Theory of Parallel and Distributed Systems
(WS2016/17)

Chapter 2
Sorting with a PRAM

Walter Unger

Lehrstuhl für Informatik 1

10:36, December 9, 2016
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Very simple Algorithm (Idea)

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| 33 | 0 1 1 0 1 0 0 1 0 0 0 0 1 0 0 1 | 7 |   |
| 41 | 1 1 1 0 1 0 0 1 0 0 0 0 1 0 1 1 | 9 | 22 |
| 26 | 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 | 5 |   |
| 59 | 1 1 1 1 1 0 1 1 0 1 1 0 1 1 1 1 | 14 |   |
| 57 | 1 1 1 1 1 0 1 1 0 1 0 0 1 1 1 1 | 13 |   |
| 52 | 1 1 1 0 1 0 1 1 0 0 0 0 1 1 1 1 | 11 |   |
| 61 | 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 | 15 |   |
| 27 | 0 1 1 0 1 0 0 0 0 0 0 0 0 1 0 0 | 6  |   |
| 49 | 1 1 1 0 1 0 0 1 0 0 0 0 1 1 1 1 | 10 |   |
| 67 | 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 | 16 |   |
| 23 | 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 | 4  |   |
| 56 | 1 1 1 0 1 0 1 1 0 1 0 0 1 1 1 1 | 12 |   |
| 14 | 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2  |   |
| 12 | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1  |   |
| 34 | 0 1 1 0 1 0 0 1 0 0 0 0 1 0 1 1 | 8  |   |

The table above represents the sequence of steps in the simple sorting algorithm.
## Very simple Algorithm (Idea)

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Very simple Sorting Algorithm

- **Idea:** Compute the position for each element.
- Compare pairwise all elements and count the number of smaller elements.
- Use $n^2$ processors.
- **Programm:** SimpleSort
  
  **Eingabe:** $s_1, \ldots, s_n$.
  
  for all $P_{i,j}$ where $1 \leq i,j \leq n$ do in parallel
  
  if $s_i > s_j$ then $P_{i,j}(1) \rightarrow R_{i,j}$ else $P_{i,j}(0) \rightarrow R_{i,j}$
  
  for all $i$ where $1 \leq i \leq n$ do in parallel
  
  for all $P_{i,j}$ where $1 \leq j \leq n$ do in parallel
  
  Processors $P_{i,j}$ bestimmen $q_i = \sum_{l=1}^{n} R_{i,l}$.
  
  $P_i(s_i) \rightarrow R_{q_i+1}$.

- **Complexity:** $T(n) = O(\log n)$ and $P(n) = n^2$.
- **Efficiency:** $\frac{O(n \log n)}{n^2 \cdot O(\log n)} = O\left(\frac{1}{n}\right)$.
- **Model:** CREW.
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  - for all $P_{i,j}$ where $1 \leq j \leq n$ **do in parallel**
    
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\]

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Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$.
- Sort parallel each block.
- Merge the blocks pairwise and parallel.

Complexity: $T(n) = O(n/P(n) \cdot \log n + \log^2 n)$.

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- Merge the blocks pairwise and parallel. $O(n/P(n) + \log n) \cdot O(\log P(n))$  

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Efficiency: $Eff(n) = \frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}$

- Is $O(1)$ for $P(n) \leq n/\log n$.  

Improved Algorithm for CREW

- Work with \( P(n) \) processors \((P(n) \leq n)\).
- Split the input in blocks of size \( O(n/P(n)) \). \( O(1) \)
- Sort parallel each block. \( O(n/P(n) \cdot \log(n/P(n))) \)
- Merge the blocks pairwise and parallel. \( O(n/P(n) + \log n) \cdot O(\log P(n)) \)

**Complexity:** \( T(n) = O(n/P(n) \cdot \log n + \log^2 n) \).

**Efficiency:** \( \text{Eff}(n) = \)

\[
\frac{O(n \log n)}{O(P(n)) \cdot O(n/P(n) \cdot \log n + \log^2 n)} = \frac{O(n \log n)}{O(n \cdot \log n + P(n) \cdot \log^2 n)}
\]

- Is \( O(1) \) for \( P(n) \leq n/\log n \).
Improved Algorithm for CREW

- Work with $P(n)$ processors ($P(n) \leq n$).
- Split the input in blocks of size $O(n/P(n))$. $O(1)$
- Sort parallel each block. $O(n/P(n) \cdot \log(n/P(n)))$
- Merge the blocks pairwise and parallel. $O(n/P(n) + \log n) \cdot O(\log P(n))$

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- Is $O(1)$ for $P(n) \leq n/\log n$. 

Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(\text{EREW})}(n) = \Theta(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
- Efficiency:

$$\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}$$

- Is $O(1)$ if $P(n) < n/\log^2 n$. 
Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(EREW)}(n) = \text{lsO}\left(\frac{n}{P(n)} + \log n \cdot \log P(n)\right)$.
- $T(n) = O\left(\frac{n}{P(n)} \cdot \log(n/P(n)) + \frac{n}{P(n)} \cdot \log P(n) + \log n \cdot \log^2 P(n)\right)$.
- $T(n) = O\left((n/P(n) + \log^2 n) \cdot \log n\right)$.
- Efficiency:

$$
\text{Eff}(n) = \frac{O(n \log n)}{O(P(n) \cdot ((n/P(n) + \log^2 n) \cdot \log n))}
$$

- Is $O(1)$ if $P(n) < n/\log^2 n$. 
Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(EREW)}(n) = \log O(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
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Improved Algorithm EREW

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Improved Algorithm EREW

- Exchange the merge algorithm.
- Recall $T_{\text{Merging}(\text{EREW})}(n) = \Omega(n/P(n) + \log n \cdot \log P(n))$.
- $T(n) = O(n/P(n) \cdot \log(n/P(n)) + O(n/P(n) \cdot \log P(n) + \log n \cdot \log^2 P(n))$
- $T(n) = O((n/P(n) + \log^2 n) \cdot \log n)$
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  \]

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Improved Algorithm EREW

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Lower Bound

Theorem:
For any parallel sorting algorithm \( Srt \) with \( P_{Srt}(n) = O(n) \) hold:
\[
T_{Srt}(n) = \Omega(\log(n)).
\]

Proof:

- Lower bound for sequential is \( \Theta(n \log n) \).
- One needs \( O(n \log n) \) comparisons.
- In each parallel step are at most \( o(n) \) comparisons possible
- Thus with less steps we have a contradiction to the lower bound for sequential

Situation at this point:

- Inefficient algorithms with: \( T(n) = O(\log n) \) and \( P(n) = n^2 \).
- Nearly efficient algorithm with: \( T(n) = O(\log^2 n) \) and \( P(n) = o(n) \).
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Basic Operation for Sorting

- **Identify basic operation for sorting.**
- Assume: sorting key is \( s_1, \ldots, s_n \).
- **Program:** `compare_exchange(i,j)`
  
  ```
  if \( s_i > s_j \) then exchange \( s_i \leftrightarrow s_j \)
  ```

- Symbolic view (Batcher):
  
  ```
  \[ x \quad \text{min}(x, y) \quad y \quad \text{max}(x, y) \]
  ```

- Basic building block for sorting networks.
- Base for Odd-Even merge
- Form this we build the optimal algorithm by Cole
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  \begin{array}{c}
    y \\
    \hline
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  \end{array}
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    \hline
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  $$ \begin{align*}
  & y & \overset{\text{max}(x, y)}{\rightarrow} \\
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  \end{align*} $$

- Basic building block for sorting networks.
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- **Form this we build the optimal algorithm by Cole**
Odd-even Merge (Definition)

- **Input**: Sequence $S = (s_1, s_2, \cdots, s_n)$. (O.E.d.A. $n$ even)
- Let $Odd(S)$ [$Even(S)$] be the elements of $S$ with odd [even] index.
- Let $S' = (s'_1, s'_2, \cdots, s'_n)$ be a second sequence.
- Then we define: $interleave(S, S') = (s_1, s'_1, s_2, s'_2, \cdots, s_n, s'_n)$.

$$T_{\text{interleave}}(n) = O(1) \text{ mit } P_{\text{interleave}}(n) = O(n)$$
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\[ T_{\text{interleave}}(n) = O(1) \text{ mit } P_{\text{interleave}}(n) = O(n) \]
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![Diagram of Odd-even Merge](attachment:diagram.png)

- $T_{interleave}(n) = O(1)$ mit $P_{interleave}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: `odd_even(S)`
  
  for all $i$ where $1 < i < n$ and $i$ even do in parallel
  
  `compare_exchange(i, i + 1).

- $T_{\text{compare\_exchange}}(n) = O(1)$ mit $P_{\text{compare\_exchange}}(n) = O(n)$
Odd-even Merge (Definition)

- Program: `odd_even(S)`
  
  For all `i` where `1 < i < n` and `i` even do in parallel:
  
  ```
  compare_exchange(i, i + 1).
  ```

- `T_{\text{compare\_exchange}}(n) = O(1)` mit `P_{\text{compare\_exchange}}(n) = O(n)`
Odd-even Merge (Definition)

- **Programm: odd_even(S)**
- for all \( i \) where \( 1 < i < n \) and \( i \) even do in parallel
  - \( \text{compare\_exchange}(i, i + 1) \).

\[
\begin{align*}
S_1 & \quad S_2 & \quad S_3 & \quad S_4 & \quad S_5 & \quad S_6 & \quad S_7 & \quad S_8 & \quad S_9 & \quad S_{10} & \quad S_{11} & \quad S_{12} & \quad S_{13} & \quad S_{14} & \quad S_{15} & \quad S_{16} \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
r_1 & \quad r_2 & \quad r_3 & \quad r_4 & \quad r_5 & \quad r_6 & \quad r_7 & \quad r_8 & \quad r_9 & \quad r_{10} & \quad r_{11} & \quad r_{12} & \quad r_{13} & \quad r_{14} & \quad r_{15} & \quad r_{16}
\end{align*}
\]

- \( T_{\text{compare\_exchange}}(n) = O(1) \) mit \( P_{\text{compare\_exchange}}(n) = O(n) \)
Odd-even Merge (Definition)

- Programm: $\text{join}_1(S, S')$
  
  $\text{odd\_even}(\text{interleave}(S, S'))$

$$T_{\text{join}_1}(n) = O(1) \text{ mit } P_{\text{join}_1}(n) = O(n)$$
Odd-even Merge (Definition)

Programm: $\text{join1}(S, S')$

$\text{odd\_even}(\text{interleave}(S, S'))$

$T_{\text{join1}}(n) = O(1)$ mit $P_{\text{join1}}(n) = O(n)$
Odd-even Merge (Definition)

- Programm: join1$(S, S')$
  - odd_even(interleave$(S, S')$)

- $T_{join1}(n) = O(1)$ mit $P_{join1}(n) = O(n)$
Sorting with Merging

- **Programm:** `odd_even_merge(S, S')`
  
  ```
  if |S| = |S'| = 1 then merge with `compare_exchange`.
  S_{odd} = odd_even_merge(odd(S), odd(S')).
  S_{even} = odd_even_merge(even(S), even(S')).
  return join1(S_{odd}, S_{even}).
  ```

- **Theorem:**
  
  \[ T_{odd\_even\_merge}(n) = O(\log n) \text{ mit } P_{odd\_even\_merge}(n) = O(n) \]

**Theorem:**

The algorithm `odd_even_merge` sorts two already sorted sequences into one.

Proof follows.
Sorting with Merging

- Programm: odd_even_merge\((S, S')\)
  
  \[
  \text{if } |S| = |S'| = 1 \text{ then merge with compare_exchange.}
  \]
  
  \[
  S_{odd} = \text{odd_even_merge}(\text{odd}(S), \text{odd}(S')).
  \]
  
  \[
  S_{even} = \text{odd_even_merge}(\text{even}(S), \text{even}(S')).
  \]
  
  return join1\((S_{odd}, S_{even})\).

- \(T_{odd\_even\_merge}(n) = O(\log n)\) mit \(P_{odd\_even\_merge}(n) = O(n)\)

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The algorithm \textit{odd\_even\_merge} sorts two already sorted sequences into one.

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- Programm: \texttt{odd\_even\_merge}(S, S')
  
  \begin{verbatim}
  if \(|S| = |S'| = 1\) then merge with \texttt{compare\_exchange}.
  \texttt{S}_{odd} = \texttt{odd\_even\_merge}(\texttt{odd}(S), \texttt{odd}(S')).
  \texttt{S}_{even} = \texttt{odd\_even\_merge}(\texttt{even}(S), \texttt{even}(S')).
  \texttt{return join1}(\texttt{S}_{odd}, \texttt{S}_{even}).
  \end{verbatim}

- \(T_{odd\_even\_merge}(n) = O(\log n)\) mit \(P_{odd\_even\_merge}(n) = O(n)\)

Theorem:
The algorithm \texttt{odd\_even\_merge} sorts two already sorted sequences into one.

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Sorting with Merging

- **Programm:** odd_even_merge($S, S'$)
  
  ```java
  if |$S| = |$S'| = 1 then merge with compare_exchange.
  
  $S_{odd} = odd\_even\_merge(odd(S), odd(S')).$
  
  $S_{even} = odd\_even\_merge(even(S), even(S')).$
  
  return join1($S_{odd}, S_{even}$).
  ```

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**Theorem:**

The algorithm `odd_even_merge` sorts two already sorted sequences into one.

Proof follows.
Sorting Networks

Theorem:
There exists a sorting algorithm with \( T(n) = O(\log^2 n) \) and \( P(n) = n \).

Proof: use divide and conquer, and merging of depth \( O(\log n) \).

Theorem:
There exists a sorting network of size \( O(n \log^2 n) \).

Proof: All calls to \textit{compare-exchange} operation are independent form the input (oblivious algorithm).
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Proof: All calls to compare exchange operation are independent from the input (oblivious algorithm).
The 0-1 Principle

Theorem:
If a sorting network $X$, resp. sorting algorithm is correct for all 0-1 inputs, then it is also correct for any input.

Proof (by contradiction):

1. Let $f(x)$ be non-decreasing function: $f(s_i) \leq f(s_j) \iff s_i \leq s_j$.
2. If $X$ sorts the sequence $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$, then if $X$ gets $(f(a_1), f(a_2), \cdots, f(a_n))$ then the output $(f(b_1), f(b_2), \cdots, f(b_n))$ is also sorted.
3. Assume $b_i > b_{i+1}$ and $f(b_i) \neq f(b_{i+1})$, then we have $f(b_i) > f(b_{i+1})$ in the “sorted” sequence $(f(b_1), f(b_2), \cdots, f(b_n))$. I.e errors may be kept under the function $f$.
4. Choose now $f$: $f(b_j) = 0$ for $b_j < b_i$ and $f(b_j) = 1$ otherwise.
5. Thus the sequence $(f(b_1), f(b_2), \cdots, f(b_n))$ is not sorted, because of $f(b_i) = 1$ and $f(b_{i+1}) = 0$.
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Correctness of the Merging

Theorem:
The algorithm \texttt{odd\_even\_merge} sorts two sorted sequences into a single one.

Proof:

- \( S \) has the form: \( S = 0^p1^{m-p} \) for some \( p \) with \( 0 \leq p \leq m \).
- \( S' \) has the form: \( S' = 0^q1^{m'-q} \) for some \( q \) with \( 0 \leq q \leq m' \).
- Thus the sequence \( S_{\text{odd}} \) has the form \( 0^\left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{q}{2} \right\rceil 1^* \).
- And \( S_{\text{even}} \) has the form \( 0^\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor 1^* \).
- Define \( d = \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{q}{2} \right\rceil - \left( \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right) \).
- Depending on \( d \) we consider three cases: \( d = 0 \), \( d = 1 \) and \( d = 2 \).
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The algorithm `odd_even_merge` sorts two sorted sequences into a single one.

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- **S** has the form: \( S = 0^p1^{m-p} \) for some \( p \) with \( 0 \leq p \leq m \).
- **S’** has the form: \( S’ = 0^q1^{m’-q} \) for some \( q \) with \( 0 \leq q \leq m’ \).
- Thus the sequence \( S_{odd} \) has the form \( 0^{\lceil p/2 \rceil + \lceil q/2 \rceil}1^* \).
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If $d = 0$: Then we have: $p$ and $q$ are even.

- The *interleave* step of *join1* has the form:
  \[
  \text{interleave}(S_{\text{odd}}, S_{\text{even}}) = (00)^{(p+q)/2} 1^{m+m'-p-q}
  \]
- The resulting sequences is already sorted.
- The *compare_exchange* step keeps the order.

If $d = 1$: Then we have: $p$ is odd and $q$ is even.

- The *interleave* step of *join1* has the form:
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- The *compare_exchange* step will exchange the 1 on position $2r$ with the 0 on position $2r + 1$. 
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**Corollary:**

The correctness of a merge network may be tested in time $O(n^2)$.

Proof: Test all inputs of the form $(0^p1^{m-p}, 0^q1^{m'-q})$.

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The test for correctness of a sorting network is NP-hard.

Proof: Literature.
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- **Aim**: Fast optimal algorithm.
- So far $T(n) = \log^2 n$ bei $P(n) = O(n)$.
- So far: Two loop for merging and sorting.
- Idea: make one loop faster, i.e. the merging in $O(1)$.
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- Choose as additional information nice splitting points for merging.
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- **Choose as additional information nice splitting points for merging.**
  - i.e choose positions which split the blocks to be merged of constants size.
- **Problem:** How to compute these points?
- **Solution is the base for the algorithm of Cole.**
Aim: Fast optimal algorithm.

So far \( T(n) = \log^2 n \) bei \( P(n) = O(n) \).

So far: Two loop for merging and sorting.

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- Before merging two sequences we will merge two sub-sequences.
- Choose as sub-sequence each $k$-th element of the original sequence.
- These sub-sequences will be used as crutch/support to do the final merging.
- I.e. these sub-sequences are used as a kind of “preview”.
- Using these crutch points we will be able to do the merging in $O(1)$ time.
- Total running time will be $O(\log n)$.
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The Merging-Tree, a View

Each processor starts with 256 elements.
The Merging-Tree, a View

Each Prozessor starts with 256 elements

- sends 4 →
- has 256 →
- ↑ each ↑
The Merging-Tree, a View

Each Prozessor starts with 256 elements
The Merging-Tree, a View
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Each processor starts with 256 elements: each processor sends 4, then 16, 64, 256 elements to another processor, which in turn sends 4, 16, 64, 256 elements to yet another processor, and so on, until all processors have merged their elements into a sorted list.
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- Note: without additional information we could not merge $J$ and $K$ in $O(1)$ time with $O(n)$ processors.
- Let $L$ be a third sequence, which will be called in the following good sampler for $J$ and $K$.
- Informal: $|L| < |J|$ and the elements of $L$ are evenly spread in $J$.
- Let $a < b$, $c$ is between $a$ and $b$ iff $a < c \leq b$.
- The rank of $e$ in $S$ is $\text{rng}(e, S) = |\{x \in S \mid x < e\}|$.
- Notation: $\text{Rng}_{A,B}$ is the function $\text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|}$ with $\text{Rng}_{A,B}(e) = \text{rng}(e, B)$ for all $e \in A$.
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Good Sampler

$$rng(e, S) = \left| \{ x \in S \mid x < e \} \right|$$ and $$Rng_{A,B} : A \mapsto \mathbb{N}^{|A|}$$ with $$Rng_{A,B}(e) = rng(e, B)$$

Definition:

We call $$L$$ a good sampler of $$J$$, iff:

- $$L$$ and $$J$$ are sorted.
- Between any $$k + 1$$ succeeding elements of $$\{-\infty\} \cup L \cup \{+\infty\}$$ are at most $$2 \cdot k + 1$$ many elements in $$J$$.

Example:

- Let $$S$$ be a sorted sequence
- Let $$S_1$$ be the sequence consisting of each forth element of $$S$$.
- Then $$S_1$$ is a good sampler of $$S$$.
- Let $$S_2$$ be the sequence consisting of each second element of $$S$$.
- Then $$S_1$$ is a good sampler of $$S_2$$.
- Example ($$k = 1$$): 1, 2, 3, 4.
- Example ($$k = 3$$): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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**Example:**

- Let \( S \) be a sorted sequence
- Let \( S_1 \) be the sequence consisting of each fourth element of \( S \).
- Then \( S_1 \) is a good sampler of \( S \).
- Let \( S_2 \) be the sequence consisting of each second element of \( S \).
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- Example \((k = 1)\): 1, 2, 3, 4.
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- Let \( S_1 \) be the sequence consisting of each forth element of \( S \).
- Then \( S_1 \) is a good sampler of \( S \).
- Let \( S_2 \) be the sequence consisting of each second element of \( S \).
- Then \( S_1 \) is a good sampler of \( S_2 \).
- Example (\( k = 1 \)): \( 1, 2, 3, 4 \).
- Example (\( k = 3 \)): \( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \).
Good Sampler

Definition:

We call \( L \) a good sampler of \( J \), iff:

- \( L \) and \( J \) are sorted.
- Between any \( k + 1 \) succeeding elements of \( \{-\infty\} \cup L \cup \{+\infty\} \) are at most \( 2 \cdot k + 1 \) many elements in \( J \).

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Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \] and \( \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\mid A \mid} \) with \( \text{Rng}_{A,B}(e) = \text{rng}(e, B) \)

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Merging using a Good Sampler

\[
\text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B)
\]

- Let \( J \), \( K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \cdots, l_s) \).
- Programm: \text{merge\_with\_help}(J, K, L)
  
  for all \( i \) where \( 1 \leq i \leq s \) do in parallel
  
  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
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  Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).

  return \( (\text{res}_1, \text{res}_2, \cdots, \text{res}_s) \).

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Merging using a Good Sampler

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## Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- Let \( J, K \) and \( L \) be sorted sequences.
- Let \( L \) be a good sampler of both \( J \) and \( K \).
- Let \( L = (l_1, l_2, \ldots, l_s) \).
- Program: \text{merge\_with\_help}(J, K, L)
  
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  Assign \( J_i = \{x \in J \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( K_i = \{x \in K \mid l_{i-1} < x \leq l_i\} \).
  
  Assign \( \text{res}_i = \text{merge}(J_i, K_i) \).

\text{return} (\text{res}_1, \text{res}_2, \ldots, \text{res}_s).

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Merging using a Good Sampler

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A ↦ \mathbb{N}^{|A|} with Rng_{A,B}(e) = rng(e, B)

- Let J, K and L be sorted sequences.
- Let L be a good sampler of both J and K.
- Let L = (l_1, l_2, \cdots, l_s).

Programm: merge_with_help(J, K, L)

for all i where 1 ≤ i ≤ s do in parallel
  Assign J_i = \{x ∈ J | l_{i-1} < x ≤ l_i\}.
  Assign K_i = \{x ∈ K | l_{i-1} < x ≤ l_i\}.
  Assign res_i = merge(J_i, K_i).

return (res_1, res_2, \cdots, res_s).

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Merging using a Good Sampler

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\lvert A \rvert} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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- Programm: merge_with_help(\( J, K, L \))
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Merging using a Good Sampler (Example)

$$rng(e, S) = |\{x \in S \mid x < e\}|$$ and $$Rng_{A,B} : A \mapsto \mathbb{N}^{|A|}$$ with $$Rng_{A,B}(e) = rng(e, B)$$

- $$K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20)$$
- $$J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21)$$
- $$L = (5, 10, 12, 17)$$

Then we have:

<table>
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<tr>
<th></th>
<th>$$K_i$$</th>
<th>$$J_i$$</th>
<th>merge($$K_i, J_i$$)</th>
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- Result: $$(1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)$$
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{ x \in S \mid x < e \}| \quad \text{and} \quad \text{Rng}_{A,B} : A \to \mathbb{N}^{|A|} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

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Then we have:

<table>
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } R_{A,B} : A \rightarrow \mathbb{N}^{|A|} \text{ with } R_{A,B}(e) = \text{rng}(e, B) \]

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Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = \left| \{ x \in S \mid x < e \} \right| \text{ and } Rng_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } Rng_{A,B}(e) = \text{rng}(e, B) \]

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- Result: \((1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)\)
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<td>(19, 20)</td>
<td>(18, 21)</td>
<td></td>
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Result: \( (1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21) \)
Merging using a Good Sampler (Example)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \quad \text{and} \quad \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{\lfloor A \rfloor} \quad \text{with} \quad \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

- \( K = (1, 4, 6, 9, 11, 12, 13, 16, 19, 20) \)
- \( J = (2, 3, 7, 8, 10, 14, 15, 17, 18, 21) \)
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Merging with good sampler (running time)

\[ \text{rng}(e, S) = |\{x \in S \mid x < e\}| \text{ and } \text{Rng}_{A,B} : A \mapsto \mathbb{N}^{|A|} \text{ with } \text{Rng}_{A,B}(e) = \text{rng}(e, B) \]

Lemma:

If \( L \) is a good sampler for \( K \) and \( J \).
If \( \text{Rng}_{L,J}, \text{Rng}_{L,K}, \text{Rng}_{K,L} \) and \( \text{Rng}_{J,L} \) is known, then we have:
\[ T_{\text{merge\_with\_help}(J,K,L)} = O(1) \text{ with } P_{\text{merge\_with\_help}(J,K,L)} = O(|J| + |K|). \]

Proof:

- The same way as in the merging introduced in the last chapter.
- Each processor uses \( \text{Rng}_{L,J} \) resp. \( \text{Rng}_{L,K} \) to know the area to read its input sequences.
- Each processor uses \( \text{Rng}_{J,L} \) and \( \text{Rng}_{K,L} \) to know the area to write its output sequence.
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Properties of Good Samplers

rng(e, S) = |{x ∈ S | x < e}| and Rng_{A,B} : A ↦ ↘ N^{|A|} with Rng_{A,B}(e) = rng(e, B)

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If X is a good sampler for X' and Y is a good sampler for Y', then merge(X, Y) is a good sampler for X' [resp. Y'].

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merge(X, Y) is not necessary a sampler for merge(X', Y').

- X = (2, 7) and X' = (2, 5, 6, 7).
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- merge(X, Y) = (1, 2, 7, 8) and merge(X', Y') = (1, 2, 3, 4, 5, 6, 7, 8).
- There are 5 elements between 2 and 7.
Properties of Good Samplers

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**Lemma:**

Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \).
Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

**Proof:**

- W.l.o.g. contain \( X \) and \( Y \) elements \(-\infty\) and \(+\infty\).
- Let \((e_1, e_2, \ldots, e_r)\) successive elements of \( \text{merge}(X, Y) \).
- W.l.o.g. let \( e_1 \in X \).
- Consider now two cases: \( e_r \in X \) and \( e_r \in Y \).
- Let in the following be

\[
x = |X \cap \{e_1, e_2, \ldots, e_r\}| \quad \text{and} \quad y = |Y \cap \{e_1, e_2, \ldots, e_r\}|.
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Properties of Good Samplers

Lemma:
Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$. Then there are at most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$.

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Let \( X \) be a good sampler for \( X' \) and let \( Y \) be a good sampler for \( Y' \). Then there are at most \( 2 \cdot r + 2 \) elements of \( \text{merge}(X', Y') \) between \( r \) successive elements of \( \text{merge}(X, Y) \).

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Properties of Good Samplers

Lemma:
Let $X$ be a good sampler for $X'$ and let $Y$ be a good sampler for $Y'$.
Then there are at most $2 \cdot r + 2$ elements of merge($X'$, $Y'$) between $r$
successive elements of merge($X$, $Y$).

Proof:

- W.l.o.g. contain $X$ and $Y$ elements $-\infty$ and $+\infty$.
- Let $(e_1, e_2, \cdots, e_r)$ successive elements of merge($X$, $Y$).
- W.l.o.g. let $e_1 \in X$.
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Let \(X\) be a good sampler for \(X'\) and let \(Y\) be a good sampler for \(Y'\). Then there are at most \(2 \cdot r + 2\) elements of merge\((X', Y')\) between \(r\) successive elements of merge\((X, Y)\).

Proof: W.l.o.g. let \(e_1 \in X\).

If: \(e_r \in X\)

- Between \(e_1\) and \(e_r\) are at most \(2(x - 1) + 1\) elements of \(X'\).
- Between \(e_1\) and \(e_r\) are at most \(2(y + 1) + 1\) elements of \(Y'\), because they are between \(y + 2\) elements of \(Y\).
- Thus we get: \(2(x - 1) + 1 + 2(y + 1) + 1 = 2 \cdot r + 2\).

Example \(x = 3\) and \(y = 2\):

\[e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X\]
Properties of Good Samplers

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\[ a \in Y \quad e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y \quad e_5 \in X \quad b \in Y \]
Properties of Good Samplers

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Proof: W.l.o.g. let \(e_1 \in X\). If: \(e_r \in Y\)

- Add \(e_0 \in Y\) with \(e_0 < e_1\) to the good sampler.
- Add \(e_{r+1} \in X\) with \(e_r < e_{r+1}\) to the good sampler.
- The elements from \(X'\) between \((e_1, e_2, \cdots, e_r)\) are between \(x + 1\) elements from \(X\).
- The elements from \(Y'\) between \((e_1, e_2, \cdots, e_r)\) are between \(y + 1\) elements from \(Y\).
- Thus we get: \(2x + 1 + 2y + 1 = 2r + 2\).

Example \(x = 2\) and \(y = 2\):

\[e_1 \in X \quad e_2 \in Y \quad e_3 \in X \quad e_4 \in Y\]
Properties of Good Samplers

Let \( (e_1, e_2, \ldots, e_r) \) successive elements of merge\((X, Y)\) and \( x = |X \cap \{e_1, e_2, \ldots, e_r\}| \) and \( y = |Y \cap \{e_1, e_2, \ldots, e_r\}| \) and

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- The elements from \( X' \) between \((e_1, e_2, \ldots, e_r)\) are between \( x + 1 \) elements from \( X \).
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  e_1 & \in X \\
  e_2 & \in Y \\
  e_3 & \in X \\
  e_4 & \in Y
\end{align*}
\]
Properties of Good Samplers

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\begin{align*}
e_0 &\in Y \\
e_1 &\in X \\
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Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition
Let reduce($X$) be the operation, which chooses from $X$ every forth element.

Lemma:
If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then reduce($\text{merge}(X, Y)$) is a good sampler for reduce($\text{merge}(X', Y')$).

Proof:
- Consider $k + 1$ successive elements $(e_1, e_2, \cdots, e_{k+1})$ of reduce($\text{merge}(X, Y)$).
- At most $4k + 1$ elements of $\text{merge}(X, Y)$ are between $e_1, e_2, \cdots, e_{k+1}$ including $e_1, e_{k+1}$.
- At most $8k + 4$ elements of $\text{merge}(X', Y')$ are between these $4k + 1$ elements.
- At most $2k + 1$ elements of reduce($\text{merge}(X', Y')$) are between $(e_1, e_2, \cdots, e_{k+1})$. 
Properties of good sampler

At most $2 \cdot r + 2$ elements of $\text{merge}(X', Y')$ between $r$ successive elements of $\text{merge}(X, Y)$

Definition

Let $\text{reduce}(X)$ be the operation, which chooses from $X$ every forth element.

Lemma:

If $X$ is a good sampler for $X'$ and $Y$ is a good sampler for $Y'$, then $\text{reduce}(\text{merge}(X, Y))$ is a good sampler for $\text{reduce}(\text{merge}(X', Y'))$.

Proof:

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Overview to the Algorithm of Cole

- We start with an explanation using a complete binary tree.
- The leaves contain the elements to be sorted.
- Interior nodes $v$ “cares” about as many elements as the number of leaves below $v$.
- A node $v$ receives from its sons sequences of already sorted sequences.
- The “length” of the sequences doubles each time.
- Node $v$ receives sequences $X_1, X_2, \ldots, X_r$ and $Y_1, Y_2, \ldots, Y_r$.
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One basic Operation of an interior Node $v$

- Receives from its sons the two sequences $X$ and $Y$.
- Computes: $val_v = \text{merge}_\text{with}_\text{help}(X, Y, val_v)$.
- Sends to its father: $\text{reduce}(val_v)$ till $v$ has sorted all received sequences.
- Sends to its father each second element from $val_v$, if $v$ is done with sorting.
- Sends to its father $val_v$, if $v$ finishes sorting two steps before.

**Example:**

<table>
<thead>
<tr>
<th>Step</th>
<th>Left</th>
<th>Right</th>
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- Thus we get the following pattern:

$$
X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_r \\
Z_1 \quad Z_2 \quad \cdots \quad Z_r \quad Z_{r+1} \quad Z_{r+2}
$$

- If a node $x$ is finished after $t$ steps, then will the father of $x$ be finished after $t + 3$ steps.
- Thus we get a running time of $3 \log n$. 
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- Each $X_i$ is a good sampler of $X_{i+1}$.
- Each $Y_i$ is a good sampler of $Y_{i+1}$.
- Each $Z_i$ is a good sampler of $Z_{i+1}$.
- Each $X_i$ is half as big as $X_{i+1}$.
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- $|X_1| = |Y_1| = |Z_1| = 1$. 

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- Each $Y_i$ is half as big as $Y_{i+1}$.
- Each $Z_i$ is half as big as $Z_{i+1}$.
- $|X_1| = |Y_1| = |Z_1| = 1.$
### Invariant

- Each $X_i$ is a good sampler of $X_{i+1}$.
- Each $Y_i$ is a good sampler of $Y_{i+1}$.
- Each $Z_i$ is a good sampler of $Z_{i+1}$.
- Each $X_i$ is half as big as $X_{i+1}$.
- Each $Y_i$ is half as big as $Y_{i+1}$.
- Each $Z_i$ is half as big as $Z_{i+1}$.
- $|X_1| = |Y_1| = |Z_1| = 1$. 

$$\begin{align*}
|X_1| &= |Y_1| = |Z_1| = 1.
\end{align*}$$
Invariant:

- Each $X_i$ is a good sampler of $X_{i+1}$.
- Each $Y_i$ is a good sampler of $Y_{i+1}$.
- Each $Z_i$ is a good sampler of $Z_{i+1}$.
- Each $X_i$ is half as big as $X_{i+1}$.
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Situation

- Running time is \( O(\log n) \).
- The inner nodes \( v \) need \(|val_v|\) many processors.
- We still have to proof that the number of processors is in \( O(n) \).
- PRAM Model has to be verified.
- Important: The computation of the values \( Rng_{X,Y} \) has to be shown.
- These values will be in the following also transmitted and updated.
Running time is $O(\log n)$.

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- Important: The computation of the values $Rng_{X,Y}$ has to be shown.
- These values will be in the following also transmitted and updated.
Computing the Ranks

- In each step will compute: `merge_with_help(X_{i+1}, Y_{i+1}, merge(X_i, Y_i))`.
- Using the Lemma from above we have: `merge(X_i, Y_i)` is a good sampler of `X_{i+1}` and `Y_{i+1}`.
- Let `L = merge(X_i, Y_i)`, `J = X_{i+1}` and `K = Y_{i+1}`.
- We have to compute: `Rng_{L,J}`, `Rng_{L,K}`, `Rng_{J,L}` and `Rng_{K,L}`.

**Invariant:**

- Let `S_1, S_2, \cdots, S_p` be a sequence of sequences at node `v`.
- Then node `c` also knows: `Rng_{S_{i+1}, S_i}` for `1 \leq i < p`.
- Furthermore for each sequence `S` is known: `Rng_{S,S}`.
Computing the Ranks

- In each step will compute: $merge\_with\_help(X_{i+1}, Y_{i+1}, merge(X_i, Y_i))$.
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Computing the Ranks

**Lemma:**

Let $S = (b_1, b_2, \ldots, b_k)$ be a sorted sequence, then we may compute the rank of $a \in S$ in time $O(1)$ using $k$ processors.

**Proof:**

- Program: `rng1(a,S)`
  - for all $P_i$ where $1 \leq i \leq k$ do in parallel
    - if $b_i < a \leq b_{i+1}$ then return $i$

- Note, the program has no write-conflicts.
- Note, it could be changed, to avoid read-conflicts.
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- Note, the program has no write-conflicts.
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Computing the Ranks

**Lemma:**

Let $S_1, S_2, S$ be two sorted sequences with $S = \text{merge}(S_1, S_2)$ and $S_1 \cap S_2 = \emptyset$. Then we may compute $\text{Rnk}_{S_1, S_2}$ and $\text{Rnk}_{S_2, S_1}$ in time $O(1)$ using $O(|S|)$ processors.

**Proof:**

- We do know $\text{Rnk}_{S, S}$, $\text{Rnk}_{S_1, S_1}$ and $\text{Rnk}_{S_2, S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
- The claim follows directly.
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Proof:

- We do know $\text{Rnk}_S, \text{Rnk}_{S_1, S_1}$ and $\text{Rnk}_{S_2, S_2}$.
- Furthermore we have: $\text{rnk}(a, S_2) = \text{rnk}(a, \text{merge}(S_1, S_2)) - \text{rnk}(a, S_1)$.
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Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$.
- Let $Y$ be a good sampler of $Y'$.
- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$, and $\text{Rnk}_{U,Y'}$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- Then we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
- Finally we compute $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$. 
Computing the Ranks

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- Let $X$ be a good sampler of $X'$.
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Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
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we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks

Lemma:

- Let \( X \) be a good sampler of \( X' \).
- Let \( Y \) be a good sampler of \( Y' \).
- Let \( U = \text{merge}(X, Y) \).
- Assume \( \text{Rnk}_{X',X} \) and \( \text{Rnk}_{Y',Y} \) are known.

Then we may compute in time \( O(1) \) using \( O(|X| + |Y|) \) processors \( \text{Rnk}_{X',U}, \text{Rnk}_{Y',U}, \text{Rnk}_{U,X'} \) and \( \text{Rnk}_{U,Y'} \).

Proof:

- First we compute \( \text{Rnk}_{X',U} \) and \( \text{Rnk}_{Y',U} \).
- Then we compute \( \text{Rnk}_{X,X'} \) and \( \text{Rnk}_{Y,Y'} \).
- Finally we compute \( \text{Rnk}_{U,X'} \) and \( \text{Rnk}_{U,Y'} \).
Computing the Ranks

**Lemma:**

- Let $X$ be a good sampler of $X'$. 
- Let $Y$ be a good sampler of $Y'$. 
- Let $U = \text{merge}(X, Y)$. 
- Assume $\text{rank}_{X', X}$ and $\text{rank}_{Y', Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{rank}_{X', U}$, $\text{rank}_{Y', U}$, $\text{rank}_{U, X'}$ and $\text{rank}_{U, Y'}$.

**Proof:**

- First we compute $\text{rank}_{X', U}$ and $\text{rank}_{Y', U}$.
- Then we compute $\text{rank}_{X, X'}$ and $\text{rank}_{Y, Y'}$.
- Finally we compute $\text{rank}_{U, X'}$ and $\text{rank}_{U, Y'}$. 

we have $\text{rank}(a, S)$ and $\text{rank}_{S_1, S_2}$ and $\text{rank}_{S_2, S_1}$.
Computing the Ranks

Lemma:

- Let $X$ be a good sampler of $X'$. 
- Let $Y$ be a good sampler of $Y'$. 
- Let $U = \text{merge}(X, Y)$. 
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$, and $\text{Rnk}_{U,Y'}$.

Proof:

- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- Then we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
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- First we compute $\text{rnk}_{X',U}$ and $\text{rnk}_{Y',U}$.
- Then we compute $\text{rnk}_{X,X'}$ and $\text{rnk}_{Y,Y'}$.
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- Let $X$ be a good sampler of $X'$.
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- Let $U = \text{merge}(X, Y)$.
- Assume $\text{Rnk}_{X',X}$ and $\text{Rnk}_{Y',Y}$ are known.

Then we may compute in time $O(1)$ using $O(|X| + |Y|)$ processors $\text{Rnk}_{X',U}$, $\text{Rnk}_{Y',U}$, $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.

**Proof:**

- First, we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- **Then** we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
- Finally, we compute $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.
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- First we compute $\text{Rnk}_{X',U}$ and $\text{Rnk}_{Y',U}$.
- Then we compute $\text{Rnk}_{X,X'}$ and $\text{Rnk}_{Y,Y'}$.
- Finally we compute $\text{Rnk}_{U,X'}$ and $\text{Rnk}_{U,Y'}$.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X', X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  Program: \(\text{Rnk}_{X', U}\)
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X'_i\) do
      \(\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\)

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \( (\text{Rnk}_{X',U}) \)

- Let \( X = (a_1, a_2, \cdots, a_k) \).
- Let w.l.o.g. \( a_0 = -\infty \) and \( a_{k+1} = +\infty \).
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- Note: \( \text{Rnk}_{X',X} \) is known.
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  Programm: \( \text{Rnk}_{X',U} \)
  for all \( i \) where \( 1 \leq i \leq k + 1 \) do in parallel
  for all \( x \in X'_i \) do
    \( \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i) \)

- Running time \( O(1) \) using \( \sum_{i=1}^{k+1} |U_i| \) processors.
Computing the Ranks (Rnk\(X', U\))

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  Programm: Rnk\(X', U\)
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X'_i\) do
      \(\text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)\)

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X'},U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
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Programm: \(\text{Rnk}_{X',U}\)

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- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks ($\text{Rnk}_{X',U}$)

- Let $X = (a_1, a_2, \cdots, a_k)$.
- Let w.l.o.g. $a_0 = -\infty$ and $a_{k+1} = +\infty$.
- Using a good sampler $X$ we split $X'$ into $X'_1, X'_2, \cdots, X'_k, X'_{k+1}$.
- Note: $\text{Rnk}_{X',X}$ is known.
- Splitting may be done in time $O(1)$ using $O(|X|)$ processors.
- Let $U_i$ be the sequence of elements of $Y$ which are between $a_{i-1}$ and $a_i$.
- Thus we get:

Programm: $\text{Rnk}_{X',U}$

for all $i$ where $1 \leq i \leq k + 1$ do in parallel

for all $x \in X'_i$ do

$rnk(x, U) = rnk(a_{i-1}, U) + rnk(x, U_i)$

Running time $O(1)$ using $\sum_{i=1}^{k+1} |U_i|$ processors.
Computing the Ranks \((\text{Rnk}_{X'}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
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- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
- Thus we get:

  \[
  \text{Programm: } \text{Rnk}_{X', U} \\
  \text{for all } i \text{ where } 1 \leq i \leq k + 1 \text{ do in parallel} \\
  \quad \text{for all } x \in X'_i \text{ do} \\
  \quad \quad \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i)
  \]

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X}, U)\)

- Let \(X = (a_1, a_2, \cdots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \cdots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X',X}\) is known.
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  Programm: \text{Rnk}_{X',U}
  for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X'_i\) do
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Computing the Ranks \( (\text{Rnk}_{X'}, U) \)

- Let \( X = (a_1, a_2, \cdots, a_k) \).
- Let w.l.o.g. \( a_0 = -\infty \) and \( a_{k+1} = +\infty \).
- Using a good sampler \( X \) we split \( X' \) into \( X_1', X_2', \cdots, X_k', X_{k+1}' \).
- Note: \( \text{Rnk}_{X', X} \) is known.
- Splitting may be done in time \( O(1) \) using \( O(|X|) \) processors.
- Let \( U_i \) be the sequence of elements of \( Y \) which are between \( a_{i-1} \) and \( a_i \).
- Thus we get:

  Program: \( \text{Rnk}_{X', U} \)
  
  for all \( i \) where \( 1 \leq i \leq k + 1 \) do in parallel
    
    for all \( x \in X_i' \) do
      \( \text{rnk}(x, U) = \text{rnk}(a_{i-1}, U) + \text{rnk}(x, U_i) \)

- Running time \( O(1) \) using \( \sum_{i=1}^{k+1} |U_i| \) processors.
Computing the Ranks \((\text{Rnk}_{X'},U)\)

- Let \(X = (a_1, a_2, \ldots, a_k)\).
- Let w.l.o.g. \(a_0 = -\infty\) and \(a_{k+1} = +\infty\).
- Using a good sampler \(X\) we split \(X'\) into \(X'_1, X'_2, \ldots, X'_k, X'_{k+1}\).
- Note: \(\text{Rnk}_{X',X}\) is known.
- Splitting may be done in time \(O(1)\) using \(O(|X|)\) processors.
- Let \(U_i\) be the sequence of elements of \(Y\) which are between \(a_{i-1}\) and \(a_i\).
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Programm: \(\text{Rnk}_{X',U}\)
for all \(i\) where \(1 \leq i \leq k + 1\) do in parallel
    for all \(x \in X'_i\) do
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```

- Running time \(O(1)\) using \(\sum_{i=1}^{k+1} |U_i|\) processors.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

- Let \(a_i \in X\).
- Let \(a'\) minimal element in \(X_{i+1}\).
- The rank of \(a_i\) in \(X'\) is the same as the rank of \(a'\) in \(X'\).
- This rank is already known.
- This may be computed in time \(O(1)\) using one processor.
Computing the Ranks ($\text{Rnk}_{X,X'}$)

- Let $a_i \in X$.
- Let $a'$ minimal element in $X_{i+1}$.
- The rank of $a_i$ in $X'$ is the same as the rank of $a'$ in $X'$.
- This rank is already known.
- This may be computed in time $O(1)$ using one processor.
Computing the Ranks \((\text{Rnk}_{X,X'})\)

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Note: We have \(\text{rnk}(a, S)\) and \(\text{Rnk}_{S_1,S_2}\) and \(\text{Rnk}_{S_2,S_1}\)
Computing the Ranks ($\text{Rnk}_X,X'$)

- Let $a_i \in X$.
- Let $a'$ minimal element in $X'_{i+1}$.
- The rank of $a_i$ in $X'$ is the same as the rank of $a'$ in $X'$.
- This rank is already known.
- This may be computed in time $O(1)$ using one processor.
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- **Note:** $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} \ X, X'$ and $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ X, X'$ is already known.
- Still to compute: $\text{Rnk} \ Y, X'$.
- $\text{Rnk} \ Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- Note: \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk} \, X, \, X'\) and \(\text{Rnk} \, Y, \, X'\).
- \(\text{Rnk} \, X, \, X'\) is already known.
- Still to compute: \(\text{Rnk} \, Y, \, X'\).
- \(\text{Rnk} \, Y, \, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- **Still to compute:** $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$.
Computing the Ranks (Rnk<sub>U,X′</sub>)

- Note: Rnk<sub>U,X′</sub> consists of Rnk<sub>X,X′</sub> and Rnk<sub>Y,X′</sub>.
- Rnk<sub>X,X′</sub> is already known.
- Still to compute: Rnk<sub>Y,X′</sub>.
- Rnk<sub>Y,X</sub> may be computed using the previous lemma.
- We compute rnk(<i>a,X′</i>) using rnk(<i>a,X</i>) and Rnk<sub>X,X′</sub>.
- Thus we compute Rnk<sub>U,X′</sub> with O(|U|) processors and time O(1).

we have rnk(<i>a,S</i>) and Rnk<sub>S₁,S₂</sub> and Rnk<sub>S₂,S₁</sub>
Computing the Ranks \((\text{Rnk}_{U,X'})\)

- Note: \(\text{Rnk}_{U,X'}\) consists of \(\text{Rnk} X, X'\) and \(\text{Rnk} Y, X'\).
- \(\text{Rnk} X, X'\) is already known.
- Still to compute: \(\text{Rnk} Y, X'\).
- \(\text{Rnk} Y, X\) may be computed using the previous lemma.
- We compute \(\text{rnk}(a, X')\) using \(\text{rnk}(a, X)\) and \(\text{Rnk}_{X,X'}\).
- Thus we compute \(\text{Rnk}_{U,X'}\) with \(O(|U|)\) processors and time \(O(1)\).
Computing the Ranks ($\text{Rnk}_{U,X'}$)

- Note: $\text{Rnk}_{U,X'}$ consists of $\text{Rnk} X, X'$ and $\text{Rnk} Y, X'$.
- $\text{Rnk} X, X'$ is already known.
- Still to compute: $\text{Rnk} Y, X'$.
- $\text{Rnk} Y, X$ may be computed using the previous lemma.
- We compute $\text{rnk}(a, X')$ using $\text{rnk}(a, X)$ and $\text{Rnk}_{X,X'}$.
- Thus we compute $\text{Rnk}_{U,X'}$ with $O(|U|)$ processors and time $O(1)$. 

we have $\text{rnk}(a, S)$ and $\text{Rnk}_{S_1,S_2}$ and $\text{Rnk}_{S_2,S_1}$.
Computing the Ranks

Consider the step

\[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).

Using the above considerations we may compute: \( \text{Rnk}_{L,J}, \text{Rnk}_{L,K}, \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).

Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

we have \( rnk(a, S) \) and \( Rnk_{S_1, S_2} \) and \( Rnk_{S_2, S_1} \)

- Consider the step
  \[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: \( Rnk_{J, X_i} \) and \( Rnk_{K, Y_i} \).

- Using the above considerations we may compute: \( Rnk_{L, J}, Rnk_{L, K}, Rnk_{J, L} \) and \( Rnk_{K, L} \).

- Still to be computed: \( Rnk_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)

- Known: \( Rnk_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( Rnk_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

- It is now easy to compute: \( Rnk_{X_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \) and \( Rnk_{Y_{i+1}, \text{reduce}(\text{merge}(X_i, Y_i))} \).

- Also easy to compute: \( Rnk_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce}(\text{merge}(X_i, Y_i))} \).
Computing the Ranks

we have \( \text{rnk}(a, S) \) and \( \text{Rnk}_{S_1, S_2} \) and \( \text{Rnk}_{S_2, S_1} \)

- Consider the step
  \[ \text{merge}_\text{with}_\text{help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]
- Using the invariant we know: \( \text{Rnk}_{J, X_i} \) and \( \text{Rnk}_{K, Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L, J} \), \( \text{Rnk}_{L, K} \), \( \text{Rnk}_{J, L} \)
  and \( \text{Rnk}_{K, L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce(merge}(X_{i+1}, Y_{i+1})), \text{reduce(merge}(X_i, Y_i))} \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
- It is now easy to compute: \( \text{Rnk}_{X_{i+1}, \text{reduce(merge}(X_i, Y_i))} \) and \( \text{Rnk}_{Y_{i+1}, \text{reduce(merge}(X_i, Y_i))} \).
- Also easy to compute: \( \text{Rnk}_{\text{merge}(X_{i+1}, Y_{i+1}), \text{reduce(merge}(X_i, Y_i))} \).
Computing the Ranks

- Consider the step
  \[\text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)):\]

- Using the invariant we know: \(Rnk_{J,X_i}\) and \(Rnk_{K,Y_i}\).

- Using the above considerations we may compute: \(Rnk_{L,J}\), \(Rnk_{L,K}\), \(Rnk_{J,L}\) and \(Rnk_{K,L}\).

- **Still to be computed:** \(Rnk_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))}\)

- Known: \(Rnk_{X_{i+1}, \text{merge}(X_i, Y_i)}\) and \(Rnk_{Y_{i+1}, \text{merge}(X_i, Y_i)}\).

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Computing the Ranks

Consider the step
\[ \text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)) \]

- Using the invariant we know: Rnk\(J, X_i\) and Rnk\(K, Y_i\).
- Using the above considerations we may compute: Rnk\(L, J\), Rnk\(L, K\), Rnk\(J, L\) and Rnk\(K, L\).
- Still to be computed: Rnk\(\text{reduce(merge}(X_{i+1}, Y_{i+1})), \text{reduce(merge}(X_i, Y_i))\)
- Known: Rnk\(X_{i+1}, \text{merge}(X_i, Y_i)\) and Rnk\(Y_{i+1}, \text{merge}(X_i, Y_i)\).
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Computing the Ranks

Consider the step

\[
\text{merge\_with\_help}(J = X_{i+1}, K = Y_{i+1}, L = \text{merge}(X_i, Y_i)):
\]

- Using the invariant we know: \( \text{Rnk}_{J,X_i} \) and \( \text{Rnk}_{K,Y_i} \).
- Using the above considerations we may compute: \( \text{Rnk}_{L,J} \), \( \text{Rnk}_{L,K} \), \( \text{Rnk}_{J,L} \) and \( \text{Rnk}_{K,L} \).
- Still to be computed: \( \text{Rnk}_{\text{reduce}(\text{merge}(X_{i+1}, Y_{i+1})), \text{reduce}(\text{merge}(X_i, Y_i))} \)
- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).
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we have \( \text{rnk}(a, S) \) and \( \text{Rnk}_{S_1, S_2} \) and \( \text{Rnk}_{S_2, S_1} \).
Computing the Ranks

we have \( rnk(a, S) \) and \( Rnk_{S_1, S_2} \) and \( Rnk_{S_2, S_1} \)

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- Using the invariant we know: \( Rnk_{J, X_i} \) and \( Rnk_{K, Y_i} \).

- Using the above considerations we may compute: \( Rnk_{L, J} \), \( Rnk_{L, K} \), \( Rnk_{J, L} \) and \( Rnk_{K, L} \).

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- Known: \( \text{Rnk}_{X_{i+1}, \text{merge}(X_i, Y_i)} \) and \( \text{Rnk}_{Y_{i+1}, \text{merge}(X_i, Y_i)} \).

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Theorem:
We may sort $n$ values on a CREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: discussed before.

Theorem:
We may sort $n$ values on a EREW PRAM using $O(n)$ processors in time $O(\log n)$.

Proof: see literature.

Theorem:
There exists a sorting network with $O(n)$ processors and depth $O(\log n)$.

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Algorithm of Cole

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Literature

Questions

- Explain the motivation behind parallel systems.
- Explain the ideas of the different sorting algorithms.
- Explain the different running times of these sorting algorithms.
- Explain the different efficiency of these sorting algorithms.
- Explain the idea of the algorithm of Cole.
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Legend

- : Not of relevance
- : implicitly used basics
- : idea of proof or algorithm
- : structure of proof or algorithm
- : Full knowledge