Kapitel 2
Sorting with a PRAM

Walter Unger

Lehrstuhl für Informatik 1

8:51 Uhr, den 28. November 2016
Inhalt 1

1. Sorting
   - Simple Sorting Algorithm
   - Improved Algorithm

2. Introduction to optimal Sorting

3. Algorithmn of Cole
   - Lower Bound
   - Batchers Sorting Algorithm
   - Sorting
   - Idea
Very simple Algorithm (Idea)

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| 34 | 12 | 14 | 56 | 23 | 67 | 49 | 27 | 61 | 52 | 57 | 59 | 26 | 41 | 33 | 22 |
### Very simple Algorithm (Idea)

| 22 | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 3   | 12  |
| 33 | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 7   | 14  |
| 41 | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 9   | 22  |
| 26 | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 5   | 23  |
| 59 | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 14  | 26  |
| 57 | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 13  | 27  |
| 52 | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 11  | 33  |
| 61 | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 15  | 34  |
| 27 | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 6   | 41  |
| 49 | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 10  | 49  |
| 67 | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 16  | 52  |
| 23 | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 4   | 56  |
| 56 | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 1   | 1   | 12  | 57  |
| 14 | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 59  |
| 12 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 61  |
| 34 | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 8   | 67  |

| 34 | 12 | 14 | 56 | 23 | 67 | 49 | 27 | 61 | 52 | 57 | 59 | 26 | 41 | 33 | 22 |
Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.

**Programm: SimpleSort**

Eingabe: $s_1, \ldots, s_n$.

for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$

for all $i$ where $1 \leq i \leq n$ do in parallel
    for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel
        Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
        $P_i(s_i) \rightarrow R_{q_i+1}$.

- **Complexity:** $T(n) = O(\log n)$ and $P(n) = n^2$.
- **Efficiency:** $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- **Model:** CREW.
Very simple Sorting Algorithm

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- **Compare pairwise all elements and count the number of smaller elements.**
- **Use** $n^2$ **processors.**
- **Programm: SimpleSort**
  - **Eingabe:** $s_1, \ldots, s_n$.
  - **for all** $P_{i,j}$ **where** $1 \leq i, j \leq n$ **do in parallel**
    - if $s_i > s_j$ **then** $P_{i,j}(1) \rightarrow R_{i,j}$ **else** $P_{i,j}(0) \rightarrow R_{i,j}$
  - **for all** $i$ **where** $1 \leq i \leq n$ **do in parallel**
    - **for all** $P_{i,j}$ **where** $1 \leq j \leq n$ **do in parallel**
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Eingabe: $s_1, \ldots, s_n$.

\begin{align*}
\text{for all } P_{i,j} \text{ where } 1 \leq i, j \leq n \text{ do in parallel} & \\
\quad \text{if } s_i > s_j \text{ then } P_{i,j}(1) \rightarrow R_{i,j} \text{ else } P_{i,j}(0) \rightarrow R_{i,j} & \\
\text{for all } i \text{ where } 1 \leq i \leq n \text{ do in parallel} & \\
\quad \text{for all } P_{i,j} \text{ where } 1 \leq j \leq n \text{ do in parallel} & \\
\quad \quad \text{Processors } P_{i,j} \text{ bestimmen } q_i = \sum_{l=1}^{n} R_{i,l}. & \\
\quad P_i(s_i) \rightarrow R_{q_i+1}. &
\end{align*}

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- Model: CREW.
Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$.
- Sort parallel each block.
- Merge the blocks pairwise and parallel.

Complexity: $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.

Efficiency: $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

Is $O(1)$ for $P(n) \leq n / \log n$. 
Improved Algorithm for CREW

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- Work with $P(n)$ processors ($P(n) \leq n$).
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- Efficiency: $Eff(n) =$

$$\frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$$

- Is $O(1)$ for $P(n) \leq n/\log n$. 

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Walter Unger 28.11.2016 8:51  WS2016/17
Improved Algorithm for CREW

- Work with \( P(n) \) processors \( (P(n) \leq n) \).
- Split the input in blocks of size \( O(n/P(n)) \). \( \mathcal{O}(1) \)
- Sort parallel each block.
- Merge the blocks pairwise and parallel.

- Complexity: \( T(n) = O(n/P(n) \cdot \log n + \log^2 n) \).
- Efficiency: \( \text{Eff}(n) = \)
  \[
  \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}
  \]

- Is \( \mathcal{O}(1) \) for \( P(n) \leq n/ \log n \).
Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$.  $O(1)$
- Sort parallel each block.  $O(n/P(n) \cdot \log(n/P(n)))$
- Merge the blocks pairwise and parallel.

Complexity:  $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.
Efficiency:  $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

Is $O(1)$ for $P(n) \leq n/\log n$. 

Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$. $O(1)$
- Sort parallel each block. $O(n/P(n) \cdot \log(n/P(n)))$
- Merge the blocks pairwise and parallel. $O(n/P(n) + \log n) \cdot O(\log P(n))$

- Complexity: $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.
- Efficiency: $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

- Is $O(1)$ for $P(n) \leq n/\log n$. 

Improved Algorithm for CREW

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- Split the input in blocks of size $O(n/P(n))$. $O(1)$
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- Merge the blocks pairwise and parallel. $O(n/P(n) + \log n) \cdot O(\log P(n))$

**Complexity:** $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.

**Efficiency:** $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n) = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

- Is $O(1)$ for $P(n) \leq n/\log n$. 
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- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$. $O(1)$
- Sort parallel each block. $O(n/P(n) \cdot \log(n/P(n)))$
- Merge the blocks pairwise and parallel. $O(n/P(n) + \log n) \cdot O(\log P(n))$

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- Is $O(1)$ for $P(n) \leq n/\log n$. 
Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(\text{EREW})}(n) = \text{lsO}(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:
  $$\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}$$
- Is $O(1)$ if $P(n) < n/\log^2 n$. 
Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}}(EREW)(n) = \Omega(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n))) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:
  \[
  \text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}
  \]
- Is $O(1)$ if $P(n) < n/\log^2 n$. 
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- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:

$$\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}$$

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- Exchange the merge algorithm.
- Recall $T_{\text{Merging(EREW)}}(n) = \text{lsO}(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:
  $$\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}$$
- Is $O(1)$ if $P(n) < n/\log^2 n$. 
Improved Algorithm EREW

- Exchange the merge algorithm.
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\[
\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}
\]

- Is \( O(1) \) if \( P(n) < n/\log^2 n \).
Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}}(\text{EREW})(n) = \Theta(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:

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- Is $O(1)$ if $P(n) < n/\log^2 n$. 
Lower Bound

Theorem:
For any parallel sorting algorithm $Srt$ with $P_{Srt}(n) = O(n)$ hold:

$$T_{Srt}(n) = \Omega(\log(n)).$$

Proof:
- Lower bound for sequential is $\Theta(n \log n)$.
- One needs $O(n \log n)$ comparisons.
- In each parallel step are at most $o(n)$ comparisons possible.
- Thus with less steps we have a contradiction to the lower bound for sequential.

Situation at this point:
- Inefficient algorithms with: $T(n) = O(\log n)$ and $P(n) = n^2$.
- Nearly efficient algorithm with: $T(n) = O(\log^2 n)$ and $P(n) = o(n)$. 
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Basic Operation for Sorting

- Identify basic operation for sorting.
- Assume: sorting key is $s_1, \ldots, s_n$.
- Program: `compare_exchange(i, j)
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
- Symbolic view (Batcher):
  - $y$ \[ \max(x, y) \]
  - $x$ \[ \min(x, y) \]
- Basic building block for sorting networks.
- Base for Odd-Even merge
- Form this we build the optimal algorithm by Cole
Basic Operation for Sorting

- Identify basic operation for sorting.
- **Assume:** sorting key is $s_1, \cdots, s_n$.
- **Programm:** 
  ```java
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
  ```
- **Symbolic view (Batcher):**
  
  $$
  \begin{array}{c}
  x \\
  \hline
  y
  \end{array}
  \overset{\text{max}(x, y)}{\rightarrow}
  \begin{array}{c}
  x \\
  \hline
  y
  \end{array}
  \overset{\text{min}(x, y)}{\rightarrow}
  $$

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  ```plaintext
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
  ```
- Symbolic view (Batcher):
  ```plaintext
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  |
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  ```
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**Programm:** \texttt{compare\_exchange(i,j)}

\begin{tabular}{l}
\textbf{if} $s_i > s_j \textbf{ then exchange } s_i \leftrightarrow s_j
\end{tabular}

- Symbolic view (Batcher):

  $$
  \begin{array}{c}
  y \\
  \hline
  \text{max}(x, y)
  \end{array}
  \quad
  \begin{array}{c}
  x \\
  \hline
  \text{min}(x, y)
  \end{array}
  $$

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Basic Operation for Sorting

- Identify basic operation for sorting.
- Assume: sorting key is $s_1, \ldots, s_n$.
- Programm: compare_exchange($i,j$)
  - if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$

Symbolic view (Batcher):

```
    y  max(x, y)
     ---
    x  min(x, y)
```

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  ```
  if $s_i > s_j$ then exchange $s_i \leftrightarrow s_j$
  ```

- **Symbolic view (Batcher)**:

  \[
  \begin{array}{c}
  x \\
  \hline
  y \\
  \end{array}
  \quad \begin{array}{c}
  \text{max}(x, y) \\
  \hline
  \text{min}(x, y)
  \end{array}
  \]

- Basic building block for sorting networks.
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- **Form this we build the optimal algorithm by Cole**
Odd-even Merge (Definition)

- **Input:** Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $interleave(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

\[
\begin{align*}
&\text{input: sequence } S = (s_1, s_2, \cdots, s_n) \\
&\text{let } Odd(S) = \{s_{2i-1} : i = 1, 2, \cdots, n/2\} \text{ and } Even(S) = \{s_{2i} : i = 1, 2, \cdots, n/2\} \\
&\text{let } S' = (s'_1, s'_2, \cdots, s'_n) \\
&\text{define: } interleave(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)
\end{align*}
\]

- $T_{interleave}(n) = O(1)$ mit $P_{interleave}(n) = O(n)$
Odd-even Merge (Definition)

- **Input:** Sequence \( S = (s_1, s_2, \cdots, s_n) \). (O.E.d.A. \( n \) even)
- Let \( Odd(S) \) [\( Even(S) \)] be the elements of \( S \) with odd [even] index.
- Let \( S' = (s'_1, s'_2, \cdots, s'_n) \) be a second sequence.
- Then we define: \( \text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n) \).

\[
\begin{align*}
& \text{s}_1 \quad \text{s}_2 \quad \text{s}_3 \quad \text{s}_4 \quad \text{s}_5 \quad \text{s}_6 \quad \text{s}_7 \quad \text{s}_8 \\
& \text{s'}_1 \quad \text{s'}_2 \quad \text{s'}_3 \quad \text{s'}_4 \quad \text{s'}_5 \quad \text{s'}_6 \quad \text{s'}_7 \quad \text{s'}_8 \\
& \text{r}_1 \quad \text{r}_2 \quad \text{r}_3 \quad \text{r}_4 \quad \text{r}_5 \quad \text{r}_6 \quad \text{r}_7 \quad \text{r}_8 \quad \text{r}_9 \quad \text{r}_{10} \quad \text{r}_{11} \quad \text{r}_{12} \quad \text{r}_{13} \quad \text{r}_{14} \quad \text{r}_{15} \quad \text{r}_{16}
\end{align*}
\]

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\[
\begin{array}{cccccccccccccccc}
s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s'_1 & s'_2 & s'_3 & s'_4 & s'_5 & s'_6 & s'_7 & s'_8 \\
\downarrow & & & & & & & & \downarrow & & & & & & & \\
\downarrow & & & & & & & & \downarrow & & & & & & & \\
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 & r_{10} & r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16}
\end{array}
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- Then we define: \( \text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n) \).

\[
\begin{array}{ccccccccccccccc}
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\downarrow & & & & & & & & & & & & & & & & \\
& r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 & r_{10} & r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16}
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$T_{\text{interleave}}(n) = O(1)$ mit $P_{\text{interleave}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: odd_even(S)
  
  for all \( i \) where \( 1 < i < n \) and \( i \) even do in parallel
  
  compare_exchange\((i, i + 1)\).

\[
\begin{align*}
S_1 & | S_2 & | S_3 & | S_4 & | S_5 & | S_6 & | S_7 & | S_8 & | S_9 & | S_{10} & | S_{11} & | S_{12} & | S_{13} & | S_{14} & | S_{15} & | S_{16} \\
r_1 & | r_2 & | r_3 & | r_4 & | r_5 & | r_6 & | r_7 & | r_8 & | r_9 & | r_{10} & | r_{11} & | r_{12} & | r_{13} & | r_{14} & | r_{15} & | r_{16}
\end{align*}
\]

- \( T_{\text{compare\_exchange}}(n) = O(1) \) mit \( P_{\text{compare\_exchange}}(n) = O(n) \)
Odd-even Merge (Definition)

- **Programm: odd_even(S)**
  
  for all $i$ where $1 < i < n$ and $i$ even do in parallel
  
  \[\text{compare\_exchange}(i, i + 1).\]

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: `odd_even(S)`
  - for all $i$ where $1 < i < n$ and $i$ even do in parallel
    - `compare_exchange(i, i + 1)`.

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: join1(S, S')

\[
\text{odd\_even(}\text{interleave}(S, S')\text{)}
\]

- \(T_{\text{join1}}(n) = O(1)\) mit \(P_{\text{join1}}(n) = O(n)\)
Odd-even Merge (Definition)

- Programm: \( \text{join1}(S, S') \)
  \( odd\_even(\text{interleave}(S, S')) \)

\[
\begin{align*}
S_1 & \quad S_2 & \quad S_3 & \quad S_4 & \quad S_5 & \quad S_6 & \quad S_7 & \quad S_8 & \quad S_9 & \quad S_{10} & \quad S_{11} & \quad S_{12} & \quad S_{13} & \quad S_{14} & \quad S_{15} & \quad S_{16} \\
\hline
r_1 & \quad r_2 & \quad r_3 & \quad r_4 & \quad r_5 & \quad r_6 & \quad r_7 & \quad r_8 & \quad r_9 & \quad r_{10} & \quad r_{11} & \quad r_{12} & \quad r_{13} & \quad r_{14} & \quad r_{15} & \quad r_{16}
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Odd-even Merge (Definition)

- Programm: join1(S, S')
  - odd_even(interleave(S, S'))

- $T_{join1}(n) = O(1)$ mit $P_{join1}(n) = O(n)$
Sorting with Merging

- Programm: odd_even_merge($S, S'$)
  
  if $|S| = |S'| = 1$ then merge with compare_exchange.
  
  $S_{odd} = odd\_even\_merge(odd(S), odd(S'))$.
  
  $S_{even} = odd\_even\_merge(even(S), even(S'))$.
  
  return $\text{join}_1(S_{odd}, S_{even})$.

- $T_{odd\_even\_merge}(n) = O(\log n)$ mit $P_{odd\_even\_merge}(n) = O(n)$

Theorem:

The algorithm $odd\_even\_merge$ sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

- **Programm: odd_even_merge(S, S')**
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  - return `join1(S_{odd}, S_{even})`.

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  \begin{align*}
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  &S_{\text{odd}} = \texttt{odd\_even\_merge}(\text{odd}(S), \text{odd}(S')). \\
  &S_{\text{even}} = \texttt{odd\_even\_merge}(\text{even}(S), \text{even}(S')). \\
  &\text{return } \texttt{join1}(S_{\text{odd}}, S_{\text{even}}).
  \end{align*}

- \(T_{\text{odd\_even\_merge}}(n) = O(\log n)\) mit \(P_{\text{odd\_even\_merge}}(n) = O(n)\)

Theorem:

The algorithm \texttt{odd\_even\_merge} sorts two already sorted sequences into one.

Proof follows.
Sorting Networks

Theorem:
There exists a sorting algorithm with $T(n) = O(\log^2 n)$ and $P(n) = n$.

Proof: use divide and conquer, and merging of depth $O(\log n)$.

Theorem:
There exists a sorting network of size $O(n \log^2 n)$.

Proof: All calls to compare_exchange operation are independent form the input (oblivious algorithm).
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**Theorem:**
There exists a sorting network of size $O(n \log^2 n)$.

Proof: All calls to `compare_exchange` operation are independent form the input (oblivious algorithm).
The 0-1 Principle

Theorem:
If a sorting network \( X \), resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

Proof (by contradiction):

- Let \( f(x) \) be non-decreasing function: \( f(s_i) \leq f(s_j) \iff s_i \leq s_j \).
- If \( X \) sorts the sequence \((a_1, a_2, \cdots, a_n)\) to \((b_1, b_2, \cdots, b_n)\), then if \( X \) gets \((f(a_1), f(a_2), \cdots, f(a_n))\) then the output \((f(b_1), f(b_2), \cdots, f(b_n))\) is also sorted.
- Assume \( b_i > b_{i+1} \) and \( f(b_i) \neq f(b_{i+1}) \), then we have \( f(b_i) > f(b_{i+1}) \) in the “sorted” sequence \((f(b_1), f(b_2), \cdots, f(b_n))\). I.e errors may be kept under the function \( f \).
- Choose now \( f: f(b_j) = 0 \) for \( b_j < b_i \) and \( f(b_j) = 1 \) otherwise.
- Thus the sequence \((f(b_1), f(b_2), \cdots, f(b_n))\) is not sorted, because of \( f(b_i) = 1 \) and \( f(b_{i+1}) = 0 \).
- This is a contradiction.
The 0-1 Principle

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If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

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- Let $f(x)$ be non-decreasing function: $f(s_i) \leq f(s_j) \iff s_i \leq s_j$.
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The 0-1 Principle

Theorem:
If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

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Correctness of the Merging

Theorem:
The algorithm odd_even_merge sorts two sorted sequences into a single one.

Proof:

- $S$ has the form: $S = 0^p1^{m-p}$ for some $p$ with $0 \leq p \leq m$.
- $S'$ has the form: $S' = 0^q1^{m'-q}$ for some $q$ with $0 \leq q \leq m'$.
- Thus the sequence $S_{odd}$ has the form $0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^*$.
- And $S_{even}$ has the form $0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^*$.
- Definiere: $d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor)$.
- Depending on $d$ we consider three cases: $d = 0$, $d = 1$ and $d = 2$. 
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If \( d = 0 \): Then we have: \( p \) and \( q \) are even.

- The \textit{interleave} step of \textit{join1} has the form:

\[
\text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{(p+q)/2} 1^{m+m'-p-q}
\]

- The resulting sequences is already sorted.
- The \textit{compare\_exchange} step keeps the order.

If \( d = 1 \): Then we have: \( p \) is odd and \( q \) is even.

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**Corollary:**
The correctness of a merge network may be tested in time $O(n^2)$.

Proof: Test all inputs of the form $(0^p1^{m-p}, 0^q1^{m'-q})$.

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The test for correctness of a sorting network is NP-hard.

Proof: Literature.
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- **Aim**: Fast optimal algorithm.
- So far \( T(n) = \log^2 n \) bei \( P(n) = O(n) \).
- So far: Two loop for merging and sorting.
- Idea: make one loop faster, i.e. the merging in \( O(1) \).
- Problem: With no further information we need \( \Theta(\log n) \) steps.
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So far: Two loop for merging and sorting.

Idea: make one loop faster, i.e. the merging in \( O(1) \).

Problem: With no further information we need \( \Theta(\log n) \) steps.

Idea: compute this additional information during the sorting.

Choose as additional information nice splitting points for merging.

I.e choose positions which split the blocks to be merged of constants size.

Problem: How to compute these points?

Solution is the base for the algorithm of Cole.
Situation

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- Before merging two sequences we will merge two sub-sequences.
- Choose as sub-sequence each $k$-th element of the original sequence.
- These sub-sequences will be used as crutch/support to do the final merging.
- I.e. these sub-sequences are used as a kind of “preview”.
- Using these crutch points we will be able to do the merging in $O(1)$ time.
- Total running time will be $O(\log n)$.
- The additional effort should be at most $O(1)$. 
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Each Prozessor starts with 256 elements
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Each processor starts with 256 elements:

- Sends 4
- Has 4
- Sends 16
- Has 256
- ↑ Each ↑

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Each Processor starts with 256 elements

sends 4 →
has 4 →
sends 16 →
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Let $L$ be a third sequence, which will be called in the following good sampler for $J$ and $K$.

Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.

Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.

The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.

Notation: $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.

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Depending on the context $\text{Rng}_{A,B}$ could also be an array with $|A|$ elements.
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } R_{A,B} : A \mapsto \mathbb{N}^{\lvert A \rvert} \text{ with } R_{A,B}(e) = \text{rng}(e, B) \]

**Definition:**

We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each forth element of \( S \).
- Then \( S_1 \) is a good sampler of \( S \).
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- Example \((k = 1)\): 1, 2, 3, 4.
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- **Example \((k = 3)\):** 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Good Sampler

Definition:

We call $L$ a good sampler of $J$, iff:

- $L$ and $J$ are sorted.
- Between any $k + 1$ succeeding elements of $\{-\infty\} \cup L \cup \{+\infty\}$ are at most $2 \cdot k + 1$ many elements in $J$.

Example:

- Let $S$ be a sorted sequence
- Let $S_1$ be the sequence consisting of each forth element of $S$.
- Then $S_1$ is a good sampler of $S$.
- Let $S_2$ be the sequence consisting of each second element of $S$.
- Then $S_1$ is a good sampler of $S_2$.
- Example ($k = 1$): $1, 2, 3, 4$.
- Example ($k = 3$): $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. 

$$rng(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{\left|A\right|} \text{ with } Rng_{A,B}(e) = rng(e, B)$$
Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Program: merge_with_help(\( J, K, L \))
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  Assign \( K_i = \{x \in K \mid l_{i-1} < x \leq l_i\} \).
  Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).

return \((\text{res}_1, \text{res}_2, \cdots, \text{res}_s)\).

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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

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  return \( (res_1, res_2, \cdots, res_s) \).

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\[
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\hline
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 \\
\hline
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & \\
\hline
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 \\
\hline
\end{array}
\]
Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \ldots, l_s) \).
- Programm: \( \text{merge\_with\_help}(J, K, L) \)
  - for all \( i \) where \( 1 \leq i \leq s \) do in parallel
    - Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
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Merging using a Good Sampler

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- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \ldots, l_s) \).
- Programm: `merge_with_help(J, K, L)`
  - for all \( i \) where \( 1 \leq i \leq s \) do in parallel
    - Assign \( J_i = \{ x \in J \mid l_{i-1} < x \leq l_i \} \).
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Let $J$, $K$ and $L$ be sorted sequences.

Let $L$ be a good sampler of both $J$ and $K$.

Let $L = (l_1, l_2, \cdots, l_s)$.

Programm: $\text{merge\_with\_help}(J, K, L)$

\begin{itemize}
  \item for all $i$ where $1 \leq i \leq s$ do in parallel
    \begin{itemize}
      \item Assign $J_i = \{x \in J \mid l_{i-1} < x \leq l_i\}$.
      \item Assign $K_i = \{x \in K \mid l_{i-1} < x \leq l_i\}$.
      \item Assign $\text{res}_i = \text{merge}(J_i, K_i)$.
    \end{itemize}
  \end{itemize}

return $(\text{res}_1, \text{res}_2, \cdots, \text{res}_s)$.

Situation:
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = \{|x \in S \mid x < e\} \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
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Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
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- $K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20)$
- $J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21)$
- $L = (5, 10, 12, 17)$

Then we have:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$K_i$</th>
<th>$J_i$</th>
<th>merge($K_i, J_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
<td>(1, 2, 3, 4)</td>
</tr>
<tr>
<td>2</td>
<td>(6, 9)</td>
<td>(7, 8, 10)</td>
<td>(6, 7, 8, 9, 10)</td>
</tr>
<tr>
<td>3</td>
<td>(11, 12)</td>
<td>$\emptyset$</td>
<td>(11, 12)</td>
</tr>
<tr>
<td>4</td>
<td>(13, 16)</td>
<td>(14, 15, 17)</td>
<td>(13, 14, 15, 16, 17)</td>
</tr>
<tr>
<td>5</td>
<td>(19, 20)</td>
<td>(18, 21)</td>
<td>(18, 19, 20, 21)</td>
</tr>
</tbody>
</table>

Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)

Then we have:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( K_i )</th>
<th>( J_i )</th>
<th>( \text{merge}(K_i, J_i) )</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
<td>(1, 2, 3, 4)</td>
</tr>
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<td>(6, 9)</td>
<td>(7, 8, 10)</td>
<td>(6, 7, 8, 9, 10)</td>
</tr>
<tr>
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<td>(11, 12)</td>
<td>( \emptyset )</td>
<td>(11, 12)</td>
</tr>
<tr>
<td>4</td>
<td>(13, 16)</td>
<td>(14, 15, 17)</td>
<td>(13, 14, 15, 16, 17)</td>
</tr>
<tr>
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<td>(19, 20)</td>
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<td>(18, 19, 20, 21)</td>
</tr>
</tbody>
</table>

Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{\text{ng}}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{\text{ng}}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( R_{\text{ng}}_{L,J}, R_{\text{ng}}_{L,K}, R_{\text{ng}}_{K,L} \) and \( R_{\text{ng}}_{J,L} \) is known, then we have:

\[ T_{\text{merge\_with\_help}(J,K,L)} = \mathcal{O}(1) \quad \text{with} \quad P_{\text{merge\_with\_help}(J,K,L)} = \mathcal{O}(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( R_{\text{ng}}_{L,J} \) resp. \( R_{\text{ng}}_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( R_{\text{ng}}_{J,L} \) and \( R_{\text{ng}}_{K,L} \) to know the area to write its output sequence.
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\lfloor |A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( \text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L} \) and \( \text{Rng}_{J,L} \) is known, then we have:
\[ T_{\text{merge\_with\_help}(J,K,L)} = O(1) \text{ with } P_{\text{merge\_with\_help}(J,K,L)} = O(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( \text{Rng}_{L,J} \) resp. \( \text{Rng}_{L,K} \) to know the area to read its input sequences.
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Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

If \( L \) is a good sampler for \( K \) and \( J \).

If \( R_{L,J}, R_{L,K}, R_{K,L} \) and \( R_{J,L} \) is known, then we have:

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Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

Proof:
- Consider \( X \) as a good sampler for \( X' \).
- Any additional element makes the good sampler just "better".

Note:

\( \text{merge}(X, Y) \) is not necessarily a sampler for \( \text{merge}(X', Y') \).
- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
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- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

$$\text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B}: A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B)$$

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{merge}(X, Y)$ is a good sampler for $X'$ [resp. $Y'$].

**Proof:**

- Consider $X$ as a good sampler for $X'$.
- Any additional element make the good sampler just "better".

**Note:**

$\text{merge}(X, Y)$ is not necessarily a sampler for $\text{merge}(X', Y')$.

- $X = (2, 7)$ and $X' = (2, 5, 6, 7)$.
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad Rng_{A,B}(e) = \text{rng}(e, B) \]

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } R_{n,g_{A,B}} : A \mapsto \mathbb{N}_{|A|} \text{ with } R_{n,g_{A,B}}(e) = \text{rng}(e, B) \]

Lemma:

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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } R_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } R_{A,B}(e) = \text{rng}(e, B) \]

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \). Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:**

- W.l.o.g. contain \( X \) and \( Y \) elements \( -\infty \) and \( +\infty \).
- Let \((e_1, e_2, \ldots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be

\[
\begin{align*}
  x &= \mid X \cap \{e_1, e_2, \ldots, e_r\} \mid \quad \text{and} \\
  y &= \mid Y \cap \{e_1, e_2, \ldots, e_r\} \mid.
\end{align*}
\]
Properties of Good Samplers

\[ rng(e, S) = \{|x \in S \mid x < e\} \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = rng(e, B) \]

**Lemma:**
Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
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**Proof:**
1. W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
2. Let \((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\).
3. W.l.o.g. let \( e_1 \in X \).
4. Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
5. Let in the following be

\[
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    x &= |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \\
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Properties of Good Samplers

Lemma:

Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$.
Then there are at most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$
successive elements of $\text{merge}(X, Y)$.

Proof:

- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of $\text{merge}(X, Y)$.
- W.l.o.g. let $e_1 \in X$.
- Consider now two cases: $e_r \in X$ and $e_r \in Y$.
- Let in the following be

$$x = |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.$$
Properties of Good Samplers

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Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$.
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Proof:
- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of $\text{merge}(X, Y)$.
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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \). Then there are at most \( 2 \cdot r + 2 \) elements of merge\((X', Y')\) between \( r \) successive elements of merge\((X, Y)\).

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Properties of Good Samplers

\[
\text{rng}(e, S) = \left| \{ x \in S \mid x < e \} \right| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } R_{A,B}(e) = \text{rng}(e, B)
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- W.l.o.g. let \( e_1 \in X \).
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- Let in the following be

  \[
  x = \left| X \cap \{ e_1, e_2, \cdots, e_r \} \right| \quad \text{and} \quad y = \left| Y \cap \{ e_1, e_2, \cdots, e_r \} \right|.
  \]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\)

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).
- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

**Example** \(x = 3\) and \(y = 2\):  
\[
e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X
\]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

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- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

**Example** \(x = 3\) and \(y = 2\):

\[e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) are

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).
- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

Example \(x = 3\) and \(y = 2\):

\[a \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \quad b \in Y\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and...

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

**Example** \(x = 2\) and \(y = 2\):

\[ e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

Lemma:

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
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- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

- \(e_0 \in Y\)
- \(e_1 \in X\)
- \(e_2 \in Y\)
- \(e_3 \in X\)
- \(e_4 \in Y\)
Properties of Good Samplers

((e_1, e_2, \cdots, e_r) \text{ successive elements of } \text{merge}(X, Y) \text{ and } x = |X \cap \{e_1, e_2, \cdots, e_r\}| \text{ and } y = |Y \cap \{e_1, e_2, \cdots, e_r\}|) \text{ and }

Lemma:

Let X be a good sampler for X' and let Y be a good sampler for Y'. Then there are at most $2 \cdot r + 2$ elements of merge(X', Y') between r successive elements of merge(X, Y).

Proof: W.l.o.g. let $e_1 \in X$. If: $e_r \in Y$

- Add $e_0 \in Y$ with $e_0 < e_1$ to the good sampler.
- Add $e_{r+1} \in X$ with $e_r < e_{r+1}$ to the good sampler.

The elements from X' between $(e_1, e_2, \cdots, e_r)$ are between $x + 1$ elements from X.

The elements from Y' between $(e_1, e_2, \cdots, e_r)$ are between $y + 1$ elements from Y.

Thus we get: $2x + 1 + 2y + 1 = 2r + 2$.

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$e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X$
Properties of Good Samplers

Lemma:

Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$. Then there are at most $2 \cdot r + 2$ elements of merge($X'$, $Y'$) between $r$ successive elements of merge($X$, $Y$).

Proof: W.l.o.g. let $e_1 \in X$. If: $e_r \in Y$

- Add $e_0 \in Y$ with $e_0 < e_1$ to the good sampler.
- Add $e_{r+1} \in X$ with $e_r < e_{r+1}$ to the good sampler.
- The elements from $X'$ between $(e_1, e_2, \cdots, e_r)$ are between $x + 1$ elements from $X$.
- The elements from $Y'$ between $(e_1, e_2, \cdots, e_r)$ are between $y + 1$ elements from $Y$.
- Thus we get: $2x + 1 + 2y + 1 = 2r + 2$.

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Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

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Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

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- The elements from \(X'\) between \((e_1, e_2, \ldots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \ldots, e_r)\) are between \(y + 1\) elements from \(Y\).

**Thus we get:** \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[e_0 \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X\]
Properties of good sampler

At most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\)

**Definition**

Let reduce\((X)\) be the operation, which chooses from \(X\) every forth element.

**Lemma:**

If \(X\) is a good sampler for \(X'\) and \(Y\) is a good sampler for \(Y'\), then reduce\((\text{merge}(X, Y))\) is a good sampler for reduce\((\text{merge}(X', Y'))\).

**Proof:**

- Consider \(k + 1\) successive elements \((e_1, e_2, \cdots, e_{k+1})\) of reduce\((\text{merge}(X, Y))\).
- At most \(4k + 1\) elements of \(\text{merge}(X, Y)\) are between \(e_1, e_2, \cdots, e_{k+1}\) including \(e_1, e_{k+1}\).
- At most \(8k + 4\) elements of \(\text{merge}(X', Y')\) are between these \(4k + 1\) elements.
- At most \(2k + 1\) elements of reduce\((\text{merge}(X', Y'))\) are between \((e_1, e_2, \cdots, e_{k+1})\).
Properties of good sampler

**Definition**

Let \( \text{reduce}(X) \) be the operation, which chooses from \( X \) every forth element.

**Lemma:**

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{reduce}(\text{merge}(X, Y)) \) is a good sampler for \( \text{reduce}(\text{merge}(X', Y')) \).

**Proof:**

- Consider \( k + 1 \) successive elements \((e_1, e_2, \cdots, e_{k+1})\) of \( \text{reduce}(\text{merge}(X, Y)) \).
- At most \( 4k + 1 \) elements of \( \text{merge}(X, Y) \) are between \( e_1, e_2, \cdots, e_{k+1} \) including \( e_1, e_{k+1} \).
- At most \( 8k + 4 \) elements of \( \text{merge}(X', Y') \) are between these \( 4k + 1 \) elements.
- At most \( 2k + 1 \) elements of \( \text{reduce}(\text{merge}(X', Y')) \) are between \( (e_1, e_2, \cdots, e_{k+1}) \).
Properties of good sampler

At most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\)

**Definition**

Let \(\text{reduce}(X)\) be the operation, which chooses from \(X\) every forth element.

**Lemma:**

If \(X\) is a good sampler for \(X'\) and \(Y\) is a good sampler for \(Y'\), then \(\text{reduce}(\text{merge}(X, Y))\) is a good sampler for \(\text{reduce}(\text{merge}(X', Y'))\).

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  - At most \(2k + 1\) elements of \(\text{reduce}(\text{merge}(X', Y'))\) are between \((e_1, e_2, \cdots, e_{k+1})\).
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:

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- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
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Properties of good sampler

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Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
- The leaves contain the elements to be sorted.
- Interior nodes \( v \) “cares” about as many elements as the number of leaves below \( v \).
- A node \( v \) receives from its sons sequences of already sorted sequences.
- The “length” of the sequences doubles each time.
- Node \( v \) receives sequences \( X_1, X_2, \ldots, X_r \) and \( Y_1, Y_2, \ldots, Y_r \).
- Node \( v \) sends to his father sequences \( Z_1, Z_2, \ldots, Z_r, Z_{r+1} \).
- Node \( v \) updates an interior help-sequence \( val_v \).
- It holds: \( |X_1| = |Y_1| = |Z_1| = 1 \).
- It holds: \( |X_i| = 2 \cdot |X_{i-1}|, \ |Y_i| = 2 \cdot |Y_{i-1}| \) and \( |Z_i| = 2 \cdot |Z_{i-1}| \).
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- Node $v$ updates an interior help-sequence $val_v$.
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Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
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One basic Operation of an interior Node $v$

- Receives from its sons the two sequences $X$ and $Y$.
- Computes: $val_v = \text{merge\_with\_help}(X, Y, val_v)$.
- Sends to its father: reduce($val_v$) till $v$ has sorted all received sequences.
- Sends to its father each second element from $val_v$, if $v$ is done with sorting.
- Sends to its father $val_v$, if $v$ finishes sorting two steps before.
- Example:

<table>
<thead>
<tr>
<th>Step</th>
<th>Left</th>
<th>Right</th>
<th>$val_v$</th>
<th>Father</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>8</td>
<td>7,8</td>
<td>⌀</td>
</tr>
<tr>
<td>2</td>
<td>3,7</td>
<td>5,8</td>
<td>3,5,7,8</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>1,3,4,7</td>
<td>2,5,6,8</td>
<td>1,2,3,4,5,6,7,8</td>
<td>4,8</td>
</tr>
<tr>
<td>4</td>
<td>1,3,4,7</td>
<td>2,5,6,8</td>
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- Thus we get the following pattern:

\[
\begin{array}{cccccc}
X_1 & X_2 & X_3 & X_4 & \cdots & X_r \\
Z_1 & Z_2 & \cdots & Z_r & Z_{r+1} & Z_{r+2}
\end{array}
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- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
- Thus we get a running time of $3 \log n$. 
Basic operation of a interior Node $\nu$

- Receives from its sons the two sequences $X$ and $Y$.
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- Sends to its father each second element from $val_{\nu}$, if $\nu$ is done with sorting.
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- Thus we get the following pattern:

\[
\begin{array}{ccccccc}
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\[
\begin{align*}
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- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
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- Each $X_i$ is a good sampler of $X_{i+1}$.
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- Each $X_i$ is half as big as $X_{i+1}$.
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Situation

- Running time is $O(\log n)$.
- The inner nodes $v$ need $|val_v|$ many processors.
- We still have to proof that the number of processors is in $O(n)$.
- PRAM Model has to be verified.
- Important: The computation of the values $Rng_{X,Y}$ has to be shown.
- These values will be in the following also transmitted and updated.
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PRAM Model has to be verified.

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Situation

- Running time is $O(\log n)$.
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Computing the Ranks

- In each step will compute: \( \text{merge}_\text{with}_\text{help}(X_{i+1}, Y_{i+1}, \text{merge}(X_i, Y_i)) \).
- Using the Lemma from above we have: \( \text{merge}(X_i, Y_i) \) is a good sampler of \( X_{i+1} \) and \( Y_{i+1} \).
- Let \( L = \text{merge}(X_i, Y_i) \), \( J = X_{i+1} \) and \( K = Y_{i+1} \).
- We have to compute: \( \text{Rng}_{L,J} \), \( \text{Rng}_{L,K} \), \( \text{Rng}_{J,L} \) and \( \text{Rng}_{K,L} \).

**Invariant:**

- Let \( S_1, S_2, \ldots, S_p \) be a sequence of sequences at node \( v \).
- Then node \( c \) also knows: \( \text{Rng}_{S_{i+1}, S_i} \) for \( 1 \leq i < p \).
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Computing the Ranks

**Lemma:**

Let \( S = (b_1, b_2, \cdots, b_k) \) be a sortierted sequence, then we may compute the rank of \( a \in S \) in time \( O(1) \) using \( k \) processors.

**Proof:**

- Programm: \( \text{rng1}(a,S) \)
  
  for all \( P_i \) where \( 1 \leq i \leq k \) do in parallel
  
  if \( b_i < a \leq b_{i+1} \) then return \( i \)

- Note, the program has no write-conflicts.

- Note, it could be changed, to avoid read-conflicts.
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Computing the Ranks

Lemma:

Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$ in time $O(1)$ using $O(|S|)$ processors.

Proof:

- We do know $\text{Rnk}_{S,S}$, $\text{Rnk}_{S_1,S_1}$ and $\text{Rnk}_{S_2,S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
- The claim follows directly.
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Computing the Ranks

**Lemma:**

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.

**Proof:**

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
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we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks

Lemma:

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Computing the Ranks (Rnk_{X',U})

- Let $X = (a_1, a_2, \cdots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \cdots, X'_k, X'_{k+1}$.
- Note: $\text{Rnk}_{X',X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

  Programm: $\text{Rnk}_{X',U}$
  for all $i$ where $1 \leq i \leq k + 1$ do in parallel
  for all $x \in X'_i$ do
    $\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)$

- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

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Computing the Ranks \( (\text{Rnk}_{X'}, U) \)

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- Using a good sampler \( X \) we split \( X' \) into \( X'_1, X'_2, \cdots, X'_k, X'_{k+1} \).
- Note: \( \text{Rnk}_{X', X} \) is known.
- Splitting may be done in time \( O(1) \) using \( O(|X|) \) processors.
- Let \( U_i \) be the sequence of elements of \( Y \) which are between \( a_{i-1} \) and \( a_i \).
- Thus we get:

  Programming: \( \text{Rnk}_{X', U} \)
  
  for all \( i \) where \( 1 \leq i \leq k + 1 \) do in parallel
  
  for all \( x \in X'_i \) do
  
  \( \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i) \)

- Running time \( O(1) \) using \( \sum_{i=1}^{k+1} |U_i| \) processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X_1', X_2', \ldots, X_k', X_{k+1}'\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  **Programm: Rnk\(_{X', U}\)**

  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

  for all \(x \in X_i'\) do

  \[
  \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
  \]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X'},U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
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- Note: \(\text{Rnk}_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

\[
\text{Programm: Rnk}_{X',U} \\
\text{for all } i \text{ where } 1 \leq i \leq k + 1 \text{ do in parallel} \\
\quad \text{for all } x \in X'_i \text{ do} \\
\quad \quad \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
\]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks ($\text{Rnk}_{X', U}$)

- Let $X = (a_1, a_2, \cdots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \cdots, X'_k, X'_{k+1}$.
- Note: $\text{Rnk}_{X', X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

Programm: $\text{Rnk}_{X', U}$

- for all $i$ where $1 \leq i \leq k + 1$ do in parallel
  - for all $x \in X'_i$ do
    - $\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)$

- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let \(w.l.o.g.\) \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X', X}\) is known.
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- Thus we get:

  **Programm:** \(\text{Rnk}_{X', U}\)
  
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  for all \(x \in X'_i\) do
  
  \[
  \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
  \]

  Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X_{i+1}'\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks \((\text{Rnk}_{X, X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
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Computing the Ranks (Rnk\(_{X,X'}\))

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note:** $\text{Rnk}_{U,X'}$ consists of $\text{Rnk}_X, X'$ and $\text{Rnk}_Y, X'$.
- $\text{Rnk}_X, X'$ is already known.
- Still to compute: $\text{Rnk}_Y, X'$.
- $\text{Rnk}_Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- Note: \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk} X, X'\) and \(\text{Rnk} Y, X'\).
- \(\text{Rnk} X, X'\) is already known.
- Still to compute: \(\text{Rnk} Y, X'\).
- \(\text{Rnk} Y, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks (Rnk_{U,X'})

- **Note:** Rnk_{U,X'} consists of Rnk X, X' and Rnk Y, X'.
- Rnk X, X' is already known.
- **Still to compute:** Rnk Y, X'.
- Rnk Y, X may be computed using the previous lemma.
- We compute rnk(a, X') using rnk(a, X) and Rnk_{X,X'}.
- Thus we compute Rnk_{U,X'} with O(|U|) processors and time O(1).
Computing the Ranks (Rnk\(_U,X'\))

- Note: Rnk\(_U,X'\) consists of Rnk \(X,X'\) and Rnk \(Y,X'\).
- Rnk \(X,X'\) is already known.
- Still to compute: Rnk \(Y,X'\).
- Rnk \(Y,X\) may be computed using the previous lemma.
- We compute rnk(\(a,X'\)) using rnk(\(a,X\)) and Rnk\(_X,X'\).
- Thus we compute Rnk\(_U,X'\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks (Rnk\(_U,X'\))

- Note: Rnk\(_{U,X'}\) consists of Rnk\(_{X,X'}\) and Rnk\(_{Y,X'}\).
- Rnk\(_{X,X'}\) is already known.
- Still to compute: Rnk\(_{Y,X'}\).
- Rnk\(_{Y,X}\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a,X')\) using \(\text{rnk}(a,X)\) and Rnk\(_{X,X'}\).
- Thus we compute Rnk\(_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks \((\text{Rnk}_{\mathcal{U},X'})\)

- Note: \(\text{Rnk}_{\mathcal{U},X'}\) consists of \(\text{Rnk} \; X, X'\) and \(\text{Rnk} \; Y, X'\).
- \(\text{Rnk} \; X, X'\) is already known.
- Still to compute: \(\text{Rnk} \; Y, X'\).
- \(\text{Rnk} \; Y, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{\mathcal{U},X'}\) with \(O(|\mathcal{U}|)\) processors and time \(O(1)\).
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]:

  - Using the invariant we know: Rnk_{J,X_i} and Rnk_{K,Y_i}.
  - Using the above considerations we may compute: Rnk_{L,J}, Rnk_{L,K}, Rnk_{J,L}, and Rnk_{K,L}.
  - Still to be computed: Rnk_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))}.
  - Known: Rnk_{X_{i+1}, \text{merge}(X_i, Y_i)} and Rnk_{Y_{i+1}, \text{merge}(X_i, Y_i)}.
  - It is now easy to compute: Rnk_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} and Rnk_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))}.
  - Also easy to compute: Rnk_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))}.
Computing the Ranks

Consider the step
\[ merge\_with\_help(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]:

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \).
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).
- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

Consider the step

\[\text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)):\]

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Still to be computed: \(\text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))}\)

Known: \(\text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)}\) and \(\text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)}\).

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Also easy to compute: \(\text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))}\).
Computing the Ranks

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$

- Consider the step $\text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i))$:
- Using the invariant we know: $\text{Rnk}_{J, X_i}$ and $\text{Rnk}_{K, Y_i}$.
- Using the above considerations we may compute: $\text{Rnk}_{L, J}$, $\text{Rnk}_{L, K}$, $\text{Rnk}_{J, L}$ and $\text{Rnk}_{K, L}$.
- **Still to be computed:** $\text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))}$
- Known: $\text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)}$ and $\text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)}$.
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Computing the Ranks

we have \( \operatorname{rnk}(a, S) \) and \( \operatorname{Rnk}_{S_1, S_2} \) and \( \operatorname{Rnk}_{S_2, S_1} \)

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  \]

- Using the invariant we know: \( \operatorname{Rnk}_{J, X_i} \) and \( \operatorname{Rnk}_{K, Y_i} \).

- Using the above considerations we may compute: \( \operatorname{Rnk}_{L, J} \), \( \operatorname{Rnk}_{L, K} \), \( \operatorname{Rnk}_{J, L} \) and \( \operatorname{Rnk}_{K, L} \).

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Computing the Ranks

Consider the step
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Algorithm of Cole

Theorem:
We may sort $n$ values on a CREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: discussed before.

Theorem:
We may sort $n$ values on a EREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: see literature.

Theorem:
There exists a sorting network with $O(n)$ processors and depth $O(\log n)$.

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Literature:

Questions

- Explain the motivation behind parallel systems.
- Explain the ideas of the different sorting algorithms.
- Explain the different running times of these sorting algorithms.
- Explain the different efficiency of these sorting algorithms.
- Explain the idea of the algorithm of Cole.
- Explain the running time of the algorithm of Cole.
- Explain the number of processors used in the algorithm of Cole.
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Legende

■ : Nicht relevant
■ : Grundlagen, die implizit genutzt werden
■ : Idee des Beweises oder des Vorgehens
■ : Struktur des Beweises oder des Vorgehens
■ : Vollständiges Wissen