Theory of Parallel and Distributed Systems
(WS2016/17)
Kapitel 2
Sorting with a PRAM

Walter Unger
Lehrstuhl für Informatik 1

11:52 Uhr, den 30. Januar 2017
### Very simple Algorithm (Idea)

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### Very simple Algorithm (Idea)

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The table above shows the sequence of operations for the very simple sorting algorithm.
Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.

**Programm: SimpleSort**

Eingabe: $s_1, \ldots , s_n$.

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel

if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$

for all $i$ where $1 \leq i \leq n$ do in parallel

for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel

Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.

$P_i(s_i) \rightarrow R_{q_i+1}$.

- **Complexity:** $T(n) = O(\log n)$ and $P(n) = n^2$.
- **Efficiency:** $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- **Model:** CREW.
Very simple Sorting Algorithm

- Idea: Compute the position for each element.
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Programm: SimpleSort

```
Eingabe: $s_1, \ldots, s_n$.
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for all $i$ where $1 \leq i \leq n$ do in parallel
  for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel
    Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
    $P_i(s_i) \rightarrow R_{q_i+1}$.
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Very simple Sorting Algorithm

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- Programm: SimpleSort
  
  Eingabe: $s_1, \ldots, s_n$.
  
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  for all $i$ where $1 \leq i \leq n$ do in parallel
  
  for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel
  
  Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
  
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- Complexity: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Efficiency: $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- Model: CREW.
Very simple Sorting Algorithm

- Idea: Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.
- Programming: SimpleSort
  
  **Eingabe:** $s_1, \ldots, s_n$.
  
  **for all** $P_{i,j}$ where $1 \leq i, j \leq n$ **do in parallel**
  
  if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$

  **for all** $i$ where $1 \leq i \leq n$ **do in parallel**
  
  **for all** $P_{i,j}$ where $1 \leq j \leq n$ **do in parallel**

  Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.

  $P_i(s_i) \rightarrow R_{q_i+1}$.

- Complexity: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Efficiency: $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- Model: CREW.
Very simple Sorting Algorithm

- Idea: Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use \( n^2 \) processors.

Programm: SimpleSort

Eingabe: \( s_1, \ldots, s_n \).

for all \( P_{i,j} \) where \( 1 \leq i, j \leq n \) do in parallel

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for all \( i \) where \( 1 \leq i \leq n \) do in parallel

for all \( P_{i,j} \) where \( 1 \leq j \leq n \) do in parallel

Processors \( P_{i,j} \) bestimmen \( q_i = \sum_{l=1}^{n} R_{i,l} \).

\( P_i(s_i) \rightarrow R_{q_i+1} \).

- Complexity: \( T(n) = O(\log n) \) and \( P(n) = n^2 \).
- Efficiency: \( \frac{O(n \log n)}{n^2 \cdot O(\log n)} = O(\frac{1}{n}) \).
- Model: CREW.
Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.
- **Compare** pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.

**Programm: SimpleSort**

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Very simple Sorting Algorithm

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- Program: SimpleSort
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- Model: CREW.
Very simple Sorting Algorithm

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- Complexity: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Efficiency: $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.

- Model: CREW.
**Improved Algorithm for CREW**

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$.
- Sort parallel each block.
- Merge the blocks pairwise and parallel.

- Complexity: $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.
- Efficiency: $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

- Is $O(1)$ for $P(n) \leq n/\log n$. 

Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
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Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging(EREW)}}(n) = \Theta(n / P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n / P(n) \cdot \log(n / P(n))) + O(n / P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n / P(n) + \log^2 n) \cdot \log n)$
- Efficiency:
  \[
  \text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n / P(n) + \log^2 n) \cdot \log n))}
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- Is $O(1)$ if $P(n) < n / \log^2 n$. 

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Walter Unger 30.1.2017 11:52  WS2016/17
Lower Bound

Theorem:
For any parallel sorting algorithm $Srt$ with $P_{Srt}(n) = O(n)$ hold:

$$T_{Srt}(n) = \Omega(\log(n)).$$

Proof:

- Lower bound for sequential is $\Theta(n \log n)$.
- One needs $O(n \log n)$ comparisons.
- In each parallel step are at most $o(n)$ comparisons possible.
- Thus with less steps we have a contradiction to the lower bound for sequential.

Situation at this point:

- Inefficient algorithms with: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Nearly efficient algorithm with: $T(n) = O(\log^2 n)$ and $P(n) = o(n)$. 
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Basic Operation for Sorting

- Identify basic operation for sorting.
- Assume: sorting key is $s_1, \ldots, s_n$.
- **Programm:** `compare_exchange(i, j)`
  
  ```
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
  ```

- Symbolic view (Batcher):
  ```
  y
  max(x, y)
  
  x
  min(x, y)
  ```

- Basic building block for sorting networks.
- Base for Odd-Even merge
- Form this we build the optimal algorithm by Cole
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- Symbolic view (Batcher):
  \[ \begin{array}{c}
  y \\
  \hline
  x
  \end{array} \quad \begin{array}{c}
  \max(x, y) \\
  \hline
  \min(x, y)
  \end{array} \]
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  $$
  \begin{align*}
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  y & \quad \text{max}(x, y)
  \end{align*}
  $$

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  \[
  \begin{array}{ccc}
  x & \cdots & y \\
  \mid & \cdots & \mid \\
  \end{array}
  \]

  \[
  \text{max}(x, y) \quad \text{min}(x, y)
  \]

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  \[
  \begin{array}{c}
  y \\
  \hline
  x \\
  \end{array}
  \begin{array}{c}
  \text{max}(x, y) \\
  \text{min}(x, y)
  \end{array}
  \]

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  \end{verbatim}

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  \begin{align*}
  y & \quad \max(x, y) \\
  \cline{1-2}
  x & \quad \min(x, y)
  \end{align*}

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  $$
  \begin{array}{c}
  y \\
  \hline
  \\hline
  x \\
  \hline
  \end{array}
  \quad \text{max}(x, y)
  \\
  \begin{array}{c}
  \hline
  \\hline
  \end{array}
  \quad \text{min}(x, y)
  $$

- Basic building block for sorting networks.
- Base for Odd-Even merge
- **Form this we build the optimal algorithm by Cole**
Odd-even Merge (Definition)

- **Input**: Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $interleave(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

\[ T_{interleave}(n) = O(1) \text{ mit } P_{interleave}(n) = O(n) \]
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- Then we define: \( \text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n) \).

\[
\begin{align*}
s_1 & \rightarrow r_1 \\
s_2 & \rightarrow r_2 \\
s_3 & \rightarrow r_3 \\
s_4 & \rightarrow r_4 \\
s_5 & \rightarrow r_5 \\
s_6 & \rightarrow r_6 \\
s_7 & \rightarrow r_7 \\
s_8 & \rightarrow r_8 \\
s'_1 & \rightarrow r_9 \\
s'_2 & \rightarrow r_{10} \\
s'_3 & \rightarrow r_{11} \\
s'_4 & \rightarrow r_{12} \\
s'_5 & \rightarrow r_{13} \\
s'_6 & \rightarrow r_{14} \\
s'_7 & \rightarrow r_{15} \\
s'_8 & \rightarrow r_{16}
\end{align*}
\]

- \( T_{\text{interleave}}(n) = O(1) \) mit \( P_{\text{interleave}}(n) = O(n) \)
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$T_{\text{interleave}}(n) = O(1)$ mit $P_{\text{interleave}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: odd_even(S)
  for all $i$ where $1 < i < n$ and $i$ even do in parallel
  compare_exchange($i$, $i + 1$).

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

- **Programm: odd_even(S)**
  
  for all $i$ where $1 < i < n$ and $i$ even do in parallel
  
  `compare_exchange(i, i + 1).`

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Odd-even Merge (Definition)

- **Programm: odd_even(S)**
  - for all $i$ where $1 < i < n$ and $i$ even do in parallel
    - $\text{compare\_exchange}(i, i + 1)$.

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

Programm: join1($S, S'$)
odd_even(interleave($S, S'$))

$T_{join1}(n) = O(1)$ mit $P_{join1}(n) = O(n)$
Odd-even Merge (Definition)

Programm: \( \text{join1}(S, S') \)  
\[
\text{odd\_even}(\text{interleave}(S, S'))
\]

\[ T_{\text{join1}}(n) = O(1) \text{ mit } P_{\text{join1}}(n) = O(n) \]
Odd-even Merge (Definition)

- Programm: join1($S, S'$)
  
  odd_even(interleave($S, S'$))

- $T_{join1}(n) = O(1)$ mit $P_{join1}(n) = O(n)$
Sorting with Merging

- Programm: odd_even_merge($S, S'$)
  
  if $|S| = |S'| = 1$ then merge with compare_exchange.
  
  $S_{odd} = odd\_even\_merge(odd(S), odd(S'))$.
  
  $S_{even} = odd\_even\_merge(even(S), even(S'))$.
  
  return join1($S_{odd}, S_{even}$).

- $T_{odd\_even\_merge}(n) = O(\log n)$ mit $P_{odd\_even\_merge}(n) = O(n)$

Theorem:

The algorithm $odd\_even\_merge$ sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

- Programm: `odd_even_merge(S, S')`
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  if |S| = |S'| = 1 then merge with compare_exchange.
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There exists a sorting algorithm with $T(n) = O(\log^2 n)$ and $P(n) = n$.

Proof: use divide and conquer, and merging of depth $O(\log n)$.

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There exists a sorting network of size $O(n \log^2 n)$.

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The 0-1 Principle

Theorem:

If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

Proof (by contradiction):

- Let $f(x)$ be non-decreasing function: $f(s_i) \leq f(s_j) \Leftrightarrow s_i \leq s_j$.
- If $X$ sorts the sequence $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$, then if $X$ gets $(f(a_1), f(a_2), \cdots, f(a_n))$ then the output $(f(b_1), f(b_2), \cdots, f(b_n))$ is also sorted.
- Assume $b_i > b_{i+1}$ and $f(b_i) \neq f(b_{i+1})$, then we have $f(b_i) > f(b_{i+1})$ in the “sorted” sequence $(f(b_1), f(b_2), \cdots, f(b_n))$. I.e errors may be kept under the function $f$.
- Choose now $f$: $f(b_j) = 0$ for $b_j < b_i$ and $f(b_j) = 1$ otherwise.
- Thus the sequence $(f(b_1), f(b_2), \cdots, f(b_n))$ is not sorted, because of $f(b_i) = 1$ and $f(b_{i+1}) = 0$.
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Correctness of the Merging

Theorem:
The algorithm `odd_even_merge` sorts two sorted sequences into a single one.

Proof:

- **S** has the form: $S = 0^p1^{m-p}$ for some $p$ with $0 \leq p \leq m$.
- **S’** has the form: $S’ = 0^q1^{m’-q}$ for some $q$ with $0 \leq q \leq m’$.
- Thus the sequence $S_{odd}$ has the form $0^{[p/2]+[q/2]}1^*$.
- And $S_{even}$ has the form $0^{[p/2]+[q/2]}1^*$.
- Definiere: $d = [p/2] + [q/2] - ([p/2] + [q/2])$
- Depending on $d$ we consider three cases: $d = 0$, $d = 1$ and $d = 2$. 
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- Thus the sequence $S_{odd}$ has the form $0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^*$.
- And $S_{even}$ has the form $0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^*$.
- Define: $d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor)$
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Theorem:
The algorithm \texttt{odd\_even\_merge} sorts two sorted sequences into a single one.

Proof:

\begin{itemize}
    \item $S$ has the form: $S = 0^p1^{m-p}$ for some $p$ with $0 \leq p \leq m$. 
    \item $S'$ has the form: $S' = 0^q1^{m'-q}$ for some $q$ with $0 \leq q \leq m'$. 
    \item Thus the sequence $S_{\text{odd}}$ has the form $0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^*$. 
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\end{itemize}
Correctness of the Merging

If $d = 0$: Then we have: $p$ and $q$ are even.
- The interleave step of $join1$ has the form:
  \[
  \text{interleave}(S_{odd}, S_{even}) = (00)^{(p+q)/2} 1^{m+m' - p - q}
  \]
- The resulting sequences is already sorted.
- The compare_exchange step keeps the order.

If $d = 1$: Then we have: $p$ is odd and $q$ is even.
- The interleave step of $join1$ has the form:
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If $d = 2$: Then we have: $p$ and $q$ are odd.
- The interleave step of $join1$ has the form:
  \[
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- The compare_exchange step will exchange the 1 on position $2r$ with the 0 on position $2r + 1$. 
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If $d = 0$: Then we have: $p$ and $q$ are even.

- The *interleave* step of *join1* has the form:

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$$\text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{\lfloor(p+q)/2\rfloor} 101^{m+m'-p-q}$$

- The *compare_exchange* step will exchange the 1 on position $2r$ with the 0 on position $2r + 1$. 
Correctness of the Merging

If \( d = 0 \): Then we have: \( p \) and \( q \) are even.

- The \textit{interleave} step of \textit{join1} has the form:
  \[ \text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{(p+q)/2}1^{m+m'-p-q} \]
- The resulting sequences is already sorted.
- The \textit{compare\_exchange} step keeps the order.

If \( d = 1 \): Then we have: \( p \) is odd and \( q \) is even.

- The \textit{interleave} step of \textit{join1} has the form:
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Testing the Correctness of a Network

Corollary:
The correctness of a merge network may be tested in time $O(n^2)$.

Proof: Test all inputs of the form $(0^p1^{m-p}, 0^q1^{m'-q})$.

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The test for correctness of a sorting network is NP-hard.

Proof: Literature.
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- **Aim:** Fast optimal algorithm.
- So far \( T(n) = \log^2 n \) bei \( P(n) = O(n) \).
- So far: Two loops for merging and sorting.
- Idea: make one loop faster, i.e. the merging in \( O(1) \).
- Problem: With no further information we need \( \Theta(\log n) \) steps.
- Idea: compute this additional information during the sorting.
- Choose as additional information nice splitting points for merging.
- I.e choose positions which split the blocks to be merged of constants size.
- Problem: How to compute these points?
- Solution is the base for the algorithm of Cole.
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- Before merging two sequences we will merge two sub-sequences.
- Choose as sub-sequence each $k$-th element of the original sequence.
- These sub-sequences will be used as crutch/support to do the final merging.
- I.e. these sub-sequences are used as a kind of “preview”.
- Using these crutch points we will be able to do the merging in $O(1)$ time.
- Total running time will be $O(\log n)$.
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The Merging-Tree, a View

Each processor starts with 256 elements.

has 256 → Each processor starts with 256 elements
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Each Processor starts with 256 elements

sends 4
has 256
↑ each ↑
The Merging-Tree, a View

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The Merging-Tree, a View

Each Processor starts with 256 elements

sends 4
has 4
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- Let $J$ and $K$ be two sorted sequences.

- Note: without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.

- Let $L$ be a third sequence, which will be called in the following good sampler for $J$ and $K$.

- Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.

- Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.

- The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.

- Notation: $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.

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Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Definition:**

We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each forth element of \( S \).
- Then \( S_1 \) is a good sampler of \( S \).
- Let \( S_2 \) be the sequence consisting of each second element of \( S \).
- Then \( S_1 \) is a good sampler of \( S_2 \).
- Example \((k = 1)\): 1, 2, 3, 4.
- Example \((k = 3)\): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
**Good Sampler**

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\text{rng}(e, S) = \left| \{x \in S \mid x < e\} \right| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B)
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- Example ($k = 1$): $1, 2, 3, 4$.
- Example ($k = 3$): $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. 

\[
\text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B)
\]
Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Definition:

We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

Example:

- Let \( S \) be a sorted sequence
- **Let \( S_1 \) be the sequence consisting of each forth element of \( S \).**
- Then \( S_1 \) is a good sampler of \( S \).
- Let \( S_2 \) be the sequence consisting of each second element of \( S \).
- Then \( S_1 \) is a good sampler of \( S_2 \).
- Example \((k = 1)\): 1, 2, 3, 4.
- Example \((k = 3)\): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Good Sampler

Definition:
We call $L$ a good sampler of $J$, iff:
- $L$ and $J$ are sorted.
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Example:
- Let $S$ be a sorted sequence
- Let $S_1$ be the sequence consisting of each forth element of $S$.
  - Then $S_1$ is a good sampler of $S$.
- Let $S_2$ be the sequence consisting of each second element of $S$.
  - Then $S_1$ is a good sampler of $S_2$.
- Example ($k = 1$): 1, 2, 3, 4.
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**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each fourth element of \( S \).
- Then \( S_1 \) is a good sampler of \( S \).
- Let \( S_2 \) be the sequence consisting of each second element of \( S \).
- Then \( S_1 \) is a good sampler of \( S_2 \).
- Example (\( k = 1 \)): 1, 2, 3, 4.
- Example (\( k = 3 \)): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

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**Example:**

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- Let \( S_1 \) be the sequence consisting of each forth element of \( S \).
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Merging using a Good Sampler

\[ rng(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = rng(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Programm: merge_with_help(\( J, K, L \))
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
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  Assign \( res_i = merge(J_i, K_i) \).

return \((res_1, res_2, \cdots, res_s)\).

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Merging using a Good Sampler

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- Program: \( \text{merge\_with\_help}(J, K, L) \)
  
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  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).  
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  Assign \( res_i = \text{merge}(J_i, K_i) \).

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Merging using a Good Sampler

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- Programm: \text{merge\_with\_help}(J, K, L)
  
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  return \((\text{res}_1, \text{res}_2, \ldots, \text{res}_s)\).

- Situation:

\[
\begin{array}{cccccccc}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 \\
\hline
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \\
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 \\
\end{array}
\]
Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Program: merge_with_help(\( J, K, L \))

  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
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Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \to \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Programm: \text{merge\_with\_help}(J, K, L)
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( K_i = \{x \in K \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).

return \( (\text{res}_1, \text{res}_2, \cdots, \text{res}_s) \).

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<td>( K_1 )</td>
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Merging using a Good Sampler (Example)

\[ rng(e, S) = |\{ x \in S \mid x < e \} | \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = rng(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)

Then we have:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( K_i )</th>
<th>( J_i )</th>
<th>( \text{merge}(K_i, J_i) )</th>
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</table>

Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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 américain Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
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Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
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\[
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<td>1</td>
<td>(1, 4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(6, 9)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(11, 12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(13, 16)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(19, 20)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } R\text{ng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } R\text{ng}_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)

Then we have:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( K_i )</th>
<th>( J_i )</th>
<th>( \text{merge}(K_i, J_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(6, 9)</td>
<td>(7, 8, 10)</td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>(14, 15, 17)</td>
<td></td>
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
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<td>1</td>
<td>(1, 4)</td>
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B) \]

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Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

rng(e, S) = |{x ∈ S | x < e}| and RngA,B : A ↦ □N |A| with RngA,B(e) = rng(e, B)

- K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20)
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<tr>
<th>i</th>
<th>Ki</th>
<th>Ji</th>
<th>merge(Ki, Ji)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
<td>(1, 2, 3, 4)</td>
</tr>
<tr>
<td>2</td>
<td>(6, 9)</td>
<td>(7, 8, 10)</td>
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</tr>
<tr>
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<td>(11, 12)</td>
<td>∅</td>
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- Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
Merging with good sampler (running time)

\[ r_{\text{ng}}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{\text{ng}}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{\text{ng}}_{A,B}(e) = r_{\text{ng}}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( R_{\text{ng}}_{L,J}, R_{\text{ng}}_{L,K}, R_{\text{ng}}_{K,L} \) and \( R_{\text{ng}}_{J,L} \) is known, then we have:

\[ T_{\text{merge\_with\_help}(J,K,L)} = O(1) \quad \text{with} \quad P_{\text{merge\_with\_help}(J,K,L)} = O(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( R_{\text{ng}}_{L,J} \) resp. \( R_{\text{ng}}_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( R_{\text{ng}}_{J,L} \) and \( R_{\text{ng}}_{K,L} \) to know the area to write its output sequence.
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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If \( L \) is a good sampler for \( K \) and \( J \).
If \( \text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L} \) and \( \text{Rng}_{J,L} \) is known, then we have:
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Merging with good sampler (running time)

rng(e, S) = |{x ∈ S | x < e}| and \( Rng_{A,B} : A \mapsto \mathbb{N}^{\lvert A \rvert} \) with \( Rng_{A,B}(e) = rng(e, B) \)

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If \( L \) is a good sampler for \( K \) and \( J \).
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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If \( L \) is a good sampler for \( K \) and \( J \).

If \( \text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L} \) and \( \text{Rng}_{J,L} \) is known, then we have:

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then merge\((X, Y)\) is a good sampler for \( X' \) [resp. \( Y' \)].

Proof:

- Consider \( X \) as a good sampler for \( X' \).
- Any additional element make the good sampler just ‘‘better’’.

Note:

merge\((X, Y)\) is not necessary a sampler for merge\((X', Y')\).

- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
- \( Y = (1, 8) \) and \( Y' = (1, 3, 4, 8) \).
- merge\((X, Y) = (1, 2, 7, 8) \) and merge\((X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[
\text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B)
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**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

**Proof:**

- W.l.o.g. contain \(X\) and \(Y\) elements \(-\infty\) and \(+\infty\).
- Let \((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\).
- W.l.o.g. let \(e_1 \in X\).
- Consider now two cases: \(e_r \in X\) and \(e_r \in Y\).

Let in the following be

\[
x = |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.
\]
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B) \]

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  \]
Properties of Good Samplers

\[ \text{rng} (e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B} (e) = \text{rng} (e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:**

- **W.l.o.g.** contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \( (e_1, e_2, \cdots, e_r) \) successive elements of \( \text{merge}(X, Y) \).
- **W.l.o.g.** let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be
  \[
  x = |X \cap \{e_1, e_2, \cdots, e_r\}| \text{ and } y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.
  \]
Properties of Good Samplers

Lemma:
Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$. Then there are at most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$.

Proof:
- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of $\text{merge}(X, Y)$.
- W.l.o.g. let $e_1 \in X$.
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Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).
- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

**Example** \(x = 3\) and \(y = 2\):

\[
e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X
\]
Properties of Good Samplers

((e_1, e_2, \cdots, e_r) successive elements of merge(X, Y) and x = |X \cap \{e_1, e_2, \cdots, e_r\}| and y = |Y \cap \{e_1, e_2, \cdots, e_r\}| and

Lemma:

Let X be a good sampler for X' and let Y be a good sampler for Y'. Then there are at most 2 \cdot r + 2 elements of merge(X', Y') between r successive elements of merge(X, Y).

Proof: W.l.o.g. let e_1 \in X.
If: e_r \in X

- Between e_1 and e_r are at most 2(x - 1) + 1 elements of X'.
- Between e_1 and e_r are at most 2(y + 1) + 1 elements of Y', because they are between y + 2 elements of Y.

Thus we get: 2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2.

Example x = 3 and y = 2:

\[ \begin{align*}
    e_1 & \in X & e_2 & \in Y & e_3 & \in X & e_4 & \in Y & e_5 & \in X
\end{align*} \]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

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- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).

Thus we get: 
\[2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2.\]

Example \(x = 3\) and \(y = 2\):

\[a \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \quad b \in Y\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

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Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example: \(x = 2\) and \(y = 2\):

\[
e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y
\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}| \) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}| \) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

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- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
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- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

**Example** \(x = 2\) and \(y = 2\):

\[e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y\]
Properties of Good Samplers

Let \( (e_1, e_2, \ldots, e_r) \) successive elements of merge\((X, Y)\) and \( x = |X \cap \{e_1, e_2, \ldots, e_r\}| \) and \( y = |Y \cap \{e_1, e_2, \ldots, e_r\}| \). Then there are at most \( 2 \cdot r + 2 \) elements of merge\((X', Y')\) between \( r \) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \( e_1 \in X \). If: \( e_r \in Y \)

- Add \( e_0 \in Y \) with \( e_0 < e_1 \) to the good sampler.
- Add \( e_{r+1} \in X \) with \( e_r < e_{r+1} \) to the good sampler.
- The elements from \( X' \) between \( (e_1, e_2, \ldots, e_r) \) are between \( x + 1 \) elements from \( X \).
- The elements from \( Y' \) between \( (e_1, e_2, \ldots, e_r) \) are between \( y + 1 \) elements from \( Y \).
- Thus we get: \( 2x + 1 + 2y + 1 = 2r + 2 \).

Example \( x = 2 \) and \( y = 2 \):

\[
\begin{align*}
e_0 &\in Y \\
e_1 &\in X \\
e_2 &\in Y \\
e_3 &\in X \\
e_4 &\in Y \\
e_5 &\in X
\end{align*}
\]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

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Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \ldots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \ldots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[ e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

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Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\).
Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \ldots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \ldots, e_r)\) are between \(y + 1\) elements from \(Y\).

Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):
\[
e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X
\]
Properties of good sampler

**Definition**

Let reduce\((X)\) be the operation, which chooses from \(X\) every forth element.

**Lemma:**

If \(X\) is a good sampler for \(X'\) and \(Y\) is a good sampler for \(Y'\), then \(\text{reduce(merge}(X, Y))\) is a good sampler for \(\text{reduce(merge}(X', Y'))\).

**Proof:**

- Consider \(k + 1\) successive elements \((e_1, e_2, \cdots, e_{k+1})\) of \(\text{reduce(merge}(X, Y))\).
- At most \(4k + 1\) elements of \(\text{merge}(X, Y)\) are between \(e_1, e_2, \cdots, e_{k+1}\) including \(e_1, e_{k+1}\).
- At most \(8k + 4\) elements of \(\text{merge}(X', Y')\) are between these \(4k + 1\) elements.
- At most \(2k + 1\) elements of \(\text{reduce(merge}(X', Y'))\) are between \((e_1, e_2, \cdots, e_{k+1})\).
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:

- Consider $k + 1$ successive elements $(e_1, e_2, \ldots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \ldots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \ldots, e_{k+1})$. 


Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

**Proof:**

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

**Proof:**

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
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- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
**Properties of good sampler**

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let reduce$(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then reduce$(\text{merge}(X, Y))$ is a good sampler for reduce$(\text{merge}(X', Y'))$.

**Proof:**

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of reduce$(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of reduce$(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and
$Y$ is a good sampler for $Y'$,
then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

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- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
- The leaves contain the elements to be sorted.
- Interior nodes \( v \) “cares” about as many elements as the number of leaves below \( v \).
- A node \( v \) receives from its sons sequences of already sorted sequences.
- The “length” of the sequences doubles each time.
- Node \( v \) receives sequences \( X_1, X_2, \ldots, X_r \) and \( Y_1, Y_2, \ldots, Y_r \).
- Node \( v \) sends to his father sequences \( Z_1, Z_2, \ldots, Z_r, Z_{r+1} \).
- Node \( v \) updates an interior help-sequence \( \text{val}_v \).
- It holds: \( |X_1| = |Y_1| = |Z_1| = 1 \).
- It holds: \( |X_i| = 2 \cdot |X_{i-1}|, |Y_i| = 2 \cdot |Y_{i-1}| \) and \( |Z_i| = 2 \cdot |Z_{i-1}| \).
Overview to the Algorithm of Cole

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Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
- The leaves contain the elements to be sorted.
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- The “length” of the sequences doubles each time.
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- Node $v$ sends to his father sequences $Z_1, Z_2, \ldots, Z_r, Z_{r+1}$.
- Node $v$ updates an interior help-sequence $val_v$.
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- A node \( v \) receives from its sons sequences of already sorted sequences.
- The “length” of the sequences doubles each time.
- Node \( v \) receives sequences \( X_1, X_2, \cdots, X_r \) and \( Y_1, Y_2, \cdots, Y_r \).
- Node \( v \) sends to his father sequences \( Z_1, Z_2, \cdots, Z_r, Z_{r+1} \).
- Node \( v \) updates an interior help-sequence \( val_v \).

- It holds: \( |X_1| = |Y_1| = |Z_1| = 1 \).
- It holds: \( |X_i| = 2 \cdot |X_{i-1}|, \quad |Y_i| = 2 \cdot |Y_{i-1}| \) and \( |Z_i| = 2 \cdot |Z_{i-1}| \).
Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
- The leaves contain the elements to be sorted.
- Interior nodes $v$ “cares” about as many elements as the number of leaves below $v$.
- A node $v$ receives from its sons sequences of already sorted sequences.
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- Receives from its sons the two sequences \( X \) and \( Y \).
- Computes: \( \text{val}_v = \text{merge \_ with \_ help}(X, Y, \text{val}_v) \).
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- Thus we get the following pattern:

$$
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & X_4 & \cdots & X_r \\
Z_1 & Z_2 & \cdots & Z_r & Z_{r+1} & Z_{r+2}
\end{array}
$$

- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
- Thus we get a running time of $3\log n$. 
Basic operation of an interior Node \( v \)

- Receives from its sons the two sequences \( X \) and \( Y \).
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\[
\begin{align*}
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- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
- Thus we get a running time of $3 \log n$. 
Invariant:

- Each $X_i$ is a good sampler of $X_{i+1}$.
- Each $Y_i$ is a good sampler of $Y_{i+1}$.
- Each $Z_i$ is a good sampler of $Z_{i+1}$.
- Each $X_i$ is half as big as $X_{i+1}$.
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The inner nodes $v$ need $|val_v|$ many processors.

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- In each step will compute: \(\text{merge}_\text{with}_\text{help}(X_{i+1}, Y_{i+1}, \text{merge}(X_i, Y_i))\).
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**Invariant:**

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Lemma:

Let $S = (b_1, b_2, \cdots, b_k)$ be a sorted sequence, then we may compute the rank of $a \in S$ in time $O(1)$ using $k$ processors.

Proof:

- **Programm: rng1(a,S)**
  
  for all $P_i$ where $1 \leq i \leq k$ do in parallel
  
  if $b_i < a \leq b_{i+1}$ then return $i$

- Note, the program has no write-conflicts.
- Note, it could be changed, to avoid read-conflicts.
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Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$ in time $O(1)$ using $O(|S|)$ processors.

Proof:
- We do know $\text{Rnk}_{S, S}$, $\text{Rnk}_{S_1, S_1}$ and $\text{Rnk}_{S_2, S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
- The claim follows directly.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
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Computing the Ranks

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- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- Then we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
- Finally we compute $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$
Computing the Ranks \((Rnk_{X',U})\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(Rnk_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  Programm: \(Rnk_{X',U}\)
  
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
  
  for all \(x \in X'_i\) do
  
  \[ rnk(x, U) = rnk(a_{i-1}, U) + rnk(x, U_i) \]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks ($\text{Rnk}_{X',U}$)

- Let $X = (a_1, a_2, \cdots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \cdots, X'_k, X'_{k+1}$.
- Note: $\text{Rnk}_{X',X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

Programm: $\text{Rnk}_{X',U}$
for all $i$ where $1 \leq i \leq k+1$ do in parallel
  for all $x \in X'_i$ do
    $\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)$

- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  \[
  \text{Programm: } \text{Rnk}_{X', U} \\
  \text{for all } i \text{ where } 1 \leq i \leq k + 1 \text{ do in parallel} \\
  \quad \text{for all } x \in X'_i \text{ do} \\
  \quad \quad \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
  \]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X',U})\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  \[
  \text{Programm: Rnk}_{X',U},
  \text{ for all } i \text{ where } 1 \leq i \leq k + 1 \text{ do in parallel}
  \]
  \[
  \text{ for all } x \in X'_i \text{ do}
  \]
  \[
  \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
  \]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X',U})\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let \(w.l.o.g. \ a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X_1', X_2', \ldots, X_k', X_{k+1}'\).
- Note: \(\text{Rnk}_{X',X} \) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  Programm: \(\text{Rnk}_{X',U}\)
  
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X_i'\) do
      \(\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\)

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks (Rnk_{X', U})

- Let $X = (a_1, a_2, \ldots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \ldots, X'_k, X'_{k+1}$.
- Note: Rnk_{X', X} is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

Programm: Rnk_{X', U}

for all $i$ where $1 \leq i \leq k + 1$ do in parallel

for all $x \in X'_i$ do

\[
\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
\]

- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.

we have rnk(a, S) and Rnk_{S_1, S_2} and Rnk_{S_2, S_1}
Computing the Ranks \( (\text{Rnk}_{X'}, U) \)

- Let \( X = (a_1, a_2, \cdots, a_k) \).
- Let w.l.o.g. \( a_0 = -\infty \) and \( a_{k+1} = +\infty \).
- Using a good sampler \( X \) we split \( X' \) into \( X'_1, X'_2, \cdots, X'_k, X'_{k+1} \).
- Note: \( \text{Rnk}_{X', X} \) is known.
- Splitting may be done in time \( O(1) \) using \( O(|X|) \) processors.
- Let \( U_i \) be the sequence of elements of \( Y \) which are between \( a_{i-1} \) and \( a_i \).
- Thus we get:

Programm: \( \text{Rnk}_{X', U} \)

for all \( i \) where \( 1 \leq i \leq k + 1 \) do in parallel

for all \( x \in X'_i \) do

\[ \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i) \]

- Running time \( O(1) \) using \( \sum_{i=1}^{k+1} |U_i| \) processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let \(w.l.o.g.\ a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

Programm: \(\text{Rnk}_{X', U}\)

for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

for all \(x \in X'_i\) do

\[\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\]

Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X_1', X_2', \ldots, X_k', X_{k+1}'\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  **Programm: \(\text{Rnk}_{X', U}\)**
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X_i'\) do
      \(\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\)

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.

we have \(\text{rnk}(a, S)\) and \(\text{Rnk}_{S_1,S_2}\) and \(\text{Rnk}_{S_2,S_1}\)
Computing the Ranks (\( \text{Rnk}_{X,X'} \))

- Let \( a_i \in X \).
- Let \( a' \) minimal element in \( X'_{i+1} \).
- The rank of \( a_i \) in \( X' \) is the same as the rank of \( a' \) in \( X' \).
- This rank is already known.
- This may be computed in time \( O(1) \) using one processor.

we have \( \text{rnk}(a, S) \) and \( \text{Rnk}_{s_1,s_2} \) and \( \text{Rnk}_{s_2,s_1} \)
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks ($\text{Rnk}_{X,X'}$)

- Let $a_i \in X$.
- Let $a'$ minimal element in $X_{i+1}'$.
- The rank of $a_i$ in $X'$ is the same as the rank of $a'$ in $X'$.
- This rank is already known.
- This may be computed in time $O(1)$ using one processor.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- **Note:** \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk} X, X'\) and \(\text{Rnk} Y, X'\).
- \(\text{Rnk} X, X'\) is already known.
- Still to compute: \(\text{Rnk} Y, X'\).
- \(\text{Rnk} Y, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note:** $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} \, X, \, X'$ and $\text{Rnk} \, Y, \, X'$.
- $\text{Rnk} \, X, \, X'$ is already known.
- Still to compute: $\text{Rnk} \, Y, \, X'$.
- $\text{Rnk} \, Y, \, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 
- We have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$.
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note**: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- **Still to compute**: $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$.

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- Note: \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk} X, X'\) and \(\text{Rnk} Y, X'\).
- \(\text{Rnk} X, X'\) is already known.
- Still to compute: \(\text{Rnk} Y, X'\).
- \(\text{Rnk} Y, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_X, X'\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note:** $\text{Rnk}_{U,X'}$ consists of $\text{Rnk } X, X'$ and $\text{Rnk } Y, X'$.
- $\text{Rnk } X, X'$ is already known.
- Still to compute: $\text{Rnk } Y, X'$.
- $\text{Rnk } Y, X$ may be computed using the previous lemma.
- **We compute** $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- **Thus we compute** $\text{Rnk}_{U,X'}$ with $O(\lvert U \rvert)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- Still to compute: $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$.
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]
- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).
- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

Consider the step
\[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \):

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce} \left( \text{merge}(X_{i+1}, Y_{i+1}) \right), \text{reduce} \left( \text{merge}(X_i, Y_i) \right)} \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce} \left( \text{merge}(X_i, Y_i) \right)} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce} \left( \text{merge}(X_i, Y_i) \right)} \).
- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

Consider the step
\[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

Using the invariant we know: Rnk\(_{J,X_i}\) and Rnk\(_{K,Y_i}\).

Using the above considerations we may compute: Rnk\(_{L,J}\), Rnk\(_{L,K}\), Rnk\(_{J,L}\) and Rnk\(_{K,L}\).

Still to be computed: Rnk\(_{\text{reduce(merge}(X_{i+1},Y_{i+1})),\text{reduce(merge}(X_i,Y_i)))}\)

Known: Rnk\(_{X_{i+1},\text{merge}(X_i,Y_i)}\) and Rnk\(_{Y_{i+1},\text{merge}(X_i,Y_i)}\).

It is now easy to compute: Rnk\(_{X_{i+1},\text{reduce(merge}(X_i,Y_i))}\) and Rnk\(_{Y_{i+1},\text{reduce(merge}(X_i,Y_i))}\).

Also easy to compute: Rnk\(_{\text{merge}(X_{i+1},Y_{i+1}),\text{reduce(merge}(X_i,Y_i)))}\).
Computing the Ranks

Consider the step
\( \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \):

- Using the invariant we know: \( Rnk_{J,X_i} \) and \( Rnk_{K,Y_i} \).
- Using the above considerations we may compute: \( Rnk_{L,J}, Rnk_{L,K}, Rnk_{J,L} \) and \( Rnk_{K,L} \).

Still to be computed: \( Rnk_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

Known: \( Rnk_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( Rnk_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

It is now easy to compute: \( Rnk_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( Rnk_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

Also easy to compute: \( Rnk_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).

- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).

- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: \( \text{Rnk}_{J, X_i} \) and \( \text{Rnk}_{K, Y_i} \).

- Using the above considerations we may compute: \( \text{Rnk}_{L, J} \), \( \text{Rnk}_{L, K} \), \( \text{Rnk}_{J, L} \) and \( \text{Rnk}_{K, L} \).

- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]
- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i)))} \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).
- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Algorithm of Cole

we have \( \text{rnk}(a, S) \) and \( \text{Rnk}_{S_1, S_2} \) and \( \text{Rnk}_{S_2, S_1} \)

**Theorem:**
We may sort \( n \) values on a CREW PRAM using \( O(n) \) processors in time \( O(\log n) \).

Proof: discussed before.

**Theorem:**
We may sort \( n \) values on a EREW PRAM using \( O(n) \) processors in time \( O(\log n) \).

Proof: see literature.

**Theorem:**
There exists a sorting network with \( O(n) \) processors and depth \( O(\log n) \).

Proof: see literature.
Algorithm of Cole

Theorem:

We may sort $n$ values on a CREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: discussed before.

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Literature

Questions

- Explain the motivation behind parallel systems.
- Explain the ideas of the different sorting algorithms.
- Explain the different running times of these sorting algorithms.
- Explain the different efficiency of these sorting algorithms.
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Legende

■ : Nicht relevant
■ : Grundlagen, die implizit genutzt werden
■ : Idee des Beweises oder des Vorgehens
■ : Struktur des Beweises oder des Vorgehens
■ : Vollständiges Wissen