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- Shows the quality of any algorithm.
- Interesting property of any problem.
- Interesting techniques to prove lower bounds.
  - No assumption about the used algorithms
  - Have to show a property for all algorithms and some inputs.
  - For all algorithms there is an input, such that the running time is at least....
  - Typically more complicated than upper bounds.
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Ideas

- Model distributed computers, connected in a cycle.
- No assumption about structure of the algorithm.
- Assume the running time is \( t \) on a cycle of length \( n \).
- Step one: Normalize the behavior of the algorithm.
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Step one: Normalize the behavior of the algorithm

- After $t$ steps a node may know the identifiers of $2t + 1$ nodes. Let
  \[ W_{s,n} = \{(x_1, x_2, ..., x_s) \mid 1 \leq x_i \leq n \land x_i = x_j \Rightarrow i = j\} \]
  be the set of possible surroundings.

- It is not necessary to color any node before step $t$:
  - Each node may simulate the behavior of the $2t + 1$ nodes in the surrounding.
  - Or each node sends also the history of colors.

- Thus after $t$ rounds node $v$ has the topological information $\zeta(v)$:
  \[ \zeta(v) = (x_1, x_2, ..., x_s) \in W_{s,n} \text{ with } s = 2t + 1. \]

- Any algorithm will use some deterministic strategy $\pi$ to find a coloring:
  \[ c(v) \leftarrow \Phi_\pi(\zeta(v)) \text{ with } \Phi_\pi : W_{s,n} \mapsto \{1, 2, ..., c_{max}\}. \]
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Step two: Extend the possible inputs

- The set of nodes is $W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\}$.
- The set of edges is $E_{s,n}$. They contain any possible edge is any cycle:

$$E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$$

- This graph $B_{s,n} = (W_{s,n}, E_{s,n})$ has $\binom{n}{s}$ nodes of degree $n - s$. Thus it has $(n - s)\binom{n}{s}$ nodes of degree $n - s$. Thus it has $(n - s)\binom{n}{s}$ nodes of degree $n - s$. Thus it has $(n - s)\binom{n}{s}$ nodes of degree $n - s$.

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If an algorithm $\pi_t$ colors any cycle of length $n$ with $c$ colors in $t$ steps, then it will define a legal coloring of $B_{s,n}$. 
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- Take this edge and extend it to a cycle of length \( n \) using the missing identifiers.
- This cycle with this order is not colored correctly.
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\[ W_{s,n} = \{(x_1, x_2, \ldots, x_s) | 1 \leq x_i \leq n\} \text{ and } E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) | x_1 \neq x_{s+1}\} \]

**Theorem (Distributed Coloring \( C_{2n} \))**

Any deterministic distributed algorithm uses \( n - 1 \) rounds to color a cycle of length \( 2n \) with 2 colors.

- Assume the algorithm runs in time \( t \leq n - 2 \).
- Then this algorithm will color the graph \( B_{2t+1,2n} \) with 2 colors.
- \( B_{2t+1,2n} \) is bipartite for \( t \leq n - 2 \).
- We will now construct the following cycle:

\[
\begin{align*}
(1, 2, 3, \ldots, 2t + 1) & \rightarrow (2, 3, 4, \ldots, 2t + 2) \\
\rightarrow (3, 4, 5, \ldots, 2t + 3) & \rightarrow (4, \ldots, 2t + 3, 1) \\
\rightarrow \ldots & \rightarrow \ldots (2t + 2, 2t + 3, 1, 2, \ldots, 2t - 1) \\
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Theorem (Parallel Coloring \( C_{2n} \))

Any deterministic parallel algorithm uses \( \log n \) rounds to color a cycle of length \( 2n \) with 2 colors.

- Assume the algorithm runs in time \( t \leq \log n \).
- The best way to collect information is doubling (see lower bound for broadcast/accumulation).
- Then we may use its strategy to construct a distributed version running in \( t \) time.
- Contradiction.
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- We want a lower bound for the 3-coloring of cycles.
- Step a) Show \( \chi(B_{2t+1,n}) \geq \log^2 n \).
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- Step b) Show \( \chi(\tilde{B}_s,n) \leq \chi(B_s,n) \).
- Step c) Use the line-graph construction.
- Step d) Show property for coloring a line-graph.
- Step e) Put everything together.
Construction of $\tilde{B}_{s,n}$

$$W_{s,n} = \{(x_1, x_2, \ldots, x_s) \mid 1 \leq x_i \leq n\} \text{ and } E_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$$

- **Remember:**
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We have: $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$. 
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Line-Graphs

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Definition (Line-Graphs)

Let \( G = (V, E) \) be an directed graph. \( DL(G) = (E, E') \) is called line-graph of \( G \), iff

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A graph \( H \) is called directed line-graph, iff a graph \( G \) exists, with \( DL(G) = H \).
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Beispiel 3

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DeBruijn network of dimension $d$

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- **DeBruijn network:**
  \[
  DB(d) = (V_{DB(d)}, E^s_{DB(d)} \cup E^{se}_{DB(d)})
  \]
  \[
  V_{DB(d)} = \{0, 1\}^d
  \]
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**Lemma**

We have: $DB(d + 1) = DL(DB(d))$ for $d \geq 1$. 

DeBruijn network of dimension $d$

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- Number of nodes: $2^d$
- Degree: $2 + 2$
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Line-Graph Properties of $\tilde{B}_{s,n}$

$\tilde{W}_{s,n} = \{(x_1, ..., x_s) \mid x_1 < ... < x_s\}$, $\tilde{E}_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_{s+1})) \mid x_1 \neq x_{s+1}\}$, $\chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n})$

Lemma

1. $\tilde{B}_{1,n}$ is the complete directed graph of $n$ nodes.
2. We have $\tilde{B}_{s+1,n} = LG(\tilde{B}_{s,n})$ for $s \geq 1$.

Proof:

1. By definition: $\tilde{E}_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_s, x_{s+1})) \mid x_1 \neq x_{s+1}\}$.
2. By construction:
   - In $\tilde{B}_{s,n}$: $(x_1, x_2, ... x_s) \rightarrow (x_2, x_3, ..., x_{s+1})$ and $(x_2, x_3, ..., x_{s+1}) \rightarrow (x_3, x_4, ..., x_{s+2})$.
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**Lemma**

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Bounds for Coloring Line-Graphs

\[ \tilde{W}_{s,n} = \{(x_1, \ldots, x_s) | x_1 < \ldots < x_s\}, \tilde{E}_{s,n} = \{((x_1, x_2, \ldots, x_s), (x_2, \ldots, x_{s+1})) | x_1 \neq x_{s+1}\}, \chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n}) \]

**Lemma**

*Let $H$ be any directed graph, then we have $\chi(DL(H)) \geq \log(\chi(H))$.*

**Proof:**

- Let $k = \chi(DL(H))$, thus we can color the nodes from $DL(H)$ with $k$ colors.
- Thus we may color the edges from $H$ with $k$ colors: $\chi'(H) \leq k$.
- For any edge $e = (\nu, \omega)$ of $H$ let $c'(e)$ be the color of $e$.
- Define now a coloring of the nodes $\nu$ of $H$:
  \[ c(\nu) = \bigcup_{\nu \in e} c'(e) \]
- This is a correct $2^k$ node-coloring of $H$.
- Thus $\chi(H) \leq 2^k = 2^{\chi(DL(H))}$.
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Bounds for Coloring Line-Graphs

\[ \tilde{W}_{s,n} = \{(x_1, ..., x_s) \mid x_1 < ... < x_s\}, \quad \tilde{E}_{s,n} = \{((x_1, x_2, ..., x_s), (x_2, ..., x_{s+1})) \mid x_1 \neq x_{s+1}\}, \quad \chi(\tilde{B}_{s,n}) \leq \chi(B_{s,n}) \]

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Lemma

We have \( \chi(\tilde{B}_s, n) \geq \log^{(s-1)} n \).

Proof:

- \( \tilde{B}_{1,n} \) is the complete directed graph of \( n \) nodes.
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Results

**Theorem**

*Any deterministic distributed algorithm needs at least $1/2(\log^* n - 1)$ rounds to color a cycle of length $n$ with 3 colors.*

**Proof:**

- We have already: $\chi(\tilde{B}_s, n) \geq \log^{(s-1)} n$, resp.:
- We have already: $\chi(\tilde{B}_{2t+1}, n) \geq \log^{(2t)} n$.
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- **NP-hard**: the “most complicated” problems for the class \( \mathcal{NP} \).

- Theory of NP-complete problems was developed, to “explain” that for many problems no polynomial time deterministic algorithm is known.

- A problem is NP-hard \( \iff \):
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Some Observations about Problems from $\mathcal{P}$

- Any problem from $\mathcal{P}$ is a candidate for a parallel algorithm.

- A problem is well to parallelize, if there is a parallel deterministic algorithm
  - which uses a polynomial number of processors
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- These class is called $\mathcal{NC}$ (Nick’s Class).

- We have by definition: $\mathcal{NC} \subseteq \mathcal{P}$.

- Important Question: $\mathcal{NC} \cong \mathcal{P}$?

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Reductions for $\mathcal{P}$

- Recall the situation for $\mathcal{NPC}$ (try to separate $\mathcal{NP}$ from $\mathcal{P}$):
  - Hard problem: stops a non-deterministic TM in polynomial time?
  - Reduction: runs deterministic in polynomial time.

- Or in other words:
  - Hard problem: a nice candidate from the “hard class”.
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- Uses the analogous technique for $\mathcal{P}$ (try to separate $\mathcal{P}$ from $\mathcal{NC}$):
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- From the parallel algorithm running deterministic in time poly-logarithmic
- we build a circuit network.
- This has poly-logarithmic depth and polynomial width.
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A problem \( X \) is called \( \mathcal{P} \)-complete, iff:

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\( \mathcal{P} \)-Complete
First Reduktion (Introduction)

Definition (Generability)
- **Input:** Set \( X \) with binary operator \( \odot \), \( T \subset X \) and \( s \in X \).
- **Output:** Is \( s \) in the closure of \( T \) in terms of \( \odot \).

- Let \( S \odot S := \{ a \odot b \mid a, b \in S \} \).
- Algorithm for Generability\((X, \odot, S, s)\) in \( \mathcal{P} \):
  - while \( S \neq S \odot S \) do \( S = S \odot S \)
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We will first show \( \mathcal{P} \)-completeness for a ternary operation.
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First Reduktion (Introduction)

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- **Input:** Set $X$ with binary operator $\circ$, $T \subset X$ and $s \in X$.
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- Let $S \circ S := \{a \circ b \mid a, b \in S\}$.
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Reduction from the halting problem of a deterministic TM.
First Reduction

Definition (Generability’)
- Input: Set \( X \) with ternary operator \( \text{next}(u, v, w) \), \( T \subset X \) and \( s \in X \).
- Output: Is \( s \) in the closure of \( T \) in terms of \( \odot \).

Definition (TM)
- Input band with positions \( 0, 1, 2, \cdot T(n) + 1 \).
- By \( c(i, j) \in \Sigma \) we denote the contents at position \( i \) at time \( j \).
- Let \( c(0, j) = c(T(n) + 1, j) = \$ \) for all time points \( j \).
- The function \( \text{trans} \) defines the transitions for the TM.
- I.e. \( c(p, t + 1) = \text{trans}(c(p - 1, t), c(p, t), c(p + 1, t)) \).
- Input given at positions \( c(p, 0) \) (\( \forall p : 1 \leq p \leq T(n) \)).
- Output placed at \( c(1, T(n)) \) where \( \# \) encodes a “true”.

Mot. Coloring Cycles P-Completeness First Reduction More Recuktions

First Reduction

Definition (Generability’)

- Input: Set $X$ with ternary operator $\text{next}(u, v, w)$, $T \subseteq X$ and $s \in X$.
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First Reduction (Generability’)

Theorem:

Generability’ is $\mathcal{P}$-complete.

Proof:

- A TM may be transformed in $\mathcal{NC}$ into the above form.
- The triple $(t, p, \text{sym})$ encodes that the contents at position $p$ and time $t$ is $\text{sym}$.
- We will now compute the input for Generability’ from the above TM:
  - $X = \{0, 1, \cdots, T(n)\} \times \{0, 1, \cdots, T(n) + 1\} \times \Sigma$.
  - $T = \{(0, i, c(0, i)) \mid 0 \leq i \leq T(n) + 1\}$
  - $s = (T(n), 1, \#)$
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\( 4:27 \) Generability 3/10

Walter Unger 30.1.2017 11:53  \( \text{WS}2016/17 \)
First Reduction (Generability’)

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First Reduction (Generability)

Theorem:
Generability ist $\mathcal{P}$-complete.

Proof:

- Reduktion von Generability’
- $X' := X \cup X^2$ ($X$ form above)
- $T = \{(0, i, c(0, i)) \mid 0 \leq i \leq T(n) + 1\}$
- $s = (T(n), 1, \#)$
- It remains to define next as a binary Operator $\circ$.
- $u \circ v := (u, v)$ and
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First Reduction (Generability)

Theorem:

Generability ist \( P \)-complete.

Proof:

- **Reduktion von Generability’**
- \( X' := X \cup X^2 \) (\( X \) form above)
- \( T = \{(0, i, c(0, i)) \mid 0 \leq i \leq T(n) + 1\} \)
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Lemma:
If \( \circ \) is associative, then is the corresponding Generability-Problem in \( NC \).

Proof:
- We transform this problem into the reachability problem on a graph \( G \).
- If \( x \circ z = y \) then generate an edge \( (x, y) \) with label \( z \).
- \( G = (X, E) \) with \( E = \{ (x, y) \mid \exists z \in X : x \circ z = y \} \)
- and \( \forall (x, y) \in E : l(x, y) := \{ z \in X \mid x \circ z = y \} \).
- If there is a path from \( a \in T \) to \( s \) using edges with labels \( b, c, d, \ldots \), then we may generate \( s \) by \( ((\cdots (a \circ b) \circ c) \circ d) \cdots ) \).
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Lemma:
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Reduktion (CVP)

Definition (CVP)

- Input: A boolean circuit with some input.
- Output: Is the output value *true*.

Theorem:
The problem CVP is \( \mathcal{P} \)-complete.

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- Reduction from the Generability Problem.
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Details for the Reduction (CVP)

- For each element \( x \) from \( X \setminus T \) do:
  - Compute pairs from \( X \times X \) which will give \( x \):
    \[
    (y_1, z_1), (y_2, z_2), (y_3, z_3), \ldots, (y_{k_x}, z_{k_x})
    \]
  - I.e. \( y_i \odot z_i = x \) for all \( 1 \leq i \leq k_x \).
  - This is one part of the circuit:
    \[
    x = \bigvee_{i=1}^{k_x} y_i \land z_i
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- Thus \( x \) will have the value true iff \( x \) may be generated.
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Reduktion (MCVP)

**Definition (MCVP)**
- Input: A boolean circuit with some input and only operators $\lor$ und $\land$.
- Output: Is the output value true.

**Theorem:**
The MCVP is $\mathcal{P}$-complete.

**Proof:**
- Similar proof to the CVP problem.
Reduktion (MCVP)

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Reduktion (TSMCVP)

**Definition (TSMCVP)**
- **Input:** A boolean circuit with some input and only operators $\lor$, $\land$ and a topological sorting of the values.
- **Output:** Is the output value *true*.

**Theorem:**
The TSMCVP is $\mathcal{P}$-complete.

**Proof:**
- Similar proof to the CVP problem.
- Note: the proof for Generability’ did contain a topological sorting.
- This was the lexicographical order of the elements $(t, p, sym)$.
- This order could easily be preserved during the following step of the reduction.
Reduktion (TSMCVP)

Definition (TSMCVP)

- Input: A boolean circuit with some input and only operators ∨, und, ∧ and a topological sorting of the values.
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Definition (CFE)

- Input: a context-free grammar $G$.
- Output: will $G$ generate the empty word $\varepsilon$.

Theorem:
The CFE is $\mathcal{P}$-complete.

Proof (Reduktion from Generability Problem):

- Let $(X, T, \odot, s)$ be the input for the Generability problem.
- Let $X$ be the non-terminals of $G$.
- Let $s$ be the start symbol.
- For each $x \in T$ generate the rule: $x \rightarrow \varepsilon$.
- If $y \odot z = x$ generate the rule: $x \rightarrow yz$.
- Note: If $G$ contains no $\varepsilon$-rules, then is CFE in $\mathcal{NC}$.
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Definition (LFMIS)

- Input: non-directed graph $G = (V, E)$.
- Output: lexicographical first maximum independent set (IS) of $G$.

Theorem:

The LFMIS is $P$-complete.

Proof (Reduction from MCVP problem)

- Consider the greedy-strategy for the LFMIS problem.
- Let $V = \{v_1, v_2, \cdots, v_n\}$ nodes for the MCVP Problems in their topological sorting.
- Let $\{v_1, v_2, \cdots, v_e\}$ be the input nodes and $v_n$ be the output node.
- We construct $G = (V', E')$ as input for LFMIS.
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3. Let $\{v_1, v_2, \cdots, v_e\}$ be the input nodes and $v_n$ be the output node.
4. We construct $G = (V', E')$ as input for LFMIS.
Continuation of the Reduction (LFMIS)

- Let $V' = \{v'_1, v''_1, v'_2, v''_2, \ldots, v'_n, v''_n\}$ be numbered from 1 till $2n$.
- The numbers of $v'_i, v''_i$ are exchanged, if
  - $v_i$ is an or-node or
  - $v_i$ is an input node with the value false.
- For all $1 \leq i \leq n$ generate an edge $\{v'_i, v''_i\}$.
- Thus only one of the nodes $v'_i, v''_i$ is in the IS.
- If $v$ is an and-node $G$ with input $u$ and $w$, then add the edges $\{v', u''\}$ and $\{v', w''\}$.
- Thus $v'$ will be in the IS iff non of the nodes $u'', w''$ are in the IS.
- If $v$ is an or-node $G$ with inputs $u$ and $w$, then add the edges $\{v'', u'\}$ and $\{v'', w'\}$.
- Thus $v''$ will be in the IS iff non of the nodes $u', w'$ are in the IS.
- Thus LFMIS is simulating correctly the boolean circuit.
Continuation of the Reduction (LFMIS)

- Let $V' = \{v'_1, v''_1, v'_2, v''_2, \ldots, v'_n, v''_n\}$ be numbered from 1 till $2n$.

- The numbers of $v'_i, v''_i$ are exchanged, if
  - $v_i$ is an or-node or
  - $v_i$ is an input node with the value $false$.

- For all $1 \leq i \leq n$ generate an edge $\{v'_i, v''_i\}$.

- Thus only one of the nodes $v'_i, v''_i$ is in the IS.

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- If $v$ is an and-node $G$ with input $u$ and $w$, then add the edges $\{v', u''\}$ and $\{v', w''\}$.
- Thus $v'$ will be in the IS iff none of the nodes $u''$, $w''$ are in the IS.
- If $v$ is an or-node $G$ with inputs $u$ and $w$, then add the edges $\{v'', u'\}$ and $\{v'', w'\}$.
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- The numbers of \( v'_i, v''_i \) are exchanged, if
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  - \( v_i \) is an input node with the value \textit{false}.

- For all \( 1 \leq i \leq n \) generate an edge \( \{v'_i, v''_i\} \).
- Thus only one of the nodes \( v'_i, v''_i \) is in the IS.

- If \( v \) is an and-node \( G \) with input \( u \) and \( w \), then add the edges \( \{v', u''\} \) and \( \{v', w''\} \).
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- Let $V' = \{v'_1, v''_1, v'_2, v''_2, \ldots, v'_n, v''_n\}$ be numbered from 1 till $2n$.
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$\exists v' \in IS \iff v$
$\exists v'' \in IS \iff \overline{v}$
Reduction (LFMC)

**Definition (LFMC)**
- Input: non-directed graph $G = (V, E)$.
- Output: lexicographical first maximum clique of $G$.

**Theorem:**
Das LFMC is $\mathcal{P}$-complete.

**Proof**
- Reduction from LFMIS problem.
- Let $G = (V, E)$ be the input for LFMIS problem.
- Then $G = (V, \overline{E})$ will be input for the LFMC problem.
Reduction (LFMC)

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Given $G = (V, E)$

Procedure DFS($v$)

if $DFI(v) = 0$ then
  $counter := counter + 1$
  $DFI(v) := counter$
  forall $w \in V : (v, w) \in E$ do
    DFS($w$)
Reduction (DFS)

**Definition (DFS)**
- Input: directed graph \( G = (V, E) \) and \( v \in V \).
- Output: The values \( DFI(w) \) of the call \( DFS(v) \) for all \( w \in V \).

**Theorem:**
The DFS is \( \mathcal{P} \)-complete.

**Proof**
- Reduction from CVP problem with \( \odot := \overline{x} \lor \overline{y} = \overline{x} \land \overline{y} \)
- It is easy to see, that this version of CVP Problem is also \( \mathcal{P} \)-complete.
- Idea: for each value of \( v \) in the input of CVP
  will be in \( G = (V, E) \) two nodes \( s \) and \( t \),
  with \( v \) is true iff \( DFI(s) < DFI(t) \).
Reduction (DFS)

**Definition (DFS)**
- Input: directed graph $G = (V, E)$ and $v \in V$.
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Reduction (DFS)

Definition (DFS)
- Input: directed graph \( G = (V, E) \) and \( v \in V \).
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Continuation of the Reduction (DFS)

- Let $v_1, v_2, \cdots, v_n$ be the nodes of the circuit.
- For each $v_i$ we will build a sub-graph $G_i$.
- These sub-graphs $G_i$ will be edge-disjoint, but not node-disjoint.
- $G_i$ and $G_j$ ($i < j$) may have common nodes $i \neq j$.
- $v_i$ has $v_{i_1}$ and $v_{i_2}$ as input nodes
- and the nodes $v_{o_1}, v_{o_2}, v_{o_3}, \cdots, v_{o_k}$ use $v_i$ as input.
- Then has $G_i$ for $k = 3$ the following structure.
- We indicate the order of the edges in the adjacency list by the number of arrow heads.
- If $v_i$ is an input node in the circuit and the nodes $v_{o_1}, v_{o_2}, v_{o_3}, \cdots, v_{o_k}$ use $v_i$ as input, then we will have a simplified graph $G_i$. This is seen as the second one.
Continuation of the Reduction (DFS)

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- $v_i$ has $v_{i1}$ and $v_{i2}$ as input nodes
- and the nodes $v_{o1}, v_{o2}, v_{o3}, \cdots, v_{ok}$ use $v_i$ as input.
- Then has $G_i$ for $k = 3$ the following structure.
- We indicate the order of the edges in the adjacency list by the number of arrow heads.
- If $v_i$ is an input node in the circuit and the nodes $v_{o1}, v_{o2}, v_{o3}, \cdots, v_{ok}$ use $v_i$ as input, then we will have a simplified graph $G_i$. This is seen as the second one.
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Continuation of the Reduction (DFS)

Let $v_1, v_2, \cdots, v_n$ be the nodes of the circuit.

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Continuation of the Reduction (DFS)

\[ \text{last}(i - 1) \]
\[ \text{first}(i) \]
\[ \text{last}(i) \]

\[ i_1 \neq i \quad i_2 \neq i \]
\[ s(i) \]
\[ t(i) \]

\[ v_i \text{ ist intern} \]

\[ i \neq o_1 \quad i \neq o_2 \quad i \neq o_3 \]
Continuation of the Reduction (DFS)

\[ \text{last}(i - 1) \]

\[ \text{first}(i) \]

\[ s(i) \]

\[ t(i) \]

\[ v_i \text{ ist Input} \]

\[ \text{last}(i) \]
Continuation of the Reduction (DFS)

- The DFS run starts at \textit{first}(1).
- After \textit{last}(i) will be the next visited node \textit{first}(i + 1).
- The order how \textit{s}(i) and \textit{t}(i) in \( G_i \) are visited, will be given by the value of \( v_i \).
- After \textit{last}(n) is visited, is each graph \( G_i \) is also visited, excluding some minor parts.
Continuation of the Reduction (DFS)

- The DFS run starts at \textit{first}(1).
- After \textit{last}(i) will be the next visited node \textit{first}(i + 1).
- The order how \textit{s}(i) and \textit{t}(i) in \textit{G}_i are visited, will be given by the value of \textit{v}_i.
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Continuation of the Reduction (DFS)

- The DFS run starts at $first(1)$.
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Lemma

We consider a DFS-run in $G$ stating in node $\text{first}(1)$:

- If $v_i$ has the value $\text{true}$, then $s(i)$ will be visited before $t(i)$ and the nodes $i\#o_1, i\#o_2, \ldots, i\#o_k$ are visited after $\text{first}(i)$ and before $\text{last}(i)$.

- If $v_i$ has the value $\text{false}$, then the node $t(i)$ will be visited before $s(i)$ and none of the nodes $i\#o_1, i\#o_2, \ldots, i\#o_k$ will be visited in the interval between $\text{first}(i)$ and $\text{last}(i)$ visits.

Proof:

- By induction:
- Start of induction, consider all input-nodes.
- Induction-step, Assume above statement holds for all graphs $G_j$ ($1 \leq j < i$).
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Continuation of the Reduction (DFS)

Lemma

We consider a DFS-run in $G$ stating in node $\text{first}(1)$:

- If $v_i$ has the value `true`, then $s(i)$ will be visited before $t(i)$ and the nodes $i \# o_1, i \# o_2, \ldots, i \# o_k$ are visited after $\text{first}(i)$ and before $\text{last}(i)$.

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- Start of induction, consider all input-nodes.

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Continuation of the Reduction (Start of Induction)

- If $v_i$ has the value \textit{true}, then we visit $s(i)$ before $t(i)$ and the nodes $i\neq o_1, i\neq o_2, \ldots, i\neq o_k$ are visited after $\text{first}(i)$ and before $\text{last}(i)$.
Continuation of the Reduction (Start of Induction)

- If $v_i$ has the value true, then we visit $s(i)$ before $t(i)$ and the nodes $i \# o_1, i \# o_2, \cdots, i \# o_k$ are visited after first($i$) and before last($i$).
Continuation of the Reduction (Start of Induction)

- If \( v_i \) has the value true, then we visit \( s(i) \) before \( t(i) \) and the nodes \( i \# o_1, i \# o_2, \ldots, i \# o_k \) are visited after \( first(i) \) and before \( last(i) \).
Continuation of the Reduction (Induction-Step)

- If \( v_i \) has the value \textit{true}, then \( s(i) \) will be visited before \( t(i) \) and the nodes \( i \# o_1, i \# o_2, \ldots, i \# o_k \) are visited after \( \text{first}(i) \) and before \( \text{last}(i) \).

- Then the nodes \( v_{i_1} \) and \( v_{i_2} \) have the value \textit{false}.

![Diagram showing nodes and edges related to the reduction process.](image-url)
Continuation of the Reduction (Induction-Step)

- If $v_i$ has the value $true$, then $s(i)$ will be visited before $t(i)$ and the nodes $i \# o_1, i \# o_2, \cdots, i \# o_k$ are visited after $first(i)$ and before $last(i)$.
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![Graph Diagram]

- $last(i-1)$
- $first(i)$
- $s(i)$
- $t(i)$
- $v_i$ ist intern
- $i \# o_1$ 
- $i \# o_2$ 
- $i \# o_3$
Continuation of the Reduction (Induction-Step)

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![Diagram showing the relationship between $v_i$, $first(i)$, $s(i)$, $t(i)$, and $last(i)$]
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![Diagram showing the relations between $first(i)$, $i\#i$, $s(i)$, $last(i)$, and $t(i)$, with notes on the visits and memberships in sets.](image-url)
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![Diagram showing the continuation of the reduction]
Continuation of the Reduction (Induction-Step)

- If $v_i$ has the value $false$, then the node $t(i)$ will be visited before $s(i)$ and none of the nodes $i \# o_1, i \# o_2, \ldots, i \# o_k$ will be visited in the interval between $first(i)$ and $last(i)$ visits.

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![Diagram](https://via.placeholder.com/150)
Continuation of the Reduction (Induction-Step)

- If \( v_i \) has the value \textit{false}, then the node \( t(i) \) will be visited before \( s(i) \) and none of the nodes \( i \neq o_1, i \neq o_2, \ldots, i \neq o_k \) will be visited in the interval between \( \text{first}(i) \) and \( \text{last}(i) \) visits.
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\[ \text{last}(i - 1) \]

\[ \text{first}(i) \]

\[ i_1 \neq i \quad i_2 \neq i \]

\[ \text{last}(i) \]

\[ t(i) \]

\[ s(i) \]
Continuation of the Reduction (Induction-Step)

- If \( v_i \) has the value \( false \), then the node \( t(i) \) will be visited before \( s(i) \) and none of the nodes \( i \# o_1, i \# o_2, \ldots, i \# o_k \) will be visited in the interval between \( first(i) \) and \( last(i) \) visits.
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![Diagram showing the relationships between nodes and visits](attachment:diagram.png)
Continuation of the Reduction (Induction-Step)

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\[
\begin{align*}
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&\downarrow \\
&first(i) \\
&\rightarrow i_1 \neq i \quad i_2 \neq i \\
&\uparrow \\
&v_i \text{ ist intern} \\
&\downarrow \\
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&\downarrow \\
&t(i) \\
&\downarrow \\
&j \neq o_1 \\
&\downarrow \\
&j \neq o_2 \\
&\downarrow \\
&j \neq o_3 \\
&\downarrow \\
&s(i)
\end{align*}
\]
Continuation of the Reduction (DFS)

- The construction is a NC-Reduction.
- The construction is the direct simulation of the operations of the circuit.
- The construction may be also given for non-directed graphs.
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Reduction (MAXFLOW)

**Definition (MAXFLOW)**

- **Input:** directed graph \( G = (V, E) \), \( s, t \in V \) and capacity function \( c : E \mapsto \mathbb{N} \).
- **Output:** Maximal flow from \( s \) to \( t \), i.e. function \( f : E \mapsto \mathbb{N} \).
  
  with: \( \forall e \in E : f(e) \leq c(e) \)
  
  and: \( \forall v \in V \setminus \{s, t\} : \sum_{e=(a,v) \in E} f(e) = \sum_{e=(v,a) \in E} f(e) \)

**Theorem:**

The MAXFLOW problem is \( \mathcal{P} \)-complete.

**Proof:**

- Reduction from the problem CVP.
- Show, even to compute the parity of a flow (PMAXFLOW), is \( \mathcal{P} \)-complete.
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Continuation of the Reduction (MAXFLOW)

- W.l.o.g. out-degree of a input node 1.
- W.l.o.g. out-degree of a node is at most 2.
- W.l.o.g. circuit is revers topological sorted, i.e. $v_0$ is the output node.
- W.l.o.g. $v_0$ is an or.
- Given is the circuit graph $G = (V, E)$.
- Input for PMAXFLOW: $G' = (V \cup \{s, t\}, E')$.
- $E \subset E'$.
- $E' \subset E \cup \{(s, v), (v, t) \mid v \in V\}$
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Continuation of the Reduction (MAXFLOW)

- \forall (i, j) \in E : c((i, j)) = 2^i.
- If the value of \( v_i \) is true then let: \( f((i, j)) = 2^i (\forall (i, j) \in E) \).
- If the value of \( v_i \) is false then let: \( f((i, j)) = 0 (\forall (i, j) \in E) \).
- Let \( d(0) = 1 \) and otherwise let \( d(i) \) be the out-degree of \( v_i \).
- Let \((k, i), (j, i) \in E\), and let \( \text{surplus}(i) := 2^k + 2^j - d(i)2^i \).
- \forall i \in V : c(s, i) = 2^i \text{ if the value of } v_i \text{ is true.}
- \forall i \in V : c(s, i) = 0 \text{ if the value of } v_i \text{ is false.}
- \forall i \in V : c(i, t) = \text{surplus}(i) \text{ if } v_i \text{ is an and-node.}
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- $\forall (i, j) \in E : c((i, j)) = 2^i$.
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- \( c(0, t) = 1 \).
Continuation of the Reduction (MAXFLOW)

- $\forall (i, j) \in E : c((i, j)) = 2^i$.
- If the value of $v_i$ is true then let: $f((i, j)) = 2^i \ (\forall (i, j) \in E)$.
- If the value of $v_i$ is false then let: $f((i, j)) = 0 \ (\forall (i, j) \in E)$.
- Let $d(0) = 1$ and otherwise let $d(i)$ be the out-degree of $v_i$.
- Let $(k, i), (j, i) \in E$, and let $\text{surplus}(i) := 2^k + 2^j - d(i)2^i$.
- $\forall i \in V : c(s, i) = 2^i$ if the value of $v_i$ is true.
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Continuation of the Reduction (MAXFLOW)

- \( \forall (i, j) \in E : c((i, j)) = 2^i \).
- If the value of \( v_i \) is \textit{true} then let: \( f((i, j)) = 2^i \ (\forall (i, j) \in E) \).
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- \( \forall i \in V : c(s, i) = 2^i \) if the value of \( v_i \) is \textit{true}.
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- $\forall (i, j) \in E : f(i, j) = c(i, j) = 2^i$ if the value of $v_i$ is $true$.
- $\forall (i, j) \in E : f(i, j) = 0$ if the value of $v_i$ is $false$.
- $f(0, t) = 1$ if $v_0$ has the value $true$.
- Let $overflow(i)$ be the difference between the current input-flow and the output-flow.
  - $f((i, t)) = overflow(i)$ if $v_i$ is an and-node.
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Continuation of the Reduction (MAXFLOW)

**Lemma**

The defined flow is optimal.

- **Use enlarging pathes from s to t:**
  - An edge $e = (i, j)$ in the path is called forward-edge if $f(e) < c(e)$.
  - An edge $e = (j, i)$ in the path is called backward-edge if $f(e) > 0$.

- Known: Flow is maximal $\iff$ there is no enlarging path.

- Assume: there is an enlarging path.
  - A path starts at $s$ with a backward-edge.
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Continuation of the Reduction (MAXFLOW)

- Thus we have three consecutive nodes $j, i, k$ with:
  - $j \neq t$.
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  - $(j, i)$ is a backward-edge.
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  - $(i, j), (i, k)$ are edges in $E'$.
  - $f((i, j)) > 0$ and $f((i, k)) < c((i, k))$.

- $v_i$ may not be a input-node.

- $v_i$ may not be an and-node, because from $j \neq t$ and $f((i, j)) > 0$ we get that all outgoing edges are full.

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  - \((i, k)\) is a forward-edge.
  - \((i, j), (i, k)\) are edges in \( E' \).
  - \( f((i, j)) > 0 \) and \( f((i, k)) < c((i, k)) \).

- \( v_i \) may not be a input-node.

- \( v_i \) may not be an and-node, because from \( j \neq t \) and \( f((i, j)) > 0 \) we get that all outgoing edges are full.

- \( v_i \) may not be an or-node, because from \( k \neq s \) and \( f((i, k)) < c((i, k)) \) we get that all outgoing edges are without flow.
Continuation of the Reduction (MAXFLOW)

- Thus we have three consecutive nodes $j, i, k$ with:
  - $j \neq t$.
  - $k \neq s$.
  - $(j, i)$ is a backward-edge.
  - $(i, k)$ is a forward-edge.
  - $(i, j), (i, k)$ are edges in $E'$.
  - $f((i, j)) > 0$ and $f((i, k)) < c((i, k))$.

- $v_i$ may not be a input-node.

- $v_i$ may not be an and-node, because from $j \neq t$ and $f((i, j)) > 0$ we get that all outgoing edges are full.

- $v_i$ may not be an or-node, because from $k \neq s$ and $f((i, k)) < c((i, k))$ be get that all outgoing edges are without flow.
Legende

- : Nicht relevant
- : Grundlagen, die implizit genutzt werden
- : Idee des Beweises oder des Vorgehens
- : Struktur des Beweises oder des Vorgehens
- : Vollständiges Wissen