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Very simple Algorithm (Idea)
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22 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
33 0 1 1 0 1 0 0 1 0 0 0 0 1 0 0 1
41 1 1 1 0 1 0 0 1 0 0 0 0 1 0 1 1
26 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1
59 1 1 1 1 1 1 0 1 1 0 1 1 0 1 1 1
57 1 1 1 1 1 0 1 1 0 1 0 0 1 1 1 1
52 1 1 1 0 1 0 1 1 0 0 0 0 1 1 1 1
61 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1
27 0 1 1 0 1 0 0 0 0 0 0 0 1 0 0 1
49 1 1 1 0 1 0 0 1 0 0 0 0 1 1 1 1
67 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1
23 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1
56 1 1 1 0 1 0 1 1 0 1 0 0 1 1 1 1
14 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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```
Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.

- Compare pairwise all elements and count the number of smaller elements.

- Use $n^2$ processors.

- Program: SimpleSort
  Eingabe: $s_1, \ldots, s_n$.
  for all $P_{i,j}$ where $1 \leq i, j \leq n$ do in parallel
    if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$
  for all $i$ where $1 \leq i \leq n$ do in parallel
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      Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
      $P_i(s_i) \rightarrow R_{q_i+1}$.

- Complexity: $T(n) = O(\log n)$ and $P(n) = n^2$.

- Efficiency: $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.

- Model: CREW.
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- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$.
- Sort parallel each block.
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Complexity: $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.

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Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}}(\text{EREW})(n) = \Theta(n/P(n) + \log n \cdot \log P(n))$. 
- $T(n) = O(n/P(n) \cdot \log(n/P(n))) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
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- Efficiency:

$$Eff(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}$$

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Lower Bound

Theorem:

For any parallel sorting algorithm $Srt$ with $P_{Srt}(n) = O(n)$ hold:

$$T_{Srt}(n) = \Omega(\log(n)).$$

Proof:

- Lower bound for sequential is $\Theta(n \log n)$.
- One needs $O(n \log n)$ comparisons.
- In each parallel step are at most $o(n)$ comparisons possible.
- Thus with less steps we have a contradiction to the lower bound for sequential.

Situation at this point:

- Inefficient algorithms with: $T(n) = O(\log n)$ and $P(n) = n^2$.
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- Inefficient algorithms with: \( T(n) = O(\log n) \) and \( P(n) = n^2 \).
- Nearly efficient algorithm with: \( T(n) = O(\log^2 n) \) and \( P(n) = o(n) \).
Basic Operation for Sorting

- **Identify basic operation for sorting.**
- **Assume:** sorting key is \( s_1, \cdots, s_n \).
- **Program:** `compare_exchange(i, j)`
  
  if \( s_i > s_j \) then exchange \( s_i \leftrightarrow s_j \)

- **Symbolic view (Batcher):**
  
  \[
  \begin{align*}
  y & \quad \cdots \quad \max(x, y) \\
  \quad \cdots \quad & \ \\
  \min(x, y) & \quad \cdots \quad x
  \end{align*}
  \]

- **Basic building block for sorting networks.**
- **Base for Odd-Even merge**
- **Form this we build the optimal algorithm by Cole**
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  \hline \\
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  - $y$ \hspace{2cm} $\max(x, y)$
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- Basic building block for sorting networks.
- Base for Odd-Even merge
- **Form this we build the optimal algorithm by Cole**
Odd-even Merge (Definition)

- **Input:** Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $interleave(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

\[ T_{interleave}(n) = O(1) \text{ mit } P_{interleave}(n) = O(n) \]
Odd-even Merge (Definition)

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- Then we define: \( \text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n) \).

\[
\begin{align*}
\text{s}_1 & \quad \text{s}_2 & \quad \text{s}_3 & \quad \text{s}_4 & \quad \text{s}_5 & \quad \text{s}_6 & \quad \text{s}_7 & \quad \text{s}_8 & \quad \text{s}'_1 & \quad \text{s}'_2 & \quad \text{s}'_3 & \quad \text{s}'_4 & \quad \text{s}'_5 & \quad \text{s}'_6 & \quad \text{s}'_7 & \quad \text{s}'_8 \\
\text{r}_1 & \quad \text{r}_2 & \quad \text{r}_3 & \quad \text{r}_4 & \quad \text{r}_5 & \quad \text{r}_6 & \quad \text{r}_7 & \quad \text{r}_8 & \quad \text{r}_9 & \quad \text{r}_{10} & \quad \text{r}_{11} & \quad \text{r}_{12} & \quad \text{r}_{13} & \quad \text{r}_{14} & \quad \text{r}_{15} & \quad \text{r}_{16}
\end{align*}
\]

- \( T_{\text{interleave}}(n) = O(1) \) mit \( P_{\text{interleave}}(n) = O(n) \)
### Odd-even Merge (Definition)

- **Input:** Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $\text{interleave}(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

![Diagram of interleave operation](image)

- $T_{\text{interleave}}(n) = O(1)$ mit $P_{\text{interleave}}(n) = O(n)$
Odd-even Merge (Definition)

- Input: Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
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$T_{\text{interleave}}(n) = O(1)$ mit $P_{\text{interleave}}(n) = O(n)$
Odd-even Merge (Definition)

- Program: `odd_even(S)`
  
  for all `i` where `1 < i < n` and `i` even do in parallel
  
  `compare_exchange(i, i + 1).`

- `T_{compare\_exchange}(n) = O(1)` mit `P_{compare\_exchange}(n) = O(n)`
Odd-even Merge (Definition)

Programm: odd_even(S)

for all \( i \) where \( 1 < i < n \) and \( i \) even do in parallel

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![Diagram](image)

$T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: $\text{join}_1(S, S')$
  
  $\text{odd\_even(\text{interleave}(S, S'))}$

- $T_{\text{join}_1}(n) = O(1)$ mit $P_{\text{join}_1}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: \( \text{join}1(S, S') \)
  \( \text{odd \_ even}(\text{interleave}(S, S')) \)

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Odd-even Merge (Definition)

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Sorting with Merging

- Programm: `odd_even_merge(S, S')`
  
  ```
  if |S| = |S'| = 1 then merge with `compare_exchange`.
  S_{odd} = odd_even_merge(odd(S), odd(S')).
  S_{even} = odd_even_merge(even(S), even(S')).
  return `join1`(S_{odd}, S_{even}).
  ```

- \( T_{odd\_even\_merge}(n) = O(\log n) \) mit \( P_{odd\_even\_merge}(n) = O(n) \)

**Theorem:**

The algorithm `odd_even_merge` sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

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  - `return join1(S_{odd}, S_{even})`

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  \]

  \[
  S_{\text{odd}} = \text{odd}_\text{even}_\text{merge}(\text{odd}(S), \text{odd}(S')).
  \]

  \[
  S_{\text{even}} = \text{odd}_\text{even}_\text{merge}(\text{even}(S), \text{even}(S')).
  \]

  \[
  \text{return } \text{join}_1(S_{\text{odd}}, S_{\text{even}}).
  \]

- \[
  T_{\text{odd}_\text{even}_\text{merge}}(n) = O(\log n) \text{ mit } P_{\text{odd}_\text{even}_\text{merge}}(n) = O(n)
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The algorithm `odd_even_merge` sorts two already sorted sequences into one.

Proof follows.
Sorting Networks

**Theorem:**
There exists a sorting algorithm with $T(n) = O(\log^2 n)$ and $P(n) = n$.

Proof: use divide and conquer, and merging of depth $O(\log n)$.

**Theorem:**
There exists a sorting network of size $O(n \log^2 n)$.

Proof: All calls to `compare_exchange` operation are independent from the input (oblivious algorithm).
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Proof: All calls to *compare_exchange* operation are independent form the input (oblivious algorithm).
The 0-1 Principle

**Theorem:**

If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

**Proof (by contradiction):**

- Let $f(x)$ be non-decreasing function: $f(s_i) \leq f(s_j) \iff s_i \leq s_j$.
- If $X$ sorts the sequence $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$, then if $X$ gets $(f(a_1), f(a_2), \cdots, f(a_n))$ then the output $(f(b_1), f(b_2), \cdots, f(b_n))$ is also sorted.
- Assume $b_i > b_{i+1}$ and $f(b_i) \neq f(b_{i+1})$, then we have $f(b_i) > f(b_{i+1})$ in the “sorted” sequence $(f(b_1), f(b_2), \cdots, f(b_n))$. I.e errors may be kept under the function $f$.
- Choose now $f$: $f(b_j) = 0$ for $b_j < b_i$ and $f(b_j) = 1$ otherwise.
- Thus the sequence $(f(b_1), f(b_2), \cdots, f(b_n))$ is not sorted, because of $f(b_i) = 1$ and $f(b_{i+1}) = 0$.
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- This is a contradiction.
Correctness of the Merging

Theorem:
The algorithm `odd_even_merge` sorts two sorted sequences into a single one.

Proof:

- \( S \) has the form: \( S = 0^p1^{m-p} \) for some \( p \) with \( 0 \leq p \leq m \).
- \( S' \) has the form: \( S' = 0^q1^{m'-q} \) for some \( q \) with \( 0 \leq q \leq m' \).
- Thus the sequence \( S_{odd} \) has the form \( 0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^* \).
- And \( S_{even} \) has the form \( 0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^* \).
- Define: \( d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor) \)
- Depending on \( d \) we consider three cases: \( d = 0 \), \( d = 1 \) and \( d = 2 \).
Correctness of the Merging

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Proof:

- $S$ has the form: $S = 0^p1^{m-p}$ for some $p$ with $0 \leq p \leq m$.
- $S'$ has the form: $S' = 0^q1^{m' - q}$ for some $q$ with $0 \leq q \leq m'$.
- Thus the sequence $S_{odd}$ has the form $0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^*$
- And $S_{even}$ has the form $0^{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}1^*$.
- Define: $d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor)$
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- Define: $d = \lceil p/2 \rceil + \lceil q/2 \rceil - (\lfloor p/2 \rfloor + \lfloor q/2 \rfloor)$
- Depending on $d$ we consider three cases: $d = 0$, $d = 1$ and $d = 2$. 


Correctness of the Merging

Theorem:
The algorithm `odd_even_merge` sorts two sorted sequences into a single one.

Proof:

- $S$ has the form: $S = 0^p 1^{m-p}$ for some $p$ with $0 \leq p \leq m$.
- $S'$ has the form: $S' = 0^q 1^{m'-q}$ for some $q$ with $0 \leq q \leq m'$.
- Thus the sequence $S_{odd}$ has the form $0^{\lceil p/2 \rceil + \lceil q/2 \rceil} 1^*$.
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If \( d = 0 \): Then we have: \( p \) and \( q \) are even.

- The \textit{interleave} step of \textit{join1} has the form:

\[
\text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{(p+q)/2}1^{m+m'-p-q}
\]

- The resulting sequences is already sorted.
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The correctness of a merge network may be tested in time $O(n^2)$.

Proof: Test all inputs of the form $(0^p1^{m-p}, 0^q1^{m'-q})$.

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Situation

- **Aim:** Fast optimal algorithm.
- So far $T(n) = \log^2 n$ bei $P(n) = O(n)$.
- So far: Two loop for merging and sorting.
- Idea: make one loop faster, i.e. the merging in $O(1)$.
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- I.e choose positions which split the blocks to be merged of constants size.
- Problem: How to compute these points?
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Before merging two sequences we will merge two sub-sequences.

Choose as sub-sequence each $k$-th element of the original sequence.

These sub-sequences will be used as crutch/support to do the final merging.

I.e. these sub-sequences are used as a kind of “preview”.

Using these crutch points we will be able to do the merging in $O(1)$ time.

Total running time will be $O(\log n)$.

The additional effort should be at most $O(1)$. 
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- has 4 →
- sends 16 →
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Let $J$ and $K$ be two sorted sequences.

Note: without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.

Let $L$ be a third sequence, which will be called in the following good sampler for $J$ and $K$.

Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.

Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.

The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.

Notation: $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.

$\text{Rng}_{A,B}$ is called the rank between $A$ and $B$.

Depending on the context $\text{Rng}_{A,B}$ could also be an array with $|A|$ elements.
Definition

- Let $J$ and $K$ be two sorted sequences.
- **Note:** without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.
- Let $L$ be a third sequence, which will be called in the following **good** sampler for $J$ and $K$.
- Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.
- Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.
- The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.
- **Notation:** $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \rightarrow \mathbb{N}^{|A|} \] with \[ R_{A,B}(e) = \text{rng}(e, B) \]

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We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each fourth element of \( S \).
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Merging using a Good Sampler

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- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \ldots, l_s) \).
- Programm: merge_with_help(\( J, K, L \))
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{ x \in J \mid l_{i-1} < x \leq l_i \} \).
  
  Assign \( K_i = \{ x \in K \mid l_{i-1} < x \leq l_i \} \).
  
  Assign \( res_i = \text{merge}(J_i, K_i) \).

  return \( (res_1, res_2, \ldots, res_s) \).

- Situation:

\[ \begin{array}{cccccccccc}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 \\
\hline
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & \\
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9
\end{array} \]
Merging using a Good Sampler

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A ↦ N^{|A|} with Rng_{A,B}(e) = rng(e, B)

- Let J, K and L be sorted sequences.
- Let L be a good sampler of both J and K.
- Let L = (l_1, l_2, ⋅⋅⋅, l_s).
- Program: merge_with_help(J, K, L)
  for all i where 1 ≤ i ≤ s do in parallel
    Assign J_i = {x ∈ J | l_{i-1} < x ≤ l_i}.  
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  return (res_1, res_2, ⋅⋅⋅, res_s).

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<table>
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Merging using a Good Sampler

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- Assign \( K_i = \{x \in K \mid l_{i-1} < x \leq l_i\} \).
- Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).

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Merging using a Good Sampler

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- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \ldots, l_s) \).

Programm: \text{merge\_with\_help}(J, K, L)

\begin{align*}
\text{for all } i \text{ where } 1 \leq i \leq s \text{ do in parallel} \\
\text{ Assign } J_i &= \{x \in J \mid l_{i-1} < x \leq l_i\}. \\
\text{ Assign } K_i &= \{x \in K \mid l_{i-1} < x \leq l_i\}. \\
\text{ Assign } res_i &= \text{merge}(J_i, K_i). \\
\text{return } (res_1, res_2, \ldots, res_s). \\
\end{align*}

Situation:

<table>
<thead>
<tr>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>( L_3 )</th>
<th>( L_4 )</th>
<th>( L_5 )</th>
<th>( L_6 )</th>
<th>( L_7 )</th>
<th>( L_8 )</th>
<th>( L_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_1 )</td>
<td>( l_2 )</td>
<td>( l_3 )</td>
<td>( l_4 )</td>
<td>( l_5 )</td>
<td>( l_6 )</td>
<td>( l_7 )</td>
<td>( l_8 )</td>
<td></td>
</tr>
<tr>
<td>( K_1 )</td>
<td>( K_2 )</td>
<td>( K_3 )</td>
<td>( K_4 )</td>
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<td>( K_6 )</td>
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</tr>
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</table>
Merging using a Good Sampler

Let $J$, $K$ and $L$ be sorted sequences.

Let $L$ be a good sampler of both $J$ and $K$.

Let $L = (l_1, l_2, \cdots, l_s)$.

Programm: \text{merge\_with\_help}(J, K, L)

\begin{align*}
\text{for all } i \text{ where } 1 \leq i \leq s \text{ do in parallel} \\
\quad \text{Assign } J_i = \{x \in J \mid l_{i-1} < x \leq l_i\}. \\
\quad \text{Assign } K_i = \{x \in K \mid l_{i-1} < x \leq l_i\}. \\
\quad \text{Assign } res_i = \text{merge}(J_i, K_i). \\
\end{align*}
return $(res_1, res_2, \cdots, res_s)$.

Situation:

\begin{array}{cccccccc}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 \\
\hline
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \\
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9
\end{array}
Merging using a Good Sampler (Example)

\[ rng(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = rng(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
- \( L = (5, 10, 12, 17) \)

Then we have:

<table>
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<tr>
<th>( i )</th>
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<tr>
<td>1</td>
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
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Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
Merging using a Good Sampler (Example)

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A ↦→ N^{|A|} with Rng_{A,B}(e) = rng(e, B)

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- J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21)
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Then we have:

<table>
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<tr>
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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\mid A\mid} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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Result: (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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<tr>
<td>1</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( Rng_{L,J}, Rng_{L,K}, Rng_{K,L} \) and \( Rng_{J,L} \) is known, then we have:
\[ T_{\text{merge\_with\_help}(J,K,L)} = O(1) \text{ with } P_{\text{merge\_with\_help}(J,K,L)} = O(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( Rng_{L,J} \) resp. \( Rng_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( Rng_{J,L} \) and \( Rng_{K,L} \) to know the area to write its output sequence.
Merging with good sampler (running time)

\[\text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B)\]

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If \(\text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L}\) and \(\text{Rng}_{J,L}\) is known, then we have:

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\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

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T_{\text{merge\_with\_help}(J,K,L)} = \mathcal{O}(1) \quad \text{with} \quad P_{\text{merge\_with\_help}(J,K,L)} = \mathcal{O}(|J| + |K|).
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Merging with good sampler (running time)

\[ \text{rng}(e, S) = \left| \{ x \in S \mid x < e \} \right| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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\[ T_{\text{merge \_ with \_ help}(J,K,L)} = O(1) \quad \text{with} \quad P_{\text{merge \_ with \_ help}(J,K,L)} = O(|J| + |K|). \]

Proof:

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- Each processor uses \( \text{Rng}_{L,J} \) resp. \( \text{Rng}_{L,K} \) to know the area to read its input sequences.
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Properties of Good Samplers

Lemma:
If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then
$\text{merge}(X, Y)$ is a good sampler for $X'$ [resp. $Y'$].

Proof:
- Consider $X$ as a good sampler for $X'$.
- Any additional element make the good sampler just "better".

Note:
$\text{merge}(X, Y)$ is not necessary a sampler for $\text{merge}(X', Y')$.
- $X = (2, 7)$ and $X' = (2, 5, 6, 7)$.
- $Y = (1, 8)$ and $Y' = (1, 3, 4, 8)$.
- $\text{merge}(X, Y) = (1, 2, 7, 8)$ and $\text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8)$.
- There are 5 elements between 2 and 7.
Properties of Good Samplers

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then merge($X, Y$) is a good sampler for $X'$ [resp. $Y'$].

Proof:

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merge($X, Y$) is not necessary a sampler for merge($X', Y'$).

- $X = (2, 7)$ and $X' = (2, 5, 6, 7)$.
- $Y = (1, 8)$ and $Y' = (1, 3, 4, 8)$.
- merge($X, Y$) = (1, 2, 7, 8) and merge($X', Y'$) = (1, 2, 3, 4, 5, 6, 7, 8).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

**Lemma:**

If \( X \) is a good sampler for \( X' \) and 
\( Y \) is a good sampler for \( Y' \), then 
merge\((X, Y)\) is a good sampler for \( X' \) [resp. \( Y' \)].

**Proof:**

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- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \text{ and } R_{g_{A,B}} : A \mapsto \mathbb{N}^{|A|} \text{ with } R_{g_{A,B}}(e) = \text{rng}(e, B) \]

Lemma:

If \( X \) is a good sampler for \( X' \) and \( Y \) is a good sampler for \( Y' \), then \( \text{merge}(X, Y) \) is a good sampler for \( X' \) [resp. \( Y' \)].

Proof:

- Consider \( X \) as a good sampler for \( X' \).
- Any additional element make the good sampler just "better".

Note:

\( \text{merge}(X, Y) \) is not necessary a sampler for \( \text{merge}(X', Y') \).

- \( X = (2, 7) \) and \( X' = (2, 5, 6, 7) \).
- \( Y = (1, 8) \) and \( Y' = (1, 3, 4, 8) \).
- \( \text{merge}(X, Y) = (1, 2, 7, 8) \) and \( \text{merge}(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8) \).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A, B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A, B}(e) = \text{rng}(e, B) \]

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Properties of Good Samplers

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad R_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad R_{A,B}(e) = \text{rng}(e, B) \]

**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).

Then there are at most \( 2 \cdot r + 2 \) elements of merge\((X', Y')\) between \( r \) successive elements of merge\((X, Y)\).

**Proof:**

- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be

\[
x = |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \cdots, e_r\}|.
\]
Properties of Good Samplers

Lemma:
Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$. Then there are at most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$.

Proof:
- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of $\text{merge}(X, Y)$.
- W.l.o.g. let $e_1 \in X$.
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Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
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- Let \( (e_1, e_2, \cdots, e_r) \) successive elements of \( \text{merge}(X, Y) \).
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x &= |X \cap \{e_1, e_2, \cdots, e_r\}| \quad \text{and} \\
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- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
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x &= |X \cap \{e_1, e_2, \cdots, e_r\}| & \text{and} \\
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Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\).
If \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).
- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

Example \(x = 3\) and \(y = 2\):

\[ e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \]
Properties of Good Samplers

Lemma:
Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$. Then there are at most $2 \cdot r + 2$ elements of merge($X'$, $Y'$) between $r$ successive elements of merge($X$, $Y$).

Proof: W.l.o.g. let $e_1 \in X$.
If: $e_r \in X$

- Between $e_1$ and $e_r$ are at most $2(x - 1) + 1$ elements of $X'$.
- Between $e_1$ and $e_r$ are at most $2(y + 1) + 1$ elements of $Y'$, because they are between $y + 2$ elements of $Y$.

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Example $x = 3$ and $y = 2$:

$$e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X$$
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

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- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).

Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

Example \(x = 3\) and \(y = 2\):

\[
\begin{align*}
a & \in Y \\
e_1 & \in X \\
e_2 & \in Y \\
e_3 & \in X \\
e_4 & \in Y \\
e_5 & \in X \\
b & \in Y
\end{align*}
\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y\]
Properties of Good Samplers

\((e_1, e_2, \ldots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \ldots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \ldots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \ldots, e_r)\) are between \(x + 1\) elements from \(X\).
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**Example** \(x = 2\) and \(y = 2\):

\[\begin{align*}
e_0 & \in Y \\
e_1 & \in X \\
e_2 & \in Y \\
e_3 & \in X \\
e_4 & \in Y
\end{align*}\]
Properties of Good Samplers

Let \((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and \(e_1 \in X\).

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

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- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
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**Example** \(x = 2\) and \(y = 2\):

\[\begin{align*}
e_0 & \in Y \\
e_1 & \in X \\
e_2 & \in Y \\
e_3 & \in X \\
e_4 & \in Y \\
e_5 & \in X
\end{align*}\]
Properties of Good Samplers

Let \((e_1, e_2, \cdots, e_r)\) successive elements of \(\text{merge}(X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and \(r \geq 1\).

Lemma:

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of \(\text{merge}(X', Y')\) between \(r\) successive elements of \(\text{merge}(X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[e_0 \in Y, \quad e_1 \in X, \quad e_2 \in Y, \quad e_3 \in X, \quad e_4 \in Y, \quad e_5 \in X\]
Properties of Good Samplers

\((e_1, e_2, \cdots, e_r)\) successive elements of merge\((X, Y)\) and \(x = |X \cap \{e_1, e_2, \cdots, e_r\}|\) and \(y = |Y \cap \{e_1, e_2, \cdots, e_r\}|\) and

**Lemma:**

Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

**Proof:** W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- **Thus we get:** \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[
\begin{align*}
& e_0 \in Y & & e_1 \in X & & e_2 \in Y & & e_3 \in X & & e_4 \in Y & & e_5 \in X
\end{align*}
\]
Properties of good sampler

Let reduce(\(X\)) be the operation, which chooses from \(X\) every forth element.

Lemma:

If \(X\) is a good sampler for \(X'\) and \(Y\) is a good sampler for \(Y'\), then reduce(merge(\(X, Y\))) is a good sampler for reduce(merge(\(X', Y'\))).

Proof:

- Consider \(k + 1\) successive elements \((e_1, e_2, \ldots, e_{k+1})\) of reduce(merge(\(X, Y\))).
- At most \(4k + 1\) elements of merge(\(X, Y\)) are between \(e_1, e_2, \ldots, e_{k+1}\) including \(e_1, e_{k+1}\).
- At most \(8k + 4\) elements of merge(\(X', Y'\)) are between these \(4k + 1\) elements.
- At most \(2k + 1\) elements of reduce(merge(\(X', Y'\))) are between \((e_1, e_2, \ldots, e_{k+1})\).
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition
Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:
If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:
- Consider $k + 1$ successive elements $(e_1, e_2, \ldots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \ldots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \ldots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of merge($X'$, $Y'$) between $r$ successive elements of merge($X$, $Y$)

Definition

Let reduce($X$) be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then reduce(merge($X$, $Y$)) is a good sampler for reduce(merge($X'$, $Y'$)).

Proof:

- Consider $k + 1$ successive elements ($e_1, e_2, \cdots, e_{k+1}$) of reduce(merge($X$, $Y$)).
- At most $4k + 1$ elements of merge($X$, $Y$) are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
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Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let reduce($X$) be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then reduce($\text{merge}(X, Y)$) is a good sampler for reduce($\text{merge}(X', Y')$).

**Proof:**

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of reduce($\text{merge}(X, Y)$).
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
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Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

**Definition**

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

**Lemma:**

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

**Proof:**

- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of $\text{reduce}(\text{merge}(X, Y))$.
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of $\text{reduce}(\text{merge}(X', Y'))$ are between $(e_1, e_2, \cdots, e_{k+1})$. 
Properties of good sampler

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<tr>
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<th>Left</th>
<th>Right</th>
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</tr>
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<tbody>
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<td>1</td>
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<td>8</td>
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<td>∅</td>
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<td>1</td>
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\begin{array}{cccccc}
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Z_1 & Z_2 & \ldots & Z_r & Z_{r+1} & Z_{r+2}
\end{array}
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- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
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 Situation

- Running time is $O(\log n)$.
- The inner nodes $v$ need $|val_v|$ many processors.
- We still have to proof that the number of processors is in $O(n)$.
- PRAM Model has to be verified.
- Important: The computation of the values $Rng_{X,Y}$ has to be shown.
- These values will be in the following also transmitted and updated.
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- In each step will compute: \textit{merge\_with\_help}(X_{i+1}, Y_{i+1}, \text{merge}(X_{i}, Y_{i}))
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- Let \(L = \text{merge}(X_{i}, Y_{i})\), \(J = X_{i+1}\) and \(K = Y_{i+1}\).
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Computing the Ranks

Lemma:
Let $S = (b_1, b_2, \ldots, b_k)$ be a sorted sequence, then we may compute the rank of $a \in S$ in time $O(1)$ using $k$ processors.

Proof:

- Program: \text{rng1}(a, S)
  
  for all $P_i$ where $1 \leq i \leq k$ do in parallel
  
  if $b_i < a \leq b_{i+1}$ then return $i$

- Note, the program has no write-conflicts.
- Note, it could be changed, to avoid read-conflicts.
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Computing the Ranks

**Lemma:**

Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$ in time $O(1)$ using $O(|S|)$ processors.

**Proof:**

- We do know $\text{Rnk}_{S, S}$, $\text{Rnk}_{S_1, S_1}$ and $\text{Rnk}_{S_2, S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
- The claim follows directly.
Computing the Ranks

**Lemma:**

Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$ in time $O(1)$ using $O(|S|)$ processors.

**Proof:**

- We do know $\text{Rnk}_{S,S}$, $\text{Rnk}_{S_1,S_1}$ and $\text{Rnk}_{S_2,S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
- The claim follows directly.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_U,X'$ and $\text{Rnk}_U,Y'$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- Then we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $R_{n,k}^{X', X}$ and $R_{n,k}^{Y', Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $R_{n,k}^{X', U}$, $R_{n,k}^{Y', U}$, $R_{n,k}^{U, X'}$ and $R_{n,k}^{U, Y'}$.

Proof:

- First we compute $R_{n,k}^{X', U}$ and $R_{n,k}^{Y', U}$.
- Then we compute $R_{n,k}^{X, X'}$ and $R_{n,k}^{Y, Y'}$.
- Finally we compute $R_{n,k}^{U, X'}$ and $R_{n,k}^{U, Y'}$. 
Computing the Ranks

**Lemma:**

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X', X}$ and $\text{Rnk}_{Y', Y}$ are known.

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- First we compute $\text{Rnk}_{X', U}$ and $\text{Rnk}_{Y', U}$.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
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- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

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we have $\text{rkn}(a, S)$ and $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$
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Lemma:

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Computing the Ranks

**Lemma:**

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{rk}_{X',X}$ and $\text{rk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{rk}_{X',U}$, $\text{rk}_{Y',U}$, $\text{rk}_{U,X'}$ and $\text{rk}_{U,Y'}$.

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- First we compute $\text{rk}_{X',U}$ and $\text{rk}_{Y',U}$.
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- Let $X$ be a good sampler of $X'$.
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- Assume $Rnk_{X',X}$ and $Rnk_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $Rnk_{X',U}$, $Rnk_{Y',U}$, $Rnk_{U,X'}$, and $Rnk_{U,Y'}$.

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- First we compute $Rnk_{X',U}$ and $Rnk_{Y',U}$.
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we have $rnk(a, S)$ and $Rnk_{S_1,S_2}$ and $Rnk_{S_2,S_1}$
Computing the Ranks ($\text{Rnk}_{X',U}$)

- Let $X = (a_1, a_2, \ldots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \ldots, X'_k, X'_{k+1}$.
- Note: $\text{Rnk}_{X',X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

Programm: $\text{Rnk}_{X',U}$

for all $i$ where $1 \leq i \leq k + 1$ do in parallel

for all $x \in X'_i$ do

$\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)$

- Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((\text{Rnk}_{X'\cup U})\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X'\cup X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

Programm: \(\text{Rnk}_{X'\cup U}\)

for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

for all \(x \in X'_i\) do

\(\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\)

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks ($\text{Rnk}_{X', U}$)

- Let $X = (a_1, a_2, \cdots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X_1', X_2', \cdots, X_k', X_{k+1}'$.
- Note: $\text{Rnk}_{X', X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
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Computing the Ranks \((\text{Rnk}_{X'}, U)\)

1. Let \(X = (a_1, a_2, \ldots, a_k)\).
2. Let \(w.l.o.g.\) \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
3. Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
4. Note: \(\text{Rnk}_{X', X}\) is known.
5. Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
6. Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
7. Thus we get:

Programm: \(\text{Rnk}_{X', U}\)

for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

for all \(x \in X'_i\) do

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• Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
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- Note: \(\text{Rnk}_{X', X}\) is known.
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- Note: \(\text{Rnk}_{X',X}\) is known.
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  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X'_{i}\) do
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- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X_1', X_2', \cdots, X_k', X_{k+1}'\).
- Note: \(\text{Rnk}_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  \[
  \text{Programm: } \text{Rnk}_{X',U} \\
  \text{for all } i \text{ where } 1 \leq i \leq k + 1 \text{ do in parallel} \\
  \quad \text{for all } x \in X_i' \text{ do} \\
  \quad \quad \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i) \\
  \]
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  **Programm:** \(\text{Rnk}_{X', U}\)

  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel

  for all \(x \in X'_i\) do

  \[\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \( (\text{Rnk}_{X,X'}) \)

- Let \( a_i \in X \).
- Let \( a' \) minimal element in \( X'_{i+1} \).
- The rank of \( a_i \) in \( X' \) is the same as the rank of \( a' \) in \( X' \).
- This rank is already known.
- This may be computed in time \( O(1) \) using one processor.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
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Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X_{i+1}'\).
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Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.

we have \(\text{rnk}(a, S)\) and \(\text{Rnk}_{S_1, S_2}\) and \(\text{Rnk}_{S_2, S_1}\)
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X'_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
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we have \(\text{rnk}(a, S)\) and \(\text{Rnk}_{S_1,S_2}\) and \(\text{Rnk}_{S_2,S_1}\)
Computing the Ranks (Rnk\(_U,X'\))

- **Note:** Rnk\(_U,X'\) consists of Rnk \(X,X'\) and Rnk \(Y,X'\).
- Rnk \(X,X'\) is already known.
- Still to compute: Rnk \(Y,X'\).
- Rnk \(Y,X\) may be computed using the previous lemma.
- We compute rnk\((a,X')\) using rnk\((a,X)\) and Rnk\(_X,X'\).
- Thus we compute Rnk\(_U,X'\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note:** $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} \ X, X'$ and $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ X, X'$ is already known.
- Still to compute: $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- **Still to compute:** $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- Still to compute: $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X'$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$.
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note**: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- Still to compute: $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- **We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.**
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks (Rnk_{U,X'})

- Note: Rnk_{U,X'} consists of Rnk X, X' and Rnk Y, X'.
- Rnk X, X' is already known.
- Still to compute: Rnk Y, X'.
- Rnk Y, X may be computed using the previous lemma.
- We compute rnk(a, X') using rnk(a, X) and Rnk_{X,X'}.
- Thus we compute Rnk_{U,X'} with O(|U|) processors and time O(1).
Computing the Ranks

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: \( \text{Rnk}_{J, X_i} \) and \( \text{Rnk}_{K, Y_i} \).

- Using the above considerations we may compute: \( \text{Rnk}_{L, J} \), \( \text{Rnk}_{L, K} \), \( \text{Rnk}_{J, L} \) and \( \text{Rnk}_{K, L} \).

- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

Consider the step
\[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).

Using the above considerations we may compute: \( \text{Rnk}_{L,J}, \text{Rnk}_{L,K}, \text{Rnk}_{J,L} \)
and \( \text{Rnk}_{K,L} \).

Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \)
Computing the Ranks

we have \( rnk(a, S) \) and \( Rnk_{S_1,S_2} \) and \( Rnk_{S_2,S_1} \)

- Consider the step
  \[ merge\_with\_help(J = X_{i+1}, K = Y_{i+1}, L = merge(X_i, Y_i)) \]

- Using the invariant we know: \( Rnk_{J,X_i} \) and \( Rnk_{K,Y_i} \).

- Using the above considerations we may compute: \( Rnk_{L,J} \), \( Rnk_{L,K} \), \( Rnk_{J,L} \)
  and \( Rnk_{K,L} \).

- Still to be computed: \( Rnk_{reduce(merge(X_{i+1}, Y_{i+1})), reduce(merge(X_i, Y_i))} \)

- Known: \( Rnk_{X_{i+1}, merge(X_i, Y_i)} \) and \( Rnk_{Y_{i+1}, merge(X_i, Y_i)} \).

- It is now easy to compute: \( Rnk_{X_{i+1}, reduce(merge(X_i, Y_i))} \) and
  \( Rnk_{Y_{i+1}, reduce(merge(X_i, Y_i))} \).

- Also easy to compute: \( Rnk_{merge(X_{i+1}, Y_{i+1}), reduce(merge(X_i, Y_i))} \).
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Computing the Ranks

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Computing the Ranks

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Theorem:

We may sort $n$ values on a CREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: discussed before.

Theorem:

We may sort $n$ values on a EREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: see literature.

Theorem:

There exists a sorting network with $O(n)$ processors and depth $O(\log n)$.

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**Algorithmn of Cole**

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Questions

- Explain the motivation behind parallel systems.
- Explain the ideas of the different sorting algorithms.
- Explain the different running times of these sorting algorithms.
- Explain the different efficiency of these sorting algorithms.
- Explain the idea of the algorithm of Cole.
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Legend

- ■: Not of relevance
- ▶️: implicitly used basics
- ◼️: idea of proof or algorithm
- ⬤: structure of proof or algorithm
- ■️: Full knowledge