | 0 | 92 | 52 | 40 | 42 | 19 | 65 | 78 | 45 | 95 | 73 | 53 | 96 | 28 | 3 | 61 | 70 | 40 | 68 | 29 | 31 | 56 | 68 | 95 | 50 | 87 | 85 | 11 | 61 | 50 | 72 | 11 | 46 | 36 |
| 29 | 77 | 61 | 53 | 72 | 33 | 87 | 92 | 28 | 55 | 95 | 48 | 19 | 4 | 62 | 52 | 7 | 77 | 34 | 63 | 90 | 10 | 97 | 21 | 42 | 5 | 71 | 58 | 36 | 65 | 59 | 87 | 91 | 18 |
| 59 | 75 | 52 | 96 | 45 | 10 | 89 | 26 | 32 | 26 | 31 | 78 | 65 | 0 | 29 | 70 | 78 | 65 | 11 | 83 | 68 | 54 | 86 | 3 | 1 | 43 | 1 | 75 | 36 | 62 | 8 | 54 | 5 | 83 | 91 |
| 1 | 91 | 2 | 46 | 91 | 62 | 2 | 54 | 46 | 96 | 89 | 51 | 16 | 76 | 84 | 56 | 91 | 0 | 54 | 69 | 20 | 6 | 2 | 8 | 72 | 62 | 3 | 49 | 60 | 75 | 23 | 23 | 46 | 93 | 12 |
| 53 | 8 | 21 | 22 | 32 | 93 | 11 | 56 | 44 | 1 | 54 | 63 | 77 | 49 | 96 | 24 | 96 | 35 | 41 | 92 | 49 | 80 | 64 | 45 | 62 | 66 | 8 | 28 | 21 | 85 | 94 | 91 | 18 | 38 | 23 |
| 63 | 48 | 63 | 37 | 56 | 96 | 74 | 40 | 15 | 71 | 48 | 25 | 47 | 71 | 94 | 35 | 50 | 8 | 24 | 31 | 66 | 34 | 90 | 93 | 39 | 96 | 69 | 62 | 29 | 61 | 87 | 4 | 0 | 97 | 28 |
| 57 | 74 | 81 | 50 | 39 | 27 | 34 | 93 | 73 | 27 | 45 | 33 | 54 | 84 | 91 | 31 | 59 | 16 | 6 | 91 | 29 | 9 | 63 | 2 | 61 | 6 | 23 | 86 | 6 | 88 | 40 | 45 | 53 | 30 | 20 |

\[
P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8 P_9 P_{10} P_{11} P_{12} P_{13} P_{14} P_{15} P_{16} P_{17} P_{18} P_{19} P_{20} P_{21} P_{22} P_{23} P_{24} P_{25} P_{26} P_{27} P_{28} P_{29} P_{30} P_{31} P_{32} P_{33} P_{34} P_{35}
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44\ 67\ 74\ 51\ 48\ 54\ 73\ 62\ 46\ 35\ 54\ 50\ 47\ 49\ 62\ 52\ 61\ 44\ 27\ 69\ 52\ 55\ 36\ 45\ 66\ 50\ 59\ 68\ 41\ 62\ 55\ 54\ 18\ 59\ 23
\]

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\]

\[
18\ 23\ 27\ 35\ 36\ 41\ 44\ 44\ 45\ 46\ 47\ 48\ 49\ 50\ 50\ 51\ 52\ 52\ 54\ 54\ 54\ 55\ 55\ 55\ 59\ 59\ 61\ 62\ 62\ 62\ 66\ 67\ 68\ 69\ 73\ 74
\]
## Example for the k-th Element

**Input/Data:**

| 30 | 67 | 94 | 47 | 28 | 59 | 92 | 93 | 51 | 24 | 67 | 4 | 41 | 30 | 41 | 23 | 70 | 97 | 3 | 76 | 57 | 85 | 26 | 20 | 67 | 73 | 68 | 69 | 41 | 1 | 71 | 55 | 17 | 59 | 50 |
| 44 | 16 | 85 | 37 | 64 | 55 | 80 | 49 | 82 | 49 | 69 | 6 | 8 | 12 | 73 | 60 | 9 | 95 | 18 | 68 | 38 | 80 | 36 | 43 | 93 | 92 | 61 | 13 | 76 | 85 | 8 | 35 | 47 | 3 | 37 |
| 42 | 24 | 52 | 12 | 94 | 54 | 2 | 63 | 6 | 3 | 56 | 95 | 29 | 89 | 3 | 64 | 83 | 19 | 85 | 92 | 70 | 67 | 56 | 87 | 87 | 10 | 31 | 72 | 61 | 92 | 59 | 54 | 15 | 69 | 22 |
| 29 | 97 | 47 | 90 | 92 | 66 | 82 | 62 | 79 | 56 | 48 | 14 | 97 | 66 | 46 | 62 | 10 | 52 | 41 | 88 | 89 | 82 | 30 | 63 | 66 | 13 | 27 | 68 | 18 | 10 | 1 | 58 | 56 | 75 | 81 |
| 81 | 25 | 84 | 51 | 48 | 92 | 73 | 90 | 43 | 35 | 62 | 21 | 8 | 20 | 27 | 79 | 78 | 75 | 91 | 61 | 52 | 19 | 70 | 19 | 92 | 57 | 59 | 1 | 66 | 28 | 75 | 22 | 7 | 45 | 5 |
| 82 | 83 | 83 | 83 | 47 | 10 | 74 | 77 | 48 | 46 | 14 | 51 | 77 | 12 | 63 | 24 | 81 | 11 | 26 | 26 | 27 | 45 | 15 | 67 | 97 | 73 | 91 | 65 | 96 | 37 | 55 | 92 | 18 | 15 | 16 |
| 22 | 91 | 74 | 75 | 13 | 54 | 27 | 19 | 78 | 15 | 16 | 50 | 2 | 41 | 79 | 46 | 35 | 0 | 21 | 1 | 77 | 13 | 35 | 1 | 23 | 70 | 89 | 84 | 82 | 91 | 2 | 77 | 12 | 23 |
| 93 | 1 | 78 | 97 | 88 | 32 | 6 | 20 | 33 | 34 | 45 | 64 | 49 | 51 | 7 | 52 | 21 | 44 | 27 | 70 | 78 | 55 | 22 | 66 | 52 | 8 | 72 | 28 | 39 | 76 | 51 | 68 | 3 | 80 | 65 |
| 30 | 0 | 92 | 52 | 40 | 42 | 19 | 65 | 78 | 45 | 95 | 73 | 53 | 96 | 28 | 3 | 61 | 70 | 40 | 68 | 29 | 31 | 56 | 68 | 95 | 50 | 87 | 85 | 11 | 61 | 50 | 72 | 11 | 46 | 36 |
| 29 | 77 | 61 | 53 | 72 | 33 | 87 | 97 | 28 | 55 | 98 | 19 | 4 | 62 | 52 | 7 | 77 | 34 | 43 | 63 | 90 | 10 | 97 | 21 | 42 | 5 | 71 | 58 | 36 | 65 | 59 | 87 | 91 | 18 |
| 59 | 75 | 52 | 96 | 45 | 10 | 89 | 26 | 32 | 26 | 31 | 78 | 65 | 0 | 29 | 70 | 78 | 65 | 11 | 83 | 68 | 54 | 86 | 3 | 1 | 43 | 1 | 75 | 36 | 62 | 8 | 54 | 5 | 83 | 91 |
| 1 | 91 | 2 | 46 | 91 | 62 | 2 | 54 | 46 | 96 | 89 | 51 | 16 | 76 | 84 | 56 | 91 | 0 | 54 | 69 | 20 | 6 | 2 | 8 | 72 | 62 | 3 | 49 | 60 | 75 | 23 | 23 | 46 | 93 | 12 |
| 53 | 8 | 21 | 22 | 32 | 93 | 11 | 56 | 44 | 1 | 54 | 63 | 77 | 49 | 96 | 24 | 96 | 35 | 41 | 92 | 49 | 80 | 64 | 45 | 62 | 66 | 8 | 28 | 21 | 85 | 94 | 91 | 18 | 38 | 23 |
| 63 | 48 | 63 | 37 | 56 | 96 | 74 | 40 | 15 | 71 | 48 | 25 | 47 | 71 | 94 | 35 | 50 | 8 | 24 | 31 | 66 | 34 | 90 | 93 | 39 | 96 | 69 | 62 | 29 | 61 | 87 | 4 | 0 | 97 | 28 |
| 57 | 74 | 81 | 50 | 39 | 27 | 34 | 93 | 73 | 27 | 45 | 33 | 54 | 84 | 91 | 31 | 59 | 16 | 6 | 91 | 29 | 9 | 63 | 2 | 61 | 6 | 23 | 86 | 6 | 88 | 40 | 45 | 53 | 30 | 20 |

\[ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{28}, P_{29}, P_{30}, P_{31}, P_{32}, P_{33}, P_{34}, P_{35} \]

\[ M: \]

| 44 | 67 | 74 | 51 | 48 | 54 | 73 | 62 | 46 | 35 | 54 | 50 | 47 | 49 | 62 | 52 | 61 | 44 | 27 | 69 | 52 | 55 | 36 | 45 | 66 | 50 | 59 | 68 | 41 | 62 | 55 | 54 | 18 | 59 | 23 |

\[ \text{sorted } M: \]

| 18 | 23 | 27 | 35 | 36 | 41 | 44 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 50 | 51 | 52 | 52 | 54 | 54 | 54 | 55 | 55 | 59 | 59 | 61 | 62 | 62 | 66 | 66 | 67 | 68 | 69 | 73 | 74 |
Parallel k-Select

Programm: ParSelect(k,S)

1: 
   \textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \ \textbf{returns} \ Select(k, S).

2: 
   S \ is \ split \ into \ \lceil |S|^{1-x} \rceil \ sub-sequences \ S_i \ with \ |S_i| \leq \lceil n^x \rceil \n
   P_i \ stores \ the \ start-address \ of \ S_i.

3: 
   \textbf{for all} \ P_i \ where \ 1 \leq i \leq \lceil n^{1-x} \rceil \ \textbf{do \ in \ parallel}
   \begin{align*}
   m_i & := \ Select(\lceil |S_i|/2 \rceil, S_i) \\
   P_i(m_1) & \rightarrow R_i.
   \end{align*}
   \begin{align*}
   \text{Assume} \ in \ the \ following \ that \ M \ is \ the \ sequence \ of \ these \ values.
   \end{align*}

4: 
   m := ParSelect(\lceil |M|/2 \rceil, M).

5: More to come!
Parallel k-Select

Programm: ParSelect(k,S)

1:  \textbf{if } |S| \leq k_1 \textbf{ then } P_1 \textbf{ returns } Select(k, S).

2:  \textit{S} is split into \left\lceil |S|^{1-x} \right\rceil \textit{sub-sequences } S_i \textit{ with } |S_i| \leq \left\lceil n^x \right\rceil \textit{.}
    \textit{P}_i \text{ stores the start-address of } S_i.

3:  \textbf{for all } P_i \textbf{ where } 1 \leq i \leq \left\lceil n^{1-x} \right\rceil \textbf{ do in parallel}
    \begin{align*}
    m_i &:= Select\left(\left\lceil |S_i|/2 \right\rceil, S_i\right) \\
    P_i(m_1) &\rightarrow R_i.
    \end{align*}
    \textbf{Assume in the following that } M \textbf{ is the sequence of these values.}

4:  m := ParSelect(\left\lceil |M|/2 \right\rceil, M).

5:  \textbf{More to come!}
Parallel k-Select

Programm: ParSelect(k,S)

1:
   if |S| ≤ k₁ then P₁ returns Select(k, S).

2:
   S is split into ⌈|S|²⁻ₓ⌉ sub-sequences Sᵢ with |Sᵢ| ≤ ⌈nˣ⌉
   Pᵢ stores the start-address of Sᵢ.

3:
   for all Pᵢ where 1 ≤ i ≤ ⌈n⁻¹⁻ₓ⌉ do in parallel
   \[ mᵢ := Select(⌈|Sᵢ|/2⌉, Sᵢ) \]
   \[ Pᵢ(m₁) \rightarrow Rᵢ. \]
   Assume in the following that M is the sequence of these values.

4:
   \[ m := ParSelect(⌈|M|/2⌉, M). \]

5: More to come!
Programm: ParSelect(k,S)
1: 
   \textbf{if } |S| \leq k_1 \textbf{ then } P_1 \textbf{ returns } Select(k, S).
2: 
   S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil
   P_i \text{ stores the start-address of } S_i.
3: 
   \textbf{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \textbf{ do in parallel}
   \hspace{1cm} m_i := Select(\lceil |S_i|/2 \rceil, S_i)
   \hspace{1cm} P_i(m_1) \to R_i.
   \hspace{1cm} \text{Assume in the following that } M \text{ is the sequence of these values.}
4: 
   m := \textit{ParSelect}(\lceil |M|/2 \rceil, M).
5: \hspace{1cm} \text{More to come!}
Programm: ParSelect(k,S)

1: \textbf{if} \ |S| \leq k_1 \ \textbf{then} \ P_1 \textbf{ returns } \text{Select}(k, S).

2: \ S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil
\ P_i \text{ stores the start-address of } S_i.

3: \textbf{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \textbf{ do in parallel}
\ \quad m_i := \text{Select}(\lceil |S_i|/2 \rceil, S_i)
\ \quad P_i(m_1) \rightarrow R_i.
\ \quad \text{Assume in the following that } M \text{ is the sequence of these values.}

4: \quad m := \text{ParSelect}(\lceil |M|/2 \rceil, M).

5: \quad \text{More to come!}
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:

Distribute $m$ via broadcast to all $P_i$. 

for all $P_i$ where $1 \leq i \leq \lfloor n^{1-x} \rfloor$ do in parallel

$L_i := \{ s \in S_i | s < m \}$

$E_i := \{ s \in S_i | s = m \}$

$G_i := \{ s \in S_i | s > m \}$

5.2:

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lfloor n^{1-x} \rfloor$.

$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lfloor n^{1-x} \rfloor$.

$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lfloor n^{1-x} \rfloor$.

Let: $l_0 = e_0 = g_0 = 0$

5.3:

Even more to come!
**Parallel k-Select**

Programm: ParSelect(k,S) Steps 5

5.1:

Distribute $m$ via broadcast to all $P_i$.

For all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

- $L_i := \{s \in S_i \mid s < m\}$
- $E_i := \{s \in S_i \mid s = m\}$
- $G_i := \{s \in S_i \mid s > m\}$

5.2:

Compute with Parallel Prefix:

- $l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
- $e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
- $g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$

5.3:

Even more to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5

5.1:
Distribute \( m \) via broadcast to all \( P_i \).

\[
\text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel}
\]

\[
L_i := \{ s \in S_i \mid s < m \}
\]

\[
E_i := \{ s \in S_i \mid s = m \}
\]

\[
G_i := \{ s \in S_i \mid s > m \}
\]

5.2:
Compute with Parallel Prefix:

\[
l_i := \sum_{j=1}^{i} |L_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil.
\]

\[
e_i := \sum_{j=1}^{i} |E_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil.
\]

\[
g_i := \sum_{j=1}^{i} |G_i| \text{ for all } 1 \leq i \leq \lceil n^{1-x} \rceil.
\]

Let: \( l_0 = e_0 = g_0 = 0 \)

5.3:

Even more to come!
Parallel k-Select

Programm: ParSelect(k,S) Steps 5+6

5.3:

Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

- \( P_i \) writes \( L_i \) in \( R_{l_{i-1}+1}, \ldots, R_{l_i} \).
- \( P_i \) writes \( E_i \) in \( R_{e_{i-1}+1}, \ldots, R_{e_i} \).
- \( P_i \) writes \( G_i \) in \( R_{g_{i-1}+1}, \ldots, R_{g_i} \).

6:

if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Parallel k-Select

Programm: ParSelect(k,S) Steps 5+6

5.3:
Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

- \( P_i \) writes \( L_i \) in \( R_{li-1+1}, \ldots, R_{li} \).
- \( P_i \) writes \( E_i \) in \( R_{ei-1+1}, \ldots, R_{ei} \).
- \( P_i \) writes \( G_i \) in \( R_{gi-1+1}, \ldots, R_{gi} \).

6:

if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)
1: \[ \text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } \text{Select}(k, S). \]
2: \[ S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil \]
   \[ P_i \text{ stores the start-address of } S_i. \]
3: \[ \text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel} \]
   \[ m_i := \text{Select}(\lceil |S_i|/2 \rceil, S_i) \]
   \[ P_i(m_1) \rightarrow R_i. \]
   Assume in the following that \( M \) is the sequence of these values
4: \[ m := \text{ParSelect}(\lceil |M|/2 \rceil, M). \]
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: $O(1)$
   \begin{itemize}
   \item if $|S| \leq k_1$ then $P_1$ returns $Select(k, S)$.
   \end{itemize}

2:
   \begin{itemize}
   \item $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   \item $P_i$ stores the start-address of $S_i$.
   \end{itemize}

3:
   \begin{itemize}
   \item for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
     \begin{itemize}
     \item $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
     \item $P_i(m_1) \rightarrow R_i$.
     \end{itemize}
   \end{itemize}
   Assume in the following that $M$ is the sequence of these values

4:
   \begin{itemize}
   \item $m := ParSelect(\lceil |M|/2 \rceil, M)$.
   \end{itemize}
Programm: ParSelect(k, S)

1: \( O(1) \)
   \[ \text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } Select(k, S). \]

2: \( O(\log_2(|S|^{1-x})) \) thus we have \( O(\log n) \)
   \[ S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil \]
   \[ P_i \text{ stores the start-address of } S_i. \]

3:
   \[
   \text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel }
   \]
   \[ m_i := Select(\lceil |S_i|/2 \rceil, S_i) \]
   \[ P_i(m_1) \rightarrow R_i. \]
   Assume in the following that \( M \) is the sequence of these values

4:
   \[ m := ParSelect(\lceil |M|/2 \rceil, M). \]
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: $O(1)$
   \textbf{if} $|S| \leq k_1$ \textbf{then} $P_1$ \textbf{returns} $Select(k,S)$.

2: $O(\log_2(|S|^{1-x}))$ thus we have $O(\log n)$
   $S$ is split into $\lceil |S|^{1-x} \rceil$ sub-sequences $S_i$ with $|S_i| \leq \lceil n^x \rceil$
   $P_i$ stores the start-address of $S_i$.

3: $O(n^x)$
   \textbf{for all} $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ \textbf{do in parallel}
   $m_i := Select(\lceil |S_i|/2 \rceil, S_i)$
   $P_i(m_1) \rightarrow R_i$.
   Assume in the following that $M$ is the sequence of these values

4: $m := ParSelect(\lceil |M|/2 \rceil, M)$. 
Parallel k-Select (Running Time)

Programm: ParSelect(k,S)

1: \(O(1)\)
   \[\text{if } |S| \leq k_1 \text{ then } P_1 \text{ returns } Select(k, S).\]

2: \(O(\log_2(|S|^{1-x}))\) thus we have \(O(\log n)\)
   \[S \text{ is split into } \lceil |S|^{1-x} \rceil \text{ sub-sequences } S_i \text{ with } |S_i| \leq \lceil n^x \rceil\]
   \(P_i\) stores the start-address of \(S_i\).

3: \(O(n^x)\)
   \[\text{for all } P_i \text{ where } 1 \leq i \leq \lceil n^{1-x} \rceil \text{ do in parallel}\]
   \[m_i := Select(\lceil |S|/2 \rceil, S_i)\]
   \[P_i(m_1) \rightarrow R_i.\]
   Assume in the following that \(M\) is the sequence of these values

4: \(T_{ParSelect}(n^{1-x})\)
   \[m := ParSelect(\lceil |M|/2 \rceil, M).\]
Programm: ParSelect(k,S) Steps 5
5.1a:
Distribute $m$ via broadcast to all $P_i$.
5.1b:
for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel
$L_i := \{ s \in S_i \mid s < m \}$
$E_i := \{ s \in S_i \mid s = m \}$
$G_i := \{ s \in S_i \mid s > m \}$
5.2:
Compute with Parallel Prefix:
$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$

Distribute $m$ via broadcast to all $P_i$.

5.1b:

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{ s \in S_i \mid s < m \}$

$E_i := \{ s \in S_i \mid s = m \}$

$G_i := \{ s \in S_i \mid s > m \}$

5.2:

Compute with Parallel Prefix:

$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$
Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$

Distribute $m$ via broadcast to all $P_i$.

5.1b: $O(|S_i|)$ thus we have $O(n^x)$

for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

- $L_i := \{s \in S_i \mid s < m\}$
- $E_i := \{s \in S_i \mid s = m\}$
- $G_i := \{s \in S_i \mid s > m\}$

5.2:

Compute with Prallel Prefix:

- $l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
- $e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
- $g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.

Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5

5.1a: $O(\log_2(n^{1-x}))$
Distribute $m$ via broadcast to all $P_i$.

5.1b: $O(|S_i|)$ thus we have $O(n^x)$
for all $P_i$ where $1 \leq i \leq \lceil n^{1-x} \rceil$ do in parallel

$L_i := \{ s \in S_i \mid s < m \}$
$E_i := \{ s \in S_i \mid s = m \}$
$G_i := \{ s \in S_i \mid s > m \}$

5.2: $O(\log_2(n^{1-x}))$

Compute with Parallel Prefix:
$l_i := \sum_{j=1}^{i} |L_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$e_i := \sum_{j=1}^{i} |E_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
$g_i := \sum_{j=1}^{i} |G_i|$ for all $1 \leq i \leq \lceil n^{1-x} \rceil$.
Let: $l_0 = e_0 = g_0 = 0$
Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5+6

5.3: Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

- \( P_i \) writes \( L_i \) in \( R_{l_{i-1}+1} \), \( \ldots \), \( R_{l_i} \).
- \( P_i \) writes \( E_i \) in \( R_{e_{i-1}+1} \), \( \ldots \), \( R_{e_i} \).
- \( P_i \) writes \( G_i \) in \( R_{g_{i-1}+1} \), \( \ldots \), \( R_{g_i} \).

6: if \( |L| \geq k \) then return \( \text{ParSelect}(k,L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Programm: ParSelect(k,S) Steps 5+6

5.3: \( O(n^x) \)

Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \) and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

- \( P_i \) writes \( L_i \) in \( R_{l_{i-1}+1}, \ldots, R_{l_i} \).
- \( P_i \) writes \( E_i \) in \( R_{e_{i-1}+1}, \ldots, R_{e_i} \).
- \( P_i \) writes \( G_i \) in \( R_{g_{i-1}+1}, \ldots, R_{g_i} \).

6:

if \( |L| \geq k \) then return \( \text{ParSelect}(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( \text{Select}(k - |L| - |E|, G) \)
Motivation and History

PRAM Introduction

Efficiency

Selection

Merging

1:54 Algorithm and Running Time 3/3

Walter Unger 30.1.2017 11:52 WS2016/17

Parallel k-Select (Running Time)

Programm: ParSelect(k,S) Steps 5+6

5.3: \( O(n^x) \)

Compute \( L = \{ s \in S \mid s < m \} \), \( E = \{ s \in S \mid s = m \} \)
and \( G = \{ s \in S \mid s > m \} \) as follows:

for all \( P_i \) where \( 1 \leq i \leq \lceil n^{1-x} \rceil \) do in parallel

\[ P_i \] writes \( L_i \) in \( R_{l_i-1+1}, \ldots, R_{l_i} \).
\[ P_i \] writes \( E_i \) in \( R_{e_i-1+1}, \ldots, R_{e_i} \).
\[ P_i \] writes \( G_i \) in \( R_{g_i-1+1}, \ldots, R_{g_i} \).

6: \( T_{ParSelect}(3 \cdot n/4) \)
if \( |L| \geq k \) then return \( ParSelect(k, L) \)
if \( |L| + |E| \geq k \) then return \( m \)
return \( Select(k - |L| - |E|, G) \)
Parallel k-Select (Running Time)

Adding all up we get:

- \( T_{ParSelect}(n) = c_1 \log n + c_2 \cdot n^x + T_{ParSelect}(n^{1-x}) + T_{ParSelect}(3/4 \cdot n) \).
- \( T_{ParSelect}(n) = O(n^x) \) with \( P_{ParSelect}(n) = O(n^{1-x}) \).

\[
Eff_{ParSelect}(n) = \frac{O(n)}{O(n^x) \cdot O(n^{1-x})} = O(1)
\]
Sequential Merging

- **Input:**
  \[A = (a_1, a_2, \cdots, a_r)\] and \[B = (b_1, b_2, \cdots, b_s)\] two sorted sequences

- **Output:**
  \[C = (c_1, c_2, \cdots, c_n)\] sorted sequence of \(A\) and \(B\) with \(n = r + s\).

- **Programm:** Merge
  
  \[
i := 1; j := 1; n := r + s
  \]
  
  \[
  \text{for } k := 1 \text{ to } n \text{ do}
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Sequential Merging

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Thus we may separate the two sequences into disjoint blocks.

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$P_i$ cares about $A_i$ and $\hat{B}_i$ if $|\hat{B}_i| \leq \lceil r/p \rceil$.

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1. Use $P(n)$ processors.
2. Each processor $P_i$ computes for $A$ [$B$] its part of size $r/P(n)$ [$s/P(n)$].
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4. Each processor computes its $A$ or $B$ block, where only he is responsible for.
5. This block has size $O(n/P(n))$.
6. Each processor merges its block into the resulting sequence.
7. Time: $O(\log n + n/P(n))$.
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$$\frac{n}{O(P(n)) \cdot O(\log n + n/P(n))}.$$ 

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Idea for Merging (EREW)

- Do some splitting into pairs of blocks of the same size.
- Rekursive splitting into pairs of blocks of the same size.
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Merging (EREW)

1. Use $P(n)$ processors.
2. Compute the median $m$ of the sequences $A$ and $B$.
3. Split the sequences $A$ and $B$ in two sub-sequences each of the “same” size ($-1 \leq |A| - |B| \leq 1$).
4. Continue recursively, till all sub-sequences are smaller than $n/P(n)$.
5. Do the merging in the same way as before.

Remaining problem: Find the median of two sequences.
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- Sequences $A$ and $B$ are sorted.
- Compute median $a$ of $A$ and median $b$ of $B$. 
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\begin{figure}
\centering
\includegraphics[width=\textwidth]{median_example}
\caption{Visualization of finding the median for two sorted sequences. The red square indicates the median value.}
\end{figure}
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### Diagram

```
4 5 10 11 20 21 22 23 27 28 29 30 31 32 33 34 35 38 39 41 42 45 46 47 48 49 50 54 55 56 58 64
1 2 3 6 7 8 9 12 13 14 15 16 17 18 19 24 25 26 36 37 40 43 44 51 52 53 57 59 60 61 62 63
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Sequences $A$ and $B$ are sorted.
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Median for two Sorted Sequences

1. Sequences $A$ and $B$ are sorted.
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2. Compute median $a$ of $A$ and median $b$ of $B$.
3. Median $a \ [b]$ splits $A \ [B]$ into half.
4. The median of $A$ and $B$ is in one block-pair of the four blocks.
5. Search recursively for the median.
Median for two Sorted Sequences

1. Sequences $A$ and $B$ are sorted.
2. Compute median $a$ of $A$ and median $b$ of $B$.
4. The median of $A$ and $B$ is in one block-pair of the four blocks.
5. Search recursively for the median.

Running time: $O(\log n)$
Running Time for Merging (EREW)

1. Use $P(n)$ processors.
2. Compute the median $m$ of the sequences $A$ and $B$.
3. Split the sequences $A$ and $B$ in two sub-sequences each.
4. Continue recursively, till all sub-sequences are smaller than $n/P(n)$.
5. Merge in the same way as before.

- Running time: $O(n/P(n) + \log(n)^2)$.
- Efficiency

$$\frac{O(n)}{O(P(n)) \cdot O(n/P(n) + \log(n)^2)} = \frac{O(n)}{O(n + P(n) \cdot \log(n)^2)}.$$ 

- Efficiency is 1 for $P(n) < \frac{n}{(\log n)^2}$.
Running Time for Merging (EREW)

1. Use $P(n)$ processors.
2. Compute the median $m$ of the sequences $A$ and $B$. $O(\log n)$
3. Split the sequences $A$ and $B$ in two sub-sequences each.
4. Continue recursively, till all sub-sequences are smaller than $n/P(n)$.
5. Merge in the same way as before.

- Running time: $O\left(n/P(n) + \log(n)^2\right)$.
- Efficiency

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Questions

- Explain the motivation behind parallel systems.
- Describe the different models of a PRAM.
- Describe idea of the k-select algorithm.
- For which problems do the running time of CWCR and EWCR algorithms differ?
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Legende

■ : Nicht relevant
■ : Grundlagen, die implizit genutzt werden
■ : Idee des Beweises oder des Vorgehens
■ : Struktur des Beweises oder des Vorgehens
■ : Vollständiges Wissen