1. **Introduction**
   - Networks
   - Permutation Networks
   - Matching on Bipartite Graphs

2. **Recall**
   - Some Basics
   - Theorem of Hall
   - Theorem of Hall

3. **Disjoint Path Lemma**
   - The Lemma
   - The Proof

4. **Routing on Mesh Networks**
   - The Problem
   - Simple Routings
   - Annexstein and Baumslag
Properties of the Networks to be considered

- Number of nodes.
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- Number of edges.
Properties of the Networks to be considered

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  - Diameter, i.e. the longest of all shortest paths.
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Easy routing
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- Easy routing
- May be the graph is based on some group-structure.
- How many graphs are in some family of networks?
Product of Graphs

**Definition:**

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. With $G \times G'$ we denote the product of $G$ and $G'$:

$$G \times G' = (V \times V', E_1 \cup E_2).$$
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Example $L(10) \times C(4)$:

![Diagram of the product of graphs](Attachment:product_of_graphs_diagram.png)
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Example $L(10) \times C(4)$:
Grid of dimension $d$

- Grids: $G(n_1, n_2, \ldots, n_d) = L(n_1) \times L(n_2) \times \cdots \times L(N_d)$ with $n_i > 1$
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- Nodecount: $\prod_{i=1}^{d} n_i$

- Degrees: $\{d, \ldots, 2 \cdot d\}$

- Edgecount: $\sum_{i=1}^{d} (n_i - 1) \prod_{j=1, j \neq i}^{d} n_j$

- Diameter: $\sum_{i=1}^{d} (n_i - 1)$

Grid: $G(14, 4)$:
Torus of dimension $d$

- Torus: $Tr(n_1, n_2, \cdots, n_d) = C(n_1) \times C(n_2) \times \cdots \times C(n_d)$ with $n_i > 1$
Torus of dimension $d$

- Torus: $Tr(n_1, n_2, \cdots, n_d) = C(n_1) \times C(n_2) \times \cdots \times C(N_d)$ with $n_i > 1$
  - Number of nodes: $\prod_{i=1}^{d} n_i$
  - Number of edges: $\prod_{i=1}^{d} n_i$
  - Degree: $2 \cdot d$
  - Diameter: $\sum_{i=1}^{d} \lfloor n_i/2 \rfloor$

- Torus: $Tr(14, 4)$:
Hypercube of dimension $d$

\[
HQ(d) = (V_{HQ(d)}, E_{HQ(d)})
\]
\[
V_{HQ(d)} = \{0, 1\}^d
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E_{HQ(d)} = \{\{w0w', w1w'\} | w0w', w1w' \in V_{HQ(d)}\}
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Note the Gray-Code.
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Number of nodes: $2^d$

Number of edges: $d \cdot 2^{d-1}$

Degree: $d$

Diameter: $d$

Note the Gray-Code.
Hypercube of dimension $d$ (alternative view)

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Cube-Connected Cycles of dimension \( d \)

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\begin{align*}

CCC(d) &= (V_{CCC(d)}, E_{CCC}^c(d) \cup E_{CCC}^h(d)) \\
V_{CCC(d)} &= \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d \\
E_{CCC}^c(d) &= \{(i, w), ((i + 1) \mod d, w)\} \mid w \in \{0, 1\}^d, 0 \leq i < d \\
E_{CCC}^h(d) &= \{((i, w0w'), (i, w1w')) \mid w' \in \{0, 1\}^{n-i-1}, w \in \{0, 1\}^i\}
\end{align*}
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Cube-Connected Cycles of dimension $d$

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Number of nodes: $d \cdot 2^d$  
Degree: 3  
Number of edges: $3 \cdot d \cdot 2^{d-1}$  
Diameter: $2 \cdot d - 2 + \lfloor d/2 \rfloor$
Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h) = \{((i, w0w'), (i, w1w')) \mid w \in \{0, 1\}^i, w' \in \{0, 1\}^{n-i-1}\}$$

$$V_{BF(d)} = \{0, 1, \cdots, d-1\} \times \{0, 1\}^d$$

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Butterfly of dimension $d$

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\begin{align*}
BF(d) &= (V_{BF(d)}, E^c_{BF(d)} \cup E^h_{nCCC(d)}) = \{(i, w0w'), (i, w1w')\} | w \in \{0,1\}^i, w' \in \{0,1\}^{n-i-1} \\
V_{BF(d)} &= \{0,1, \ldots, d-1\} \times \{0,1\}^d \\
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Number of nodes: $d \cdot 2^d$
Degree: 4
Number of edges: $d \cdot 2^{d+1}$
Diameter: $d + \lfloor d/2 \rfloor$
DeBruijn network of dimension $d$

- **DeBruijn network:**
  
  $$DB(d) = (V_{DB(d)}, E^s_{DB(d)} \cup E^{se}_{DB(d)})$$
  
  $$V_{DB(d)} = \{0, 1\}^d$$
  
  $$E^s_{DB(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}$$
  
  $$E^{se}_{DB(d)} = \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}$$
DeBruijn network of dimension $d$

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  \[
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DeBruijn network of dimension $d$

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  \[
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  \]
DeBruijn network of dimension $d$

- DeBruijn network:
  \[
  DB(d) = (V_{DB(d)}, E^{s}_{DB(d)} \cup E^{se}_{DB(d)})
  \]
  \[
  V_{DB(d)} = \{0, 1\}^d
  \]
  \[
  E^{s}_{DB(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{DB(d)}\}
  \]
  \[
  E^{se}_{DB(d)} = \{(aw, wb) \mid a \in \{0, 1\}, b = 1 - a, aw, wb \in V_{DB(d)}\}
  \]

- Number of nodes: $2^d$
- Degree: $2 + 2$
- Number of edges: $2^{d+1}$
- Diameter: $d$
Shuffle-Exchange network of dimension \(d\)

- **Shuffle-Exchange network:**
  \[
  \begin{align*}
  SE(d) &= (V_{SE(d)}, E_{SE(d)}^s \cup E_{SE(d)}^e) \\
  V_{SE(d)} &= \{0, 1\}^d \\
  E_{SE(d)}^s &= \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)}\} \\
  E_{SE(d)}^e &= \{(wa, wb) \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}
  \end{align*}
  \]
Shuffle-Exchange network of dimension $d$

- **Shuffle-Exchange network:**

  $$SE(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})$$

  $$V_{SE(d)} = \{0, 1\}^d$$

  $$E^s_{SE(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}$$

  $$E^e_{SE(d)} = \{(wa, wb) \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}$$
Shuffle-Exchange network of dimension \( d \)

- Shuffle-Exchange network:
  \[
  SE(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})
  \]
  \[
  V_{SE(d)} = \{0, 1\}^d
  \]
  \[
  E^s_{SE(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}
  \]
  \[
  E^e_{SE(d)} = \{(wa, wb) \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}
  \]
Shuffle-Exchange network of dimension $d$

- Shuffle-Exchange network:
  \[
  \text{SE}(d) = (V_{SE(d)}, E^s_{SE(d)} \cup E^e_{SE(d)})
  \]
  \[
  V_{SE(d)} = \{0, 1\}^d
  \]
  \[
  E^s_{SE(d)} = \{(aw, wa) \mid a \in \{0, 1\}, aw, wa \in V_{SE(d)}\}
  \]
  \[
  E^e_{SE(d)} = \{(wa, wb) \mid a \in \{0, 1\}, b = 1 - a, wa, wb \in V_{SE(d)}\}
  \]

- Number of nodes: $2^d$
- Degree: $2 + 2$
- Number of edges: $2^{d+1}$
- Diameter: $2 \cdot d - 1$
Recall Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E^c_{BF(d)} \cup E^h_{BF(d)})$$
$$V_{BF(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d$$
$$E^c_{BF(d)} = \{\{(i, w), ((i + 1) \mod d, w)\} \mid w \in \{0, 1\}^d, 0 \leq i < d\}$$
$$E^h_{BF(d)} = \{\{(i, w0w'), ((i + 1) \mod d, w1w')\} \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i\}$$

Number of nodes: $d \cdot 2^d$
Degree: 4
Number of edges: $d \cdot 2^{d+1}$
Diameter: $d + \lfloor d/2 \rfloor$
Recall Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h)$$

$$V_{BF(d)} = \{0, 1, \ldots, d - 1\} \times \{0, 1\}^d$$

$$E_{BF(d)}^c = \{((i, w), ((i + 1) \mod d, w)) \mid w \in \{0, 1\}^d, 0 \leq i < d\}$$

$$E_{BF(d)}^h = \{((i, w0w'), ((i + 1) \mod d, w1w')) \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i\}$$

Number of nodes: $d \cdot 2^d$
Degree: 4
Number of edges: $d \cdot 2^{d+1}$
Diameter: $d + \lfloor d/2 \rfloor$
Unwrapped Butterfly of dimension $d$

$$BF(d) = (V_{BF(d)}, E_{BF(d)}^c \cup E_{BF(d)}^h)$$

$$V_{BF(d)} = \{0, \cdots, d\} \times \{0, 1\}^d$$

$$E_{BF(d)}^c = \{(i, w), (i + 1, w)\} \mid w \in \{0, 1\}^d, 0 \leq i < d$$

$$E_{BF(d)}^h = \{(i, w0w'), (i + 1, w1w')\} \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i, 0 \leq i < d$$

Number of nodes: $(d + 1) \cdot 2^d$

Number of edges: $d \cdot 2^{d+1}$
Permutation network

\[
PN(d) = (V_{PN(d)}, E^c_{PN(d)} \cup E^h_{PN(d)})
\]

\[
V_{PN(d)} = \{1, 2, \ldots, d, -1, -2, \ldots, -d\} \times \{0, 1\}^d
\]

\[
E^c_{PN(d)} = \{(i, w), (i + 1, w) \mid w \in \{0, 1\}^d, 1 \leq i < d\}
\]

\[
\bigcup \{(1, w), (-1, w) \mid w \in \{0, 1\}^d\}
\]

\[
\bigcup \{(-i, w), (-i - 1, w) \mid w \in \{0, 1\}^d, 1 \leq i < d\}
\]

\[
E^h_{PN(d)} = \{(i, w0w'), (i + 1, w1w') \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i\}
\]

\[
\bigcup \{(1, w0w'), (-1, w1w') \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i\}
\]

\[
\bigcup \{(-i, w0w'), (-i - 1, w1w') \mid w \in \{0, 1\}^{n-i-1}, w' \in \{0, 1\}^i\}
\]
Large Example Permutation network
Extended Permutation network

\[ PN(n, d) = (V_{PN(n,d)}, E_{PN(d)}^c \cup E_{PN(n,d)}^h) \]

\[ V_{PN(d)} = \{1, 2, \cdots, d, -1, -2, \cdots, -d\} \times \{0, \ldots, n - 1\}^d \]

\[ E_{PN}^c(n,d) = \{\{(i, w), (i + 1, w)\} | w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d\} \]

\[ E_{PN(d)}^h = \{\{(1, w0w'), (i + 1, w1w')\} | w \in \{0, \ldots, n - 1\}^{n - i - 1}, w' \in \{0, \ldots, n - 1\}^{i}\} \]

\[ E_{PN(n,d)}^h = \{\{(1, w0w'), (-1, w1w')\} | w \in \{0, \ldots, n - 1\}^{n - i - 1}, w' \in \{0, \ldots, n - 1\}^{i}\} \]

\[ E_{PN(n,d)}^h = \{\{(-i, w0w'), (-i - 1, w1w')\} | w \in \{0, \ldots, n - 1\}^{n - i - 1}, w' \in \{0, \ldots, n - 1\}^{i}\} \]

The 2d \cdot n^d nodes of (n, d)-PN are partitioned into 2d levels and n^d columns.
Extended Permutation network

\[
PN(n, d) = (V_{PN(n,d)}, E_{PN(n,d)}^c \cup E_{PN(n,d)}^h)
\]
\[
V_{PN(d)} = \{1, 2, \ldots, d, -1, -2, \ldots, -d\} \times \{0, \ldots, n-1\}^d
\]
\[
E_{PN(n,d)}^c = \{\{(i, w), (i + 1, w)\} | w \in \{0, \ldots, n-1\}^d, 1 \leq i < d\}
\]
\[
E_{PN(n,d)}^h = \{\{(i, w0w'), (i + 1, w1w')\} | w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i\}
\]
\[
\cup \{\{(1, w0w'), (−1, w1w')\} | w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i\}
\]
\[
\cup \{\{(-i, w0w'), (-i - 1, w1w')\} | w \in \{0, \ldots, n-1\}^*, w' \in \{0, \ldots, n-1\}^i\}
\]

- The 2d \cdot n^d nodes of (n, d)-PN are partitioned into 2d levels and n^d columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
Extended Permutation network

\[ \text{PN}(n, d) = (V_{\text{PN}(n,d)}, E_{\text{PN}(n,d)}^c \cup E_{\text{PN}(n,d)}^h) \]
\[ V_{\text{PN}(n,d)} = \{1, 2, \cdots, d, -1, -2, \cdots, -d\} \times \{0, \ldots, n-1\}^d \]
\[ E_{\text{PN}(n,d)}^c = \{((i, w), (i + 1, w)) | w \in \{0, \ldots, n-1\}^d, 1 \leq i < d\} \]
\[ \bigcup \{((1, w), (-1, w)) | w \in \{0, \ldots, n-1\}^d\} \]
\[ \bigcup \{((-i, w), (-i - 1, w)) | w \in \{0, \ldots, n-1\}^d, 1 \leq i < d\} \]
\[ E_{\text{PN}(n,d)}^h = \{((i, w0w'), (i + 1, w1w')) | w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i\} \]
\[ \bigcup \{(1, w0w'), (-1, w1w') | w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i\} \]
\[ \bigcup \{((-i, w0w'), (-i - 1, w1w')) | w \in \{0, \ldots, n-1\}^i, w' \in \{0, \ldots, n-1\}^i\} \]

- The 2d \( \cdot \) \( n^d \) nodes of \((n, d)\)-PN are partitioned into 2d levels and \( n^d \) columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
- Columns are labeled with strings of length \( d \) over \{0, 1, \ldots, n - 1\}. 

Extended Permutation network

\[ PN(n, d) = (V_{PN(n,d)}, E^c_{PN(n,d)} \cup E^h_{PN(n,d)}) \]

\[ V_{PN(d)} = \{1, 2, \ldots, d, -1, -2, \ldots, -d\} \times \{0, \ldots, n - 1\}^d \]

\[ E^c_{PN(n,d)} = \{(i, w), (i + 1, w)\} \mid w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d \]

\[ E^h_{PN(n,d)} = \{(1, w0w'), (i + 1, w1w')\} \mid w \in \{0, \ldots, n - 1\}^{n-i-1}, w' \in \{0, \ldots, n - 1\}^i \}

\[ \cup \{(1, w0w'), (-1, w1w')\} \mid w \in \{0, \ldots, n - 1\}^{n-i-1}, w' \in \{0, \ldots, n - 1\}^i \]

\[ \cup \{(-i, w0w'), (-i - 1, w1w')\} \mid w \in \{0, \ldots, n - 1\}^*, w' \in \{0, \ldots, n - 1\}^i \]

- The 2\(d \cdot n^d\) nodes of \((n, d)\)-PN are partitioned into 2\(d\) levels and \(n^d\) columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
- Columns are labeled with strings of length \(d\) over \(\{0, 1, \ldots, n - 1\}\).
- The parameter \(d\) is called the **dimension** of the network.
Extended Permutation network

\[
PN(n, d) = (V_{PN(n,d)}, E^c_{PN(n,d)} \cup E^h_{PN(n,d)})
\]

\[
V_{PN(d)} = \{1, 2, \ldots, d, -1, -2, \ldots, -d\} \times \{0, \ldots, n - 1\}^d
\]

\[
E^c_{PN(n,d)} = \{(i, w), (i + 1, w) \mid w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d\}
\]

\[
E^h_{PN(n,d)} = \{(1, w0w'), (i + 1, w1w') \mid w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d\} \cup \{(1, w0w'), (w', -1, w') \mid w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d\} \cup \{(-i, w0w'), (-i - 1, w') \mid w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d\}
\]

- The $2d \cdot n^d$ nodes of $(n, d)$-PN are partitioned into $2d$ levels and $n^d$ columns.
- Levels are numbered $-d, \ldots, -1, 1, \ldots, d$.
- Columns are labeled with strings of length $d$ over $\{0, 1, \ldots, n - 1\}$.
- The parameter $d$ is called the dimension of the network.
- The nodes on level $-d$ [resp. $d$] are called inputs [resp. outputs].
Extended Permutation network

\[ \text{PN}(n, d) = (V_{\text{PN}(n,d)}, E^c_{\text{PN}(d)} \cup E^h_{\text{PN}(n,d)}) \]
\[ V_{\text{PN}(d)} = \{1, 2, \cdots, d, -1, -2, \cdots, -d\} \times \{0, ..., n - 1\}^d \]
\[ E^c_{\text{PN}(n,d)} = \{\{(i, w), (i + 1, w)\} | w \in \{0, ..., n - 1\}^d, 1 \leq i < d\} \]
\[ \cup \{\{(1, w), (-1, w)\} | w \in \{0, ..., n - 1\}^d\} \]
\[ \cup \{\{(-i, w), (-i - 1, w)\} | w \in \{0, ..., n - 1\}^d, 1 \leq i < d\} \]
\[ E^h_{\text{PN}(n,d)} = \{\{(i, w0w'), (i + 1, w1w')\} | w \in \{0, ..., n-1\}^{n-i-1}, w' \in \{0, ..., n-1\}^i\} \]
\[ \cup \{\{(1, w0w'), (-1, w1w')\} | w \in \{0, ..., n-1\}^{n-i-1}, w' \in \{0, ..., n-1\}^i\} \]
\[ \cup \{\{(-i, w0w'), (-i - 1, w1w')\} | w \in \{0, ..., n-1\}^{*}, w' \in \{0, ..., n-1\}^i\} \]

- The 2\(d \cdot n^d\) nodes of (\(n, d\))-PN are partitioned into 2\(d\) levels and \(n^d\) columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
- Columns are labeled with strings of length \(d\) over \(\{0, 1, \ldots, n - 1\}\).
- The parameter \(d\) is called the dimension of the network.
- The nodes on level \(-d\) [resp. \(d\)] are called inputs [resp. outputs].
- Each input (output) is identified with its column label.
Extended Permutation network

\[ PN(n, d) \quad = \quad (V_{PN(n,d)}, E^c_{PN(n,d)} \cup E^h_{PN(n,d)}) \]

\[ V_{PN(d)} \quad = \quad \{1, 2, \cdots, d, -1, -2, \cdots, -d\} \times \{0, \ldots, n-1\}^d \]

\[ E^c_{PN(n,d)} \quad = \quad \{(i, w), (i+1, w)\} \mid w \in \{0, \ldots, n-1\}^d, 1 \leq i < d \]

\[ \bigcup \{((1, w), (-1, w)) \mid w \in \{0, \ldots, n-1\}^d\} \]

\[ \bigcup \{((-i, w), (-i-1, w)) \mid w \in \{0, \ldots, n-1\}^d, 1 \leq i < d\} \]

\[ E^h_{PN(n,d)} \quad = \quad \{(i, w0w'), (i+1, w1w')\} \mid w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i \]

\[ \bigcup \{(1, w0w'), (-1, w1w')\} \mid w \in \{0, \ldots, n-1\}^{n-i-1}, w' \in \{0, \ldots, n-1\}^i \}

\[ \bigcup \{((-i, w0w'), (-i-1, w1w')\} \mid w \in \{0, \ldots, n-1\}^*, w' \in \{0, \ldots, n-1\}^i \} \]

- The 2\(d\) \cdot n^d\ nodes of (\(n, d\))-PN are partitioned into 2\(d\) levels and \(n^d\) columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
- Columns are labeled with strings of length \(d\) over \(\{0, 1, \ldots, n-1\}\).
- The parameter \(d\) is called the dimension of the network.
- The nodes on level \(-d\) [resp. \(d\)] are called inputs [resp. outputs].
- Each input (output) is identified with its column label.
- Permutation networks have a recursive structure.
Extended Permutation network

\[ PN(n,d) = (V_{PN(n,d)}, E_{PN}^c \cup E_{PN}^h) \]
\[ V_{PN(d)} = \{1, 2, \ldots, d, -1, -2, \ldots, -d\} \times \{0, \ldots, n - 1\}^d \]
\[ E_{PN(n,d)}^c = \{(i, w), (i + 1, w)\} | w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d \]
\[ \cup \{(1, w), (-1, w)\} | w \in \{0, \ldots, n - 1\}^d \]
\[ \cup \{(-i, w), (-i - 1, w)\} | w \in \{0, \ldots, n - 1\}^d, 1 \leq i < d \]
\[ E_{PN(n,d)}^h = \{(i, w0w'), (i + 1, w1w')\} | w \in \{0, \ldots, n - 1\}^{n-i-1}, w' \in \{0, \ldots, n - 1\}^i \]
\[ \cup \{(1, w0w'), (-1, w1w')\} | w \in \{0, \ldots, n - 1\}^{n-i-1}, w' \in \{0, \ldots, n - 1\}^i \]
\[ \cup \{(-i, w0w'), (-i - 1, w1w')\} | w \in \{0, \ldots, n - 1\}^*, w' \in \{0, \ldots, n - 1\}^i \]

- The \(2d \cdot n^d\) nodes of \((n, d)\)-PN are partitioned into \(2d\) levels and \(n^d\) columns.
- Levels are numbered \(-d, \ldots, -1, 1, \ldots, d\).
- Columns are labeled with strings of length \(d\) over \(\{0, 1, \ldots, n - 1\}\).
- The parameter \(d\) is called the dimension of the network.
- The nodes on level \(-d\) [resp. \(d\)] are called inputs [resp. outputs].
- Each input (output) is identified with its column label.
- Permutation networks have a recursive structure.
- The Permutation network \((n, 1)\) is complete (all possible connections).
Definition (Bipartite graph)

A graph \( G = (V, E) \) is called bipartite if there exist \( U, W \subset V \) with \( U \cup W = V \) and \( \forall e \in E : \exists u \in U, w \in W : e = \{u, w\} \).

Definition (Matching)

For a given Graph \( G = (V, E) \), a matching \( M \subset E \) is a set of non-incident edges, i.e., \( \forall e, f \in M : e \cap f = \emptyset \).

Definition (Perfect matching)

A matching \( M \) is called perfect, if it contains all nodes from \( G \): \( \forall v \in V \exists e \in M : v \in e \).
Theorem of Hall

Definition

Let $G = (V_1, V_2, E)$ be a bipartite graph, and $A \subseteq V_1$. We denote:

$$\Gamma(A) = \{v \in V_2 \mid (v, w) \in E, w \in A\}.$$
Theorem of Hall

Definition

Let $G = (V_1, V_2, E)$ be a bipartite graph, and $A \subseteq V_1$. We denote:

$$\Gamma(A) = \{v \in V_2 \mid (v, w) \in E, w \in A\}.$$  

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exits a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$
Theorem of Hall

Definition

Let $G = (V_1, V_2, E)$ be a bipartite graph, and $A \subseteq V_1$. We denote:

$$\Gamma(A) = \{ v \in V_2 \mid (v, w) \in E, w \in A \}.$$

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exits a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$

Corollary

Every regular bipartite Graph $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$ contains a complete matching.
Proof (Hall)

Theorem (Hall)

Let \( G = (V_1, V_2, E) \) be a bipartite graph. There exits a complete matching from \( V_1 \) to \( V_2 \), iff for each \( A \subseteq V_1 \) we have

\[
|\Gamma(A)| \geq |A|.
\]

\[\Rightarrow\] simple:
Proof (Hall)

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exists a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$ 

simple:

- Let $M$ be a matching with $|M| = |V_1|$ and let $A \subseteq V_1$ arbitrary.
Proof (Hall)

Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exits a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$ 

simple:

- Let $M$ be a matching with $|M| = |V_1|$ and let $A \subseteq V_1$ arbitrary.
- $|\Gamma(A)| = |\{v \in V_2 \mid (v, w) \in E, w \in A\}|.$
Proof (Hall)

**Theorem (Hall)**

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exists a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

$$|\Gamma(A)| \geq |A|.$$

---

simple:

- Let $M$ be a matching with $|M| = |V_1|$ and let $A \subseteq V_1$ arbitrary.
- $|\Gamma(A)| = |\{v \in V_2 \mid (v, w) \in E, w \in A\}|$.
- $|\Gamma(A)| \geq |\{v \in V_2 \mid (v, w) \in M, w \in A\}|$. 
Proof (Hall)

Theorem (Hall)

Let \( G = (V_1, V_2, E) \) be a bipartite graph. There exits a complete matching from \( V_1 \) to \( V_2 \), iff for each \( A \subseteq V_1 \) we have

\[ |\Gamma(A)| \geq |A|. \]

\[ \implies \text{ simple:} \]

- Let \( M \) be a matching with \( |M| = |V_1| \) and let \( A \subset V_1 \) arbitrary.
- \( |\Gamma(A)| = |\{ v \in V_2 \mid (v, w) \in E, w \in A\}|. \)
- \( |\Gamma(A)| \geq |\{ v \in V_2 \mid (v, w) \in M, w \in A\}|. \)
- \( |\Gamma(A)| \geq |A|. \)
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Theorem (Hall)

Let $G = (V_1, V_2, E)$ be a bipartite graph. There exists a complete matching from $V_1$ to $V_2$, iff for each $A \subseteq V_1$ we have

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- $|A_1' \cup \{a\}| > |A_2'|$. 
Definition (Edge coloring)

Let $G = (V, E)$ be a graph. $\psi : E \rightarrow \{1, ..., k\}$ is an edge coloring if every pair of incident edges $e_1, e_2$ is colored in different colors, i.e., $\psi(e_1) \neq \psi(e_2)$. 
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Definition

The Edge-Colouring-Problem for a graph $G$ corresponds to the node-colouring of $L(G)$: 
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**Theorem (Vizing 1965)**

$\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$. 
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**Theorem**  
$\chi'(G) \geq \omega(L(G)) \geq \Delta(G)$. 
Definition (Regular graphs)
A graph is called $n$-regular, $n \in \mathbb{N}$, if all nodes have the same degree $n$.

Theorem
A bipartite $n$-regular graph $G = (V_1 \cup V_2, E)$ has an edge coloring with $n$ colors.
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Any bipartite graph with degree \( \Delta \) is \( \Delta \) edge-colourable (Running-Time \( O(nm) \)).
Edge-Colouring II

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* Any graph with degree \( \Delta \) is \( \Delta + 1 \) edge-colourable (Running-Time \( O(nm) \)).
Simple Proof

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- Now Hall’s theorem implies that $G$ has a perfect matching $M$.
- The edges of $M$ get assigned color $n - 1$.
- The remaining graph is $n - 1$-regular and, by our induction hypothesis, can be colored with the remaining colors $\{0, 1, \ldots, n - 2\}$.
Proof (König)

Theorem (König)

Any bipartite graph with degree $\Delta$ is $\Delta$ edge-colourable (Running-Time $O(nm)$).

- Show how to colour an edge $(a, b)$ in $O(n)$ time.
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- Observe now the graph $H_{a,b}$, who consists only of edges coloured with $c_a, c_b$. 
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- Running-Time: store for each node and colour the corresponding edge.
Disjoint Path Lemma

Lemma (Disjoint Path Lemma)

For every permutation $\pi : \{0, 1, \ldots, n - 1\}^d \rightarrow \{0, 1, \ldots, n - 1\}^d$, there is a collection of $n^d$ node disjoint paths in $(n, d)$-PN that, for every $a \in \{0, 1, \ldots, n - 1\}^d$, contains a path $W_a$ connecting input $a$ with output $\pi(a)$. 
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- Induction over \( d \).
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Proof:

- Induction over $d$.
- **Base Case:** $d = 1$: This case is trivially true since the inputs and the outputs are completely connected in $P(n, 1)$.
- **Induction step:** $(d - 1) \rightarrow d$. 
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- **Idea is:** Recall the recursive description of \((n, d)\)-PN.
Proof (Recursive Description)

- An input/output-pair \((a, \pi(a))\) that should be connected by a path is called a request.
Proof (Recursive Description)

- An input/output-pair \((a, \pi(a))\) that should be connected by a path is called a request.

- For each request, we choose a subnetwork \(B^{(i)}, i \in \{0, 1, ..., n - 1\}\), through which the request is routed.
Proof (Recursive Step)

The choices of the subnetworks satisfy the following properties:

1. Each input of each subnetwork is used by exactly one of the requests.
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Thus, these requests can be routed along disjoint paths in $B^{(i)}$ by our induction hypothesis, so that the Disjoint Path Lemma follows.

We have to show how to choose the subnetworks for the requests.
Towards this end, we define the following bipartite conflict graph:
Proof (by Conflict Graph)

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\[ G_\pi = (\{u_x \mid x \in \{0, 1, \ldots, n - 1\}^{d-1}\} \cup \{v_x \mid x \in \{0, 1, \ldots, n - 1\}^{d-1}\}, E_{\pi}) \]
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Proof (by Conflict Graph)

- Towards this end, we define the following bipartite conflict graph:
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- Analogously, edges incident to the same node \(v_x\) represent an output conflict and the corresponding requests should be routed through different subnetworks as well.
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- $G_{\pi}$ is $n$-regular and bipartite.
Proof (by Conflict Graph)

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- By the Coloring Lemma, \( E_\pi \) can be colored with colors \(0, 1, \ldots, n-1\).
- For any \(i \in \{0, \ldots, n-1\}\), the edges of color \(i\) build a matching in \( G_\pi \) and, hence, the corresponding requests do not have input or output conflict.
- If all requests of color \(i \in \{0, \ldots, n-1\}\) are routed through subnetwork \(B^{(i)}\) then the Properties 1 and 2 are satisfied.
Proof (by Conflict Graph)

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- If all requests of color $i \in \{0, \ldots, n-1\}$ are routed through subnetwork $B^{(i)}$ then the Properties 1 and 2 are satisfied.
- This completes the proof of the Disjoint Path Lemma.
The Routing Problem

Definition (Permutation routing problem)

Let $G = (V, E)$ be a network. A permutation routing problem is defined by a permutation $\pi : V \to V$. Each node $v \in V$ has a message (packet) that shall be routed to node $\pi(v)$. 

Note: We use the synchronous congestion model from Peleg’s book: In each step, each edge can forward one packet in each direction.
The Routing Problem

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Note: We use the synchronous congestion model from Peleg’s book: In each step, each edge can forward one packet in each direction.
On $M(n, d)$, sending a packet from a source to a destination can be done by using dimension-by-dimension routing, that is, first the packet is routed to the target position with respect to dimension 0, then with respect to dimension 1, and so on.

On the two-dimensional array $M(n, 2)$, this approach is also called row-column routing as a packet is first routed to the target position in the row and then to the target position in the column.
On the hypercube $M(2, d)$, the paths chosen by dimension-by-dimension routing are called **bit-fixing paths**.
In the following, let $D$ denote the diameter of the network.

**Observation**

*Every permutation $\pi$ can be routed along dimension-by-dimension paths in at most $D$ steps on $M(n,1)$ and $M(n,2)$.*
More Routing on Meshes

In the following, let $D$ denote the diameter of the network.

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Every permutation $\pi$ can be routed along dimension-by-dimension paths in at most $D$ steps on $M(n,1)$ and $M(n,2)$.

**Lemma**

Consider $M(n,3)$. There is a permutation $\pi$ such that every packet routing algorithm using dimension-by-dimension paths needs at least $\Omega(D^2)$ steps for routing $\pi$. 
Idea for Proof of Lemma

- A Grid
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- Exchange between $a_i$'s
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- Exchange between $h_i$'s
Idea for Proof of Lemma

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- Exchange between \( b_i \)'s
- Exchange between \( c_i \)'s
- Exchange between \( d_i \)'s
- Exchange between \( e_i \)'s
- Exchange between \( f_i \)'s
- Exchange between \( g_i \)'s
- Exchange between \( h_i \)'s
- Red edge is always used in both directions.
More Routing on Meshes

In the following, let $D$ denote the diameter of the network.

**Observation**

*Every permutation $\pi$ can be routed along dimension-by-dimension paths in at most $D$ steps on $M(n, 1)$ and $M(n, 2)$.***

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*Consider $M(n, 3)$. There is a permutation $\pi$ such that every packet routing algorithm using dimension-by-dimension paths needs at least $\Omega(D^2)$ steps for routing $\pi$.***

**Question:**

Can one achieve time complexity $O(D)$ on meshes of dimension $d > 2$?
More Routing on Meshes

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**Idea:** Translate the routing algorithm for permutation networks into an efficient algorithm for mesh networks.
Notation (d-dimensional mesh of side length n)

Let $n \geq 1$ and $d \geq 0$ be integers. The $d$-dimensional mesh of side length $n$, denoted $M(n, d)$, is the graph $G(\{0, 1, ..., n - 1\}^d, E)$ with

$$E = \{ \{a, b\} \mid \exists i \in \{0, 1, ..., d - 1\} : |a_i - b_i| = 1 \text{ and } a_j = b_j, \text{ for } j \neq i \}.$$

- $M(n, d)$ has $n^d$ nodes and $d \cdot n^d - d \cdot n^{d-1}$ edges.
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- \(M(n, d)\) has \(n^d\) nodes and \(d \cdot n^d - d \cdot n^{d-1}\) edges.
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- For fixed numbers \(i \in \{0, 1, \ldots, d - 1\}, \ell \in \{0, 1, \ldots, n - 1\}\), the subgraph \(M(n, d)\mid_{\{a \in \{0,1,\ldots,n-1\}^d \mid a_i = \ell\}}\) is isomorphic to \(M(n, d - 1)\).
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- For a fixed vector $b \in \{0, 1, \ldots, n-1\}^{d-1}$, the subgraph $M(n, d)|_{\{a \in \{0, 1, \ldots, n-1\}^d \mid a = ib\}}$ is isomorphic to $M(n, 1)$. 
Example of the Decomposition

Illustration of the decomposition of $M(n, d)$ into:

$n$ submeshes $M_0, ..., M_{n-1}$ and
Example of the Decomposition

Illustration of the decomposition of $M(n, d)$ into:

$n$ submeshes $M_0, ..., M_{n-1}$ and one of the columns $A_b$: 

![Diagram showing the decomposition of a mesh into submeshes and a column.](image)
Example of the Decomposition

Illustration of the decomposition of $M(n, d)$ into:

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![Diagram showing the decomposition of a mesh into submeshes](image-url)
Annexstein and Baumslag

Theorem (Annexstein and Baumslag 1990)

\( M(n, d) \) can route a permutation in time \( O(n \cdot d) = O(D) \).

Proof We 'simulate' the \((n, d)\)-PN on \( M(n, d) \).

- Decompose \( M(n, d) \) into \( n \) submeshes \( M_0, M_1, ..., M_{n-1} \) of dimension \( d - 1 \) by fixing the last digit of the label, that is, for \( i \in \{0, 1, ..., n - 1\} \),

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- These submeshes are connected by one-dimensional meshes (columns), one for each \(d - 1\)-dimensional vector \(b \in \{0, 1, \ldots, n - 1\}^{d-1}\), namely

\[ A_b := M(n, d)|_{\{a | a = ib\}}. \]
Example of the Decomposition
Proof

Analogous to the algorithm for permutation networks, we color the requests (packets) with $n$ colors. The following algorithm then performs the routing:

- Packets with color $i$ route from their sources to submesh $M_i$ (inside the corresponding column $A_b$)
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- In each submesh \( M_i \): Each packet is routed from the position isomorphic to its source to the position isomorphic to its destination. (The permutation routing problem in \( M_i \) is solved recursively.)
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- Let \( T(n, d) \) be the routing time for \( M(n,d) \).
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- Solving the recurrence gives $T(n, d) = (2d - 1)(n - 1) \leq 2D$. 


