Theory of Parallel and Distributed Systems (WS2016/17)
Kapitel 2
Sorting with a PRAM

Walter Unger

Lehrstuhl für Informatik 1

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Very simple Algorithm (Idea)

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### Very simple Algorithm (Idea)

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### Very simple Algorithm (Idea)

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| 67| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |   |
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|   | 34| 12| 14| 56| 23| 67| 49| 27| 61| 52| 57| 59| 26| 41| 33| 22|

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Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.
- **Programm:** SimpleSort
  
  **Eingabe:** $s_1, \ldots, s_n$.
  
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
  
  if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$
  
  for all $i$ where $1 \leq i \leq n$ do in parallel
  
  for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel
  
  Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
  
  $P_i(s_i) \rightarrow R_{q_i+1}$.

- Complexity: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Efficiency: $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- Model: CREW.
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Processors \( P_{i,j} \) bestimmen \( q_i = \sum_{l=1}^{n} R_{i,l} \).

\( P_{i}(s_i) \rightarrow R_{q_i+1} \).

- **Complexity:** \( T(n) = O(\log n) \) and \( P(n) = n^2 \).
- **Efficiency:** \( \frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right) \).
- **Model:** CREW.
Very simple Sorting Algorithm

- Idea: Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use \( n^2 \) processors.

**Programm: SimpleSort**

**Eingabe:** \( s_1, \ldots, s_n \).

**for all** \( P_{i,j} \) where \( 1 \leq i, j \leq n \) **do in parallel**

\[
\text{if } s_i > s_j \text{ then } P_{i,j}(1) \to R_{i,j} \text{ else } P_{i,j}(0) \to R_{i,j}
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Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(\text{EREW})}(n) = \Theta(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
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  \[
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Lower Bound

Theorem:

For any parallel sorting algorithm \( Srt \) with \( P_{Srt}(n) = O(n) \) hold:

\[
T_{Srt}(n) = \Omega(\log(n)).
\]

Proof:

- Lower bound for sequential is \( \Theta(n \log n) \).
- One needs \( O(n \log n) \) comparisons.
- In each parallel step are at most \( o(n) \) comparisons possible.
- Thus with less steps we have a contradiction to the lower bound for sequential

Situation at this point:

- Inefficient algorithms with: \( T(n) = O(\log n) \) and \( P(n) = n^2 \).
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Basic Operation for Sorting

- Identify basic operation for sorting.
- Assume: sorting key is $s_1, \cdots, s_n$.
- Programm: compare_exchange$(i,j)$
  
  \[
  \text{if } s_i > s_j \text{ then exchange } s_i \leftrightarrow s_j
  \]
- Symbolic view (Batcher):

\[
\begin{array}{c}
  y \\
  \hline \\
  \text{max}(x, y)
\end{array}
\quad
\begin{array}{c}
  \hline \\
  x \\
  \text{min}(x, y)
\end{array}
\]

- Basic building block for sorting networks.
- Base for Odd-Even merge
- Form this we build the optimal algorithm by Cole
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**Programm:** `compare_exchange(i,j)`

```plaintext
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```

- **Symbolic view (Batcher):**

```
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\end{array}
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- Symbolic view (Batcher):
  
  $$
  \begin{array}{c}
  y \\
  \hline
  max(x, y) \\
  \hline
  min(x, y) \\
  \end{array}
  $$

- Basic building block for sorting networks.
- Base for Odd-Even merge
- Form this we build the optimal algorithm by Cole
Basic Operation for Sorting

- Identify basic operation for sorting.
- Assume: sorting key is $s_1, \ldots, s_n$.
- **Programm:** `compare_exchange(i, j)`
  
  ```
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
  ```

- Symbolic view (Batcher):
  
  ```
  y ______________ max(x, y) 
  
  x ______________ min(x, y)
  ```

- Basic building block for sorting networks.
- Base for Odd-Even merge
- **Form this we build the optimal algorithm by Cole**
Odd-even Merge (Definition)

- Input: Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $\text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

$$T_{\text{interleave}}(n) = O(1) \text{ mit } P_{\text{interleave}}(n) = O(n)$$
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$T_{interleave}(n) = O(1)$ mit $P_{interleave}(n) = O(n)$
Odd-even Merge (Definition)

- **Programm: odd_even(S)**
  - for all \( i \) where \( 1 < i < n \) and \( i \) even do in parallel
    - \( \text{compare}_\text{exchange}(i, i + 1) \).

\[ S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6 \quad S_7 \quad S_8 \quad S_9 \quad S_{10} \quad S_{11} \quad S_{12} \quad S_{13} \quad S_{14} \quad S_{15} \quad S_{16} \]

\[ r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5 \quad r_6 \quad r_7 \quad r_8 \quad r_9 \quad r_{10} \quad r_{11} \quad r_{12} \quad r_{13} \quad r_{14} \quad r_{15} \quad r_{16} \]

- \( T_{\text{compare}_\text{exchange}}(n) = O(1) \) mit \( P_{\text{compare}_\text{exchange}}(n) = O(n) \)
Odd-even Merge (Definition)

- Programm: odd_even($S$)
  
  for all $i$ where $1 < i < n$ and $i$ even do in parallel
  
  compare_exchange($i, i + 1$).

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
 Odd-even Merge (Definition)

- Programm: `odd_even(S)`
  - for all `i` where `1 < i < n` and `i` even do in parallel
    - `compare_exchange(i, i + 1)`.

- `T_{compare_exchange}(n) = O(1)` mit `P_{compare_exchange}(n) = O(n)`
Odd-even Merge (Definition)

Programm: $join_1(S, S')$

\[odd\_even(\text{interleave}(S, S'))\]

$T_{join_1}(n) = O(1)$ mit $P_{join_1}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: \( \text{join1}(S, S') \)
  
  \[ \text{odd\_even}(\text{interleave}(S, S')) \]

- \( T_{\text{join1}}(n) = O(1) \) mit \( P_{\text{join1}}(n) = O(n) \)
Odd-even Merge (Definition)

- Programm: join1($S, S'$)
  
  $odd\_even(interleave(S, S'))$

- $T_{join1}(n) = O(1)$ mit $P_{join1}(n) = O(n)$
Sorting with Merging

- Programm: \( \text{odd\_even\_merge}(S, S') \)
  
  - if \(|S| = |S'| = 1\) then merge with \(\text{compare\_exchange}\).
  - \(S_{\text{odd}} = \text{odd\_even\_merge}(\text{odd}(S), \text{odd}(S'))\).
  - \(S_{\text{even}} = \text{odd\_even\_merge}(\text{even}(S), \text{even}(S'))\).
  - \(\text{return } \text{join}\_1(S_{\text{odd}}, S_{\text{even}})\).

- \(T_{\text{odd\_even\_merge}}(n) = O(\log n)\) mit \(P_{\text{odd\_even\_merge}}(n) = O(n)\)

Theorem:

The algorithm \(\text{odd\_even\_merge}\) sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

- Programm: `odd_even_merge(S, S')`
  
  \[
  \begin{array}{l}
  \text{if } |S| = |S'| = 1 \text{ then merge with } \text{compare} \_	ext{exchange}.
  \\
  S_{\text{odd}} = \text{odd} \_	ext{even} \_	ext{merge}(\text{odd}(S), \text{odd}(S')).
  \\
  S_{\text{even}} = \text{odd} \_	ext{even} \_	ext{merge}(\text{even}(S), \text{even}(S')).
  \\
  \text{return } \text{join1}(S_{\text{odd}}, S_{\text{even}}).
  \end{array}
  \]

- \(T_{\text{odd} \_	ext{even} \_	ext{merge}}(n) = O(\log n)\) mit \(P_{\text{odd} \_	ext{even} \_	ext{merge}}(n) = O(n)\)

Theorem:

The algorithm `odd_even_merge` sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

- **Programm:** odd_even_merge($S, S'$)
  
  if $|S| = |S'| = 1$ then merge with `compare_exchange`.
  
  $S_{odd} = odd\_even\_merge(odd(S), odd(S'))$.
  
  $S_{even} = odd\_even\_merge(even(S), even(S'))$.
  
  return $join1(S_{odd}, S_{even})$.

- $T_{odd\_even\_merge}(n) = O(log n)$ mit $P_{odd\_even\_merge}(n) = O(n)$

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  return `join1`(S_{odd}, S_{even}).
  ```

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  return join(S_{odd}, S_{even}).`

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Theorem:
There exists a sorting algorithm with \( T(n) = O(\log^2 n) \) and \( P(n) = n \).

Proof: use divide and conquer, and merging of depth \( O(\log n) \).

Theorem:
There exists a sorting network of size \( O(n \log^2 n) \).

Proof: All calls to \texttt{compare\_exchange} operation are independent form the input (oblivious algorithm).
Sorting Networks

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The 0-1 Principle

**Theorem:**
If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

**Proof (by contradiction):**
- Let $f(x)$ be non-decreasing function: $f(s_i) \leq f(s_j) \iff s_i \leq s_j$.
- If $X$ sorts the sequence $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$, then if $X$ gets $(f(a_1), f(a_2), \ldots, f(a_n))$ then the output $(f(b_1), f(b_2), \ldots, f(b_n))$ is also sorted.
- Assume $b_i > b_{i+1}$ and $f(b_i) \neq f(b_{i+1})$, then we have $f(b_i) > f(b_{i+1})$ in the “sorted” sequence $(f(b_1), f(b_2), \ldots, f(b_n))$. I.e errors may be kept under the function $f$.
- Choose now $f$: $f(b_j) = 0$ for $b_j < b_i$ and $f(b_j) = 1$ otherwise.
- Thus the sequence $(f(b_1), f(b_2), \ldots, f(b_n))$ is not sorted, because of $f(b_i) = 1$ and $f(b_{i+1}) = 0$.
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- Assume \( b_i > b_{i+1} \) and \( f(b_i) \neq f(b_{i+1}) \), then we have \( f(b_i) > f(b_{i+1}) \) in the “sorted” sequence \( (f(b_1), f(b_2), \cdots, f(b_n)) \). I.e errors may be kept under the function \( f \).

- Choose now \( f: f(b_j) = 0 \) for \( b_j < b_i \) and \( f(b_j) = 1 \) otherwise.

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Correctness of the Merging

Theorem:
The algorithm odd_even_merge sorts two sorted sequences into a single one.

Proof:

- $S$ has the form: $S = 0^p1^{m-p}$ for some $p$ with $0 \leq p \leq m$.
- $S'$ has the form: $S' = 0^q1^{m'-q}$ for some $q$ with $0 \leq q \leq m'$.
- Thus the sequence $S_{odd}$ has the form $0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^*$.
- And $S_{even}$ has the form $0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^*$.
- Define: $d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor)$
- Depending on $d$ we consider three cases: $d = 0$, $d = 1$ and $d = 2$. 
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Correctness of the Merging

If \( d = 0 \): Then we have: \( p \) and \( q \) are even.
- The \textit{interleave} step of \textit{join}\textsubscript{1} has the form:
  \[
  \text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{(p+q)/2} 1^{m+m'-p-q}
  \]
- The resulting sequences is already sorted.
- The \textit{compare\_exchange} step keeps the order.

If \( d = 1 \): Then we have: \( p \) is odd and \( q \) is even.
- The \textit{interleave} step of \textit{join}\textsubscript{1} has the form:
  \[
  \text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{\lfloor(p+q)/2\rfloor} 01^{m+m'-p-q}
  \]
- The resulting sequences is already sorted.

If \( d = 2 \): Then we have: \( p \) and \( q \) are odd.
- The \textit{interleave} step of \textit{join}\textsubscript{1} has the form:
  \[
  \text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{\lfloor(p+q)/2\rfloor} 101^{m+m'-p-q}
  \]
- The \textit{compare\_exchange} step will exchange the 1 on position \( 2r \) with the 0 on position \( 2r + 1 \).
Correctness of the Merging

If $d = 0$: Then we have: $p$ and $q$ are even.
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  \]
- The resulting sequences is already sorted.

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Corollary:
The correctness of a merge network may be tested in time $O(n^2)$.

Proof: Test all inputs of the form $(0^p1^{m-p}, 0^q1^{m'-q})$.

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- **Aim:** Fast optimal algorithm.
- So far $T(n) = \log^2 n$ bei $P(n) = O(n)$.
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- Idea: make one loop faster, i.e. the merging in $O(1)$.
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I.e. these sub-sequences are used as a kind of “preview”.

Using these crutch points we will be able to do the merging in $O(1)$ time.

Total running time will be $O(\log n)$.

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- Let $J$ and $K$ be two sorted sequences.
- Note: without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.
- Let $L$ be a third sequence, which will be called in the following **good sampler** for $J$ and $K$.
- Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.
- Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.
- The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.
- Notation: $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.
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- Depending on the context $\text{Rng}_{A,B}$ could also be an array with $|A|$ elements.
Definition

- Let $J$ and $K$ be two sorted sequences.
- Note: without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.
- Let $L$ be a third sequence, which will be called in the following good sampler for $J$ and $K$.
- Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.
- Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.
- The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.
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Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{\vert A \vert} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

**Definition:**

We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each forth element of \( S \).
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- Example \((k = 1)\): \(1, 2, 3, 4\).
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Merging using a Good Sampler

\[ rng(e, S) = \{|x \in S \mid x < e\} \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = rng(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Programm: \( \text{merge\_with\_help}(J, K, L) \) 
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel 
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( K_i = \{x \in K \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( res_i = \text{merge}(J_i, K_i) \).
  
  return \( (res_1, res_2, \cdots, res_s) \).

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Merging using a Good Sampler

rng(e, S) = |\{x \in S \mid x < e\}| and Rng_{A,B}: A \mapsto \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

- Let J, K and L be sorted sequences.
- Let L be a good sampler of both J and K.
- Let L = (l_1, l_2, \cdots, l_s).
- Programm: merge_with_help(J, K, L)
  for all i where 1 \leq i \leq s do in parallel
    Assign J_i = \{x \in J \mid l_{i-1} < x \leq l_i\}.
    Assign K_i = \{x \in K \mid l_{i-1} < x \leq l_i\}.
    Assign res_i = merge(J_i, K_i).
  return (res_1, res_2, \cdots, res_s).

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- Let \( J, K \) and \( L \) be sorted sequences.
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- Program: merge_with_help(\( J, K, L \))
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
    Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
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    Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).
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\text{rng}(e, S) = \left| \{ x \in S \mid x < e \} \right| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B)
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  \begin{align*}
  &\text{for all } i \text{ where } 1 \leq i \leq s \text{ do in parallel} \\
  &\quad \text{Assign } J_i = \{ x \in J \mid l_{i-1} < x \leq l_i \}. \\
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  &\quad \text{Assign } \text{res}_i = \text{merge}(J_i, K_i). \\
  &\text{return } (\text{res}_1, \text{res}_2, \cdots, \text{res}_s).
  \end{align*}

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Programm: \texttt{merge\_with\_help}(J, K, L)

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    Assign \( res_i = \text{merge}(J_i, K_i) \).
\end{verbatim}

\textbf{return} \( (res_1, res_2, \cdots, res_s) \).

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<th>( L_7 )</th>
<th>( L_8 )</th>
<th>( L_9 )</th>
</tr>
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<tbody>
<tr>
<td>( i )</td>
<td>( l_1 )</td>
<td>( l_2 )</td>
<td>( l_3 )</td>
<td>( l_4 )</td>
<td>( l_5 )</td>
<td>( l_6 )</td>
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<td></td>
</tr>
<tr>
<td>( K )</td>
<td>( K_1 )</td>
<td>( K_2 )</td>
<td>( K_3 )</td>
<td>( K_4 )</td>
<td>( K_5 )</td>
<td>( K_6 )</td>
<td>( K_7 )</td>
<td>( K_8 )</td>
<td>( K_9 )</td>
</tr>
</tbody>
</table>
Merging using a Good Sampler

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A → \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

- Let J, K and L be sorted sequences.
- Let L be a good sampler of both J and K.
- Let L = (l_1, l_2, \cdots, l_s).

Programm: merge_with_help(J, K, L)

for all i where 1 \leq i \leq s do in parallel

Assign \( J_i = \{x \in J | l_{i-1} < x \leq l_i\} \).
Assign \( K_i = \{x \in K | l_{i-1} < x \leq l_i\} \).
Assign \( res_i = merge(J_i, K_i) \).

return \( (res_1, res_2, \cdots, res_s) \).

Situation:

<table>
<thead>
<tr>
<th></th>
<th>L_1</th>
<th>L_2</th>
<th>L_3</th>
<th>L_4</th>
<th>L_5</th>
<th>L_6</th>
<th>L_7</th>
<th>L_8</th>
<th>L_9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>l_1</td>
<td>l_2</td>
<td>l_3</td>
<td>l_4</td>
<td>l_5</td>
<td>l_6</td>
<td>l_7</td>
<td>l_8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>K_1</td>
<td>K_2</td>
<td>K_3</td>
<td>K_4</td>
<td>K_5</td>
<td>K_6</td>
<td>K_7</td>
<td>K_8</td>
<td>K_9</td>
</tr>
</tbody>
</table>
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \rightarrow \mathbb{N}^{\left|A\right|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)
- Then we have:

<table>
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<tr>
<th>( i )</th>
<th>( K_i )</th>
<th>( J_i )</th>
<th>( \text{merge}(K_i, J_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 )</td>
<td>( 2 )</td>
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<td>2</td>
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<td>( 11 )</td>
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</tr>
<tr>
<td>7</td>
<td>( 20 )</td>
<td>( 21 )</td>
<td>( 22 )</td>
</tr>
</tbody>
</table>

- Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

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<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
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</table>

Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
Merging using a Good Sampler (Example)

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A → \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
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Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A ↦→ \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

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<tr>
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<td></td>
<td></td>
</tr>
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- Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
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<td></td>
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<tr>
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Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)

Then we have:

| \( i \) | \( K_i \)     | \( J_i \)     | merge(\( K_i, J_i \)) |
|-------|--------|--------|-----------------
| 1     | (1, 4) | (2, 3) |                  |
| 2     | (6, 9) | (7, 8, 10) |                |
| 3     | (11, 12) | \( \emptyset \) |              |
| 4     | (13, 16) | (14, 15, 17) |            |
| 5     | (19, 20) | (18, 21) |                  |

Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

Let $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$ and $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$.

- $K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20)$
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<td>(6, 7, 8, 9, 10)</td>
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<tr>
<td>5</td>
<td>(19, 20)</td>
<td>(18, 21)</td>
<td>(18, 19, 20, 21)</td>
</tr>
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Result: $(1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)$
Merging using a Good Sampler (Example)

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A → \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

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- J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21)
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<td>(11, 12)</td>
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</tr>
<tr>
<td>4</td>
<td>(13, 16)</td>
<td>(14, 15, 17)</td>
<td>(13, 14, 15, 16, 17)</td>
</tr>
<tr>
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<td>(18, 21)</td>
<td>(18, 19, 20, 21)</td>
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Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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<td>1</td>
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<td>(1, 2, 3, 4)</td>
</tr>
<tr>
<td>2</td>
<td>(6, 9)</td>
<td>(7, 8, 10)</td>
<td>(6, 7, 8, 9, 10)</td>
</tr>
<tr>
<td>3</td>
<td>(11, 12)</td>
<td>( \emptyset )</td>
<td>(11, 12)</td>
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<tr>
<td>4</td>
<td>(13, 16)</td>
<td>(14, 15, 17)</td>
<td>(13, 14, 15, 16, 17)</td>
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<tr>
<td>5</td>
<td>(19, 20)</td>
<td>(18, 21)</td>
<td>(18, 19, 20, 21)</td>
</tr>
</tbody>
</table>

Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( R_{L,J}, R_{L,K}, R_{K,L} \) and \( R_{J,L} \) is known, then we have:
\[ T_{\text{merge with help}(J,K,L)} = O(1) \quad \text{with} \quad P_{\text{merge with help}(J,K,L)} = O(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( R_{L,J} \) resp. \( R_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( R_{J,L} \) and \( R_{K,L} \) to know the area to write its output sequence.
Merging with good sampler (running time)

Lemma:
If $L$ is a good sampler for $K$ and $J$.
If $Rng_{L,J}$, $Rng_{L,K}$, $Rng_{K,L}$ and $Rng_{J,L}$ is known, then we have:

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$$rng(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = rng(e, B)$$
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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Merging with good sampler (running time)

\[ \text{rng}(e, S) = \left| \{ x \in S \mid x < e \} \right| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\left| A \right|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( L \) is a good sampler for \( K \) and \( J \).
If \( \text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L} \text{ and } \text{Rng}_{J,L} \) is known, then we have:
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**Proof:**

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- Each processor uses \( \text{Rng}_{L,J} \) resp. \( \text{Rng}_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( \text{Rng}_{J,L} \) and \( \text{Rng}_{K,L} \) to know the area to write its output sequence.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \rightarrow \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

**Proof:**

- Consider \( X \) as a good sampler for \( X' \).
- Any additional element makes the good sampler just “better”.

**Note:**

\( \text{merge}(X, Y) \) is not necessary a sampler for \( \text{merge}(X', Y') \).

- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
- \( Y = (1, 8) \) and \( Y' = (1, 3, 4, 8) \).
- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = | \{ x \in S \mid x < e \} | \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{\lfloor A \rfloor} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

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- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
- \( Y = (1, 8) \) and \( Y' = (1, 3, 4, 8) \).
- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

**Proof:**

- Consider \( X \) as a good sampler for \( X' \).
- Any additional element makes the good sampler just "better".

**Note:**

\( \text{merge}(X, Y) \) is not necessarily a sampler for \( \text{merge}(X', Y') \).

- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
- \( Y = (1, 8) \) and \( Y' = (1, 3, 4, 8) \).
- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

Lemma:

If $X$ is a good sampler for $X'$ and 
$Y$ is a good sampler for $Y'$, then 
merge($X$, $Y$) is a good sampler for $X'$ [resp. $Y'$].

Proof:

- Consider $X$ as a good sampler for $X'$.
- Any additional element make the good sampler just "better".

Note:

merge($X$, $Y$) is not necessary a sampler for merge($X'$, $Y'$).

- $X = (2, 7)$ and $X' = (2, 5, 6, 7)$.
- $Y = (1, 8)$ and $Y' = (1, 3, 4, 8)$.
- merge($X$, $Y$) = (1, 2, 7, 8) and merge($X'$, $Y'$) = (1, 2, 3, 4, 5, 6, 7, 8).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

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Properties of Good Samplers

Lemma:
If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{merge}(X, Y)$ is a good sampler for $X'$ [resp. $Y'$].

Proof:
- Consider $X$ as a good sampler for $X'$.
- Any additional element makes the good sampler just "better".

Note:
$\text{merge}(X, Y)$ is not necessarily a sampler for $\text{merge}(X', Y')$.
- $X = (2, 7)$ and $X' = (2, 5, 6, 7)$.
- $Y = (1, 8)$ and $Y' = (1, 3, 4, 8)$.
- $\text{merge}(X, Y) = (1, 2, 7, 8)$ and $\text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8)$.
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \] and \( \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \) with \( \text{Rng}_{A,B}(e) = \text{rng}(e, B) \)

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

**Proof:**

- Consider \( X \) as a good sampler for \( X' \).
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\( \text{merge}(X, Y) \) is not necessary a sampler for \( \text{merge}(X', Y') \).

- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
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- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- **There are 5 elements between 2 and 7.**
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).

Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:**

- W.l.o.g. contain \( X \) and \( Y \) elements \( -\infty \) and \( +\infty \).
- Let \((e_1, e_2, \cdots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be
  \[
  x = |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.
  \]
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**
Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
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- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \((e_1, e_2, \cdots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be
  \[
  x = \left| X \cap \{ e_1, e_2, \cdots, e_r \} \right| \quad \text{and} \quad
  y = \left| Y \cap \{ e_1, e_2, \cdots, e_r \} \right|.\]
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{n, A, B} : A \mapsto \mathbb{N} |A| \quad \text{with} \quad R_{n, A, B}(e) = \text{rng}(e, B) \]

Lemma:

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \). Then there are at most \( 2 \cdot r + 2 \) elements of merge\((X', Y')\) between \( r \) successive elements of merge\((X, Y)\).

Proof:

- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \( (e_1, e_2, \cdots, e_r) \) successive elements of merge\((X, Y)\).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be
  \[
  x = |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.
  \]
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:**

- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \((e_1, e_2, \ldots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be

  \[
  x = |X \cap \{e_1, e_2, \ldots, e_r\}| \quad \text{and}
  \]
  \[
  y = |Y \cap \{e_1, e_2, \ldots, e_r\}|.
  \]
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).

Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

Proof:

- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty \) and \(+\infty \).
- Let \((e_1, e_2, \cdots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be

\[
\begin{align*}
x &= |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \\
y &= |Y \cap \{e_1, e_2, \cdots, e_r\}|.
\end{align*}
\]
Properties of Good Samplers

Lemma:

Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$.
Then there are at most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$.

Proof:

- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of $\text{merge}(X, Y)$.
- W.l.o.g. let $e_1 \in X$.
- Consider now two cases: $e_r \in X$ and $e_r \in Y$.
- Let in the following be

$$x = |X \cap \{e_1, e_2, \cdots, e_r\}|$$

and

$$y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.$$
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) among the elements of merge\((X, Y)\).

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).

Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

**Example** \(x = 3\) and \(y = 2\):

\(e_1 \in X\) \hspace{1cm} e_2 \in Y\) \hspace{1cm} e_3 \in X\) \hspace{1cm} e_4 \in Y\) \hspace{1cm} e_5 \in X\)
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\).
If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).

Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

**Example** \(x = 3\) and \(y = 2\):

\[ e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).

Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

Example \(x = 3\) and \(y = 2\):

\[
\begin{align*}
a & \in Y \\
e_1 & \in X \\
e_2 & \in Y \\
e_3 & \in X \\
e_4 & \in Y \\
e_5 & \in X \\
b & \in Y
\end{align*}
\]
Properties of Good Samplers

Let \( (e_1, e_2, \ldots, e_r) \) successive elements of \( \text{merge}(X, Y) \) and \( x = |X \cap \{e_1, e_2, \ldots, e_r\}| \) and \( y = |Y \cap \{e_1, e_2, \ldots, e_r\}| \) and

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \). Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:** W.l.o.g. let \( e_1 \in X \). If: \( e_r \in Y \)

- Add \( e_0 \in Y \) with \( e_0 < e_1 \) to the good sampler.
- Add \( e_{r+1} \in X \) with \( e_r < e_{r+1} \) to the good sampler.
- The elements from \( X' \) between \( (e_1, e_2, \ldots, e_r) \) are between \( x + 1 \) elements from \( X \).
- The elements from \( Y' \) between \( (e_1, e_2, \ldots, e_r) \) are between \( y + 1 \) elements from \( Y \).
- Thus we get: \( 2x + 1 + 2y + 1 = 2r + 2 \).

**Example** \( x = 2 \) and \( y = 2 \):

\[
e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y
\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

**Example** \(x = 2\) and \(y = 2\):

\[
e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y
\]
Properties of Good Samplers

(e₁, e₂, · · · , eᵣ) successive elements of merge(X, Y) and x = |X ∩ {e₁, e₂, · · · , eᵣ}| and y = |Y ∩ {e₁, e₂, · · · , eᵣ}| and

Lemma:

Let X be a good sampler for X′ and let Y be a good sampler for Y′. Then there are at most 2 · r + 2 elements of merge(X′, Y′) between r successive elements of merge(X, Y).

Proof: W.l.o.g. let e₁ ∈ X. If: eᵣ ∈ Y

- Add e₀ ∈ Y with e₀ < e₁ to the good sampler.
- Add eᵣ₊₁ ∈ X with eᵣ < eᵣ₊₁ to the good sampler.
- The elements from X′ between (e₁, e₂, · · · , eᵣ) are between x + 1 elements from X.
- The elements from Y′ between (e₁, e₂, · · · , eᵣ) are between y + 1 elements from Y.
- Thus we get: 2x + 1 + 2y + 1 = 2r + 2.

Example x = 2 and y = 2:

e₀ ∈ Y  e₁ ∈ X  e₂ ∈ Y  e₃ ∈ X  e₄ ∈ Y  e₅ ∈ X
Properties of Good Samplers

(e₁, e₂, ⋯, e_r) successive elements of merge(X, Y) and x = |X ∩ {e₁, e₂, ⋯, e_r}| and y = |Y ∩ {e₁, e₂, ⋯, e_r}|.

Lemma:
Let X be a good sampler for X′ and let Y be a good sampler for Y′. Then there are at most 2 ⋅ r + 2 elements of merge(X′, Y′) between r successive elements of merge(X, Y).

Proof: W.l.o.g. let e₁ ∈ X. If: e_r ∈ Y
- Add e₀ ∈ Y with e₀ < e₁ to the good sampler.
- Add e_r₊₁ ∈ X with e_r < e_r₊₁ to the good sampler.
- The elements from X′ between (e₁, e₂, ⋯, e_r) are between x + 1 elements from X.
- The elements from Y′ between (e₁, e₂, ⋯, e_r) are between y + 1 elements from Y.
- Thus we get: 2x + 1 + 2y + 1 = 2r + 2.

Example x = 2 and y = 2:

\[ \begin{align*} 
  e₀ & \in Y \\
n  e₁ & \in X \\
n  e₂ & \in Y \\
n  e₃ & \in X \\
n  e₄ & \in Y \\
n  e₅ & \in X 
\end{align*} \]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

Lemma:

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \ldots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \ldots, e_r)\) are between \(y + 1\) elements from \(Y\).

Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\(e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X\)
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:

- Consider $k + 1$ successive elements $(e_1, e_2, \ldots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \ldots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \ldots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**
Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**
If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

**Proof:**
- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
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Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

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<th>$val_v$</th>
<th>Father</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>8</td>
<td>7,8</td>
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</tr>
<tr>
<td>2</td>
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Lemma:
Let $S = (b_1, b_2, \cdots, b_k)$ be a sorted sequence, then we may compute the rank of $a \in S$ in time $O(1)$ using $k$ processors.

Proof:
- Program: `rng1(a,S)`
  - for all $P_i$ where $1 \leq i \leq k$ do in parallel
    - if $b_i < a \leq b_{i+1}$ then return $i$
- Note, the program has no write-conflicts.
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Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$ in time $O(1)$ using $O(|S|)$ processors.

Proof:

- We do know $\text{Rnk}_{S,S}$, $\text{Rnk}_{S_1,S_1}$ and $\text{Rnk}_{S_2,S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
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Computing the Ranks

**Lemma:**
- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{rank}_{X',X}$ and $\text{rank}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{rank}_{X',U}$, $\text{rank}_{Y',U}$, $\text{rank}_{U,X'}$ and $\text{rank}_{U,Y'}$.

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- First we compute $\text{rank}_{X',U}$ and $\text{rank}_{Y',U}$.
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Computing the Ranks

Lemma:

- Let \( X \) be a good sampler of \( X' \).
- Let \( Y \) be a good sampler of \( Y' \).
- Let \( U = \text{merge}(X, Y) \).
- Assume \( \text{Rnk}_{X', X} \) and \( \text{Rnk}_{Y', Y} \) are known.

Then we may compute in time \( O(1) \) using \( O(|X| + |Y|) \) processors \( \text{Rnk}_{X', U} \), \( \text{Rnk}_{Y', U} \), \( \text{Rnk}_{U, X'} \) and \( \text{Rnk}_{U, Y'} \).

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- First we compute \( \text{Rnk}_{X', U} \) and \( \text{Rnk}_{Y', U} \).
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- Let $X$ be a good sampler of $X'$.
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Computing the Ranks \((Rnk_{X',U})\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(Rnk_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  Programm: \(Rnk_{X',U}\)
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
  for all \(x \in X'_i\) do
  \[\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

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for all \( i \) where \( 1 \leq i \leq k + 1 \) do in parallel
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- Let $X = (a_1, a_2, \ldots, a_k)$.
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- Note: $\text{Rnk}_{X',X}$ is known.
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- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((Rnk_{X'}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
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  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

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  \]

  

  Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X,X'})\):

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks \((\text{Rnk}_{X,X'}).\)

- Let \(a_i \in X\).
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We have \(\text{rnk}(a,S)\) and \(\text{Rnk}_{S_1,S_2}\) and \(\text{Rnk}_{S_2,S_1}\).
Computing the Ranks \((\text{Rnk}_X, X')\)

- Let \(a_i \in X\).
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- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks (Rnk_X, X')

- Let \( a_i \in X \).
- Let \( a' \) minimal element in \( X_{i+1}' \).
- The rank of \( a_i \) in \( X' \) is the same as the rank of \( a' \) in \( X' \).
- **This rank is already known.**
- This may be computed in time \( O(1) \) using one processor.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks \((Rnk_{U,X'})\)

- **Note:** \(Rnk_{U,X'}\) consists of \(Rnk\,X,X'\) and \(Rnk\,Y,Y'\).
- \(Rnk\,X,X'\) is already known.
- Still to compute: \(Rnk\,Y,Y'\).
- \(Rnk\,Y,X\) may be computed using the previous lemma.
- We compute \(rnk(a,X')\) using \(rnk(a,X)\) and \(Rnk_{X,X'}\).
- Thus we compute \(Rnk_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks \((Rnk_{U,X'})\)

- **Note:** \(Rnk_{U,X'}\) consists of \(Rnk\ X, X'\) and \(Rnk\ Y, X'\).

- \(Rnk\ X, X'\) is already known.

- Still to compute: \(Rnk\ Y, X'\).

- \(Rnk\ Y, X\) may be computed using the previous lemma.

- We compute \(rnk(a, X')\) using \(rnk(a, X)\) and \(Rnk_{X,X'}\).

- Thus we compute \(Rnk_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk } X, X'$ and $\text{Rnk } Y, X'$.
- $\text{Rnk } X, X'$ is already known.
- Still to compute: $\text{Rnk } Y, X'$.
- $\text{Rnk } Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk } X, X'$ and $\text{Rnk } Y, X'$.
Computing the Ranks (Rnk_{U,X'})

- Note: Rnk_{U,X'} consists of Rnk X, X' and Rnk Y, X'.
- Rnk X, X' is already known.
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- We compute rnk(a, X') using rnk(a, X) and Rnk_{X,X'}.
- Thus we compute Rnk_{U,X'} with O(|U|) processors and time O(1).
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- Note: \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk}_{X,X'}\) and \(\text{Rnk}_{Y,Y'}\).
- \(\text{Rnk}_{X,X'}\) is already known.
- Still to compute: \(\text{Rnk}_{Y,Y'}\).
- \(\text{Rnk}_{Y,X}\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a,X')\) using \(\text{rnk}(a,X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} \ X, X'$ and $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ X, X'$ is already known.
- Still to compute: $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).

- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \)
  and \( \text{Rnk}_{K,L} \).

- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and
  \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

Consider the step
\[ \text{merge} \_ \text{with} \_ \text{help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \):

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J}, \text{Rnk}_{L,K}, \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}\_\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))) \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
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Computing the Ranks

- Consider the step $\text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i))$:

- Using the invariant we know: $\text{Rnk}_{J, X_i}$ and $\text{Rnk}_{K, Y_i}$.

- Using the above considerations we may compute: $\text{Rnk}_{L, J}$, $\text{Rnk}_{L, K}$, $\text{Rnk}_{J, L}$ and $\text{Rnk}_{K, L}$.

- Still to be computed: $\text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))}$

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- Also easy to compute: $\text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))}$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$.
Computing the Ranks

- Consider the step \( \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \):

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- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1}))}, \text{reduce}(\text{merge}(X_i, Y_i))} \)

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Computing the Ranks

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- Known: \( Rnk_{X_{i+1}, merge(X_i,Y_i)} \) and \( Rnk_{Y_{i+1}, merge(X_i,Y_i)}. \)

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Algorithm of Cole

**Theorem:**
We may sort \( n \) values on a CREW PRAM using \( O(n) \) processors in time \( O(\log n) \).

Proof: discussed before.

**Theorem:**
We may sort \( n \) values on a EREW PRAM using \( O(n) \) processors in time \( O(\log n) \).

Proof: see literature.

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There exists a sorting network with \( O(n) \) processors and depth \( O(\log n) \).

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Literatur:

Questions

- Explain the motivation behind parallel systems.
- Explain the ideas of the different sorting algorithms.
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- Explain the different efficiency of these sorting algorithms.
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Legende

■ : Nicht relevant
■ : Grundlagen, die implizit genutzt werden
■ : Idee des Beweises oder des Vorgehens
■ : Struktur des Beweises oder des Vorgehens
■ : Vollständiges Wissen