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Current Situation:
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We now want to devise **local-control** algorithms.  
Each node decides on the next step by some local information.
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- The number of steps $T$ taken by an algorithm to deliver all packets is referred to as **routing time**.
Oblivious routing

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- Every packet with source $u$ and destination $v$ is sent along the path $P_{u,v}$.

Example: bit-fixing paths on the hypercube
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Lower Bound by Borodin and Hopcroft

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- Let $n$ denote the number of nodes and $\Delta$ the maximum degree of $G$. 

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General NW.
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Lower Bound by Borodin and Hopcroft

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There exists a permutation $\pi : V \to V$ and an edge $e^* \in E$ such that at least

$$\sqrt{n/(2\Delta^2)} = \Omega(\sqrt{n}/\Delta)$$

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This is very bad news about deterministic oblivious routing:
- The time complexity for permutation routing under this paradigm is lower bounded by $\Omega(\sqrt{n}/\Delta)$, which is polynomial in $n$.
- Even a small diameter, say logarithmic in $n$, does not help.
Definition

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Proof of the lower bound by Borodin and Hopcroft
Definition

- For $v \in V$, let $W_v = \{P_{v,u} \mid u \in V\}$.
- For a positive number $t$, a node $v \in V$, and an edge $e \in E$, we say that $e$ is $t$-popular for $v$ if at least $t$ paths from $W_v$ contain $e$. 

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Outline of the proof:

- First, we prove a lemma showing that, for any given node $v \in V$, there are "many" edges that are "quite popular" for $v$. 
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- Then we use the lemma to show that there is an edge $e^*$ that is "quite popular" for "many" nodes, that is, $e^*$ is $t$-popular for $t$ different nodes, for $t = \Omega(\sqrt{n}/\Delta)$. 
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- Given this, we will be able to construct a permutation $\pi$ such that $t$ of the paths selected by $\pi$ contain $e^*$, which proves the lower bound.
Proof of the lower bound by Borodin and Hopcroft

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- For $v \in V$, and $e \in E$, define

$$A_{v,e}(t) = \begin{cases} 1 & \text{if } e \text{ is } t\text{-popular for } v, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$
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- For $e \in E$, let $A_e(t) = \sum_{v \in V} A_{v,e}(t)$ denote the column sum of $e$. 
One Lemma for the Proof of the lower bound

Lemma

∀v ∈ V and t ≤ (n − 1)/Δ : A_v(t) ≥ \frac{n}{2Δt}.

Proof of lemma:
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\[ \forall v \in V \text{ and } t \leq (n - 1)/\Delta : A_v(t) \geq \frac{n}{2\Delta t}. \]

Proof of lemma:

- Let \( Q \subseteq V \) be the set of nodes from which there is a path to \( v \) that contains only edges that are \( t \)-popular for \( v \).
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Lemma

∀v ∈ V and t ≤ (n − 1)/Δ : A_v(t) ≥ \frac{n}{2Δt}.

Proof of lemma:

- Let Q ⊆ V be the set of nodes from which there is a path to v that contains only edges that are t-popular for v.
- Let L = V − Q and B = E ∩ (L × Q), that is, B is the set of those edges connecting a node in L with a node in Q.
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**Lemma**

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- It holds
One Lemma for the Proof of the lower bound

**Lemma**

\[ \forall v \in V \text{ and } t \leq \frac{n - 1}{\Delta} : A_v(t) \geq \frac{n}{2\Delta t}. \]

**Proof of lemma:**

- Let \( Q \subseteq V \) be the set of nodes from which there is a path to \( v \) that contains only edges that are \( t \)-popular for \( v \).

- Let \( L = V - Q \) and \( B = E \cap (L \times Q) \), that is, \( B \) is the set of those edges connecting a node in \( L \) with a node in \( Q \).

- It holds

  - \( |B| \cdot (t - 1) \geq |L| \) because, for each node \( u \in L \), the path \( P_{v,u} \) leads through at least one edge in \( B \) and these edges are not \( t \)-popular so that each of them can be contained in at most \( t - 1 \) paths from \( W_v \).
Lemma

∀v ∈ V and t ≤ (n − 1)/Δ : A_v(t) ≥ \frac{n}{2\Delta t}.

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- Let Q ⊆ V be the set of nodes from which there is a path to v that contains only edges that are t-popular for v.
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  - |B| · (t − 1) ≥ |L| because, for each node u ∈ L, the path P_{v,u} leads through at least one edge in B and these edges are not t-popular so that each of them can be contained in at most t − 1 paths from \mathcal{W}_v.
  - |B| ≤ Δ |Q| as each node in Q has at most Δ incident edges.
Proof of the lemma

Combining the two equations, we obtain

\[ \Delta |Q| (t - 1) \geq |L| = n - |Q| , \]
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and, hence,

$$|Q| \geq \frac{n}{\Delta t}.$$
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Combining the two equations, we obtain

\[ |B| \cdot (t - 1) \geq |L| \] and \[ |B| \leq \Delta |Q| \]

which implies

\[ \Delta |Q| (t - 1) \geq |L| = n - |Q| , \]

and, hence,

\[ \Delta |Q| t \geq n . \]

Next we will show \(|Q| \leq 2A_v(t)\) which completes the proof of the lemma as it implies

\[ A_v(t) \geq \frac{|Q|}{2} \geq \frac{n}{2\Delta t} . \]
Proof of the lemma

Let $E'$ denote the set of edges that are $t$-popular for $v$. To complete the proof of the lemma, we have to show $|Q| \leq 2|E'| = 2A_v(t)$. 

Show: $|Q| \leq 2A_v(t)$
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such that at least one of the edges incident to $v$ is contained in at least $(n - 1)/\Delta \geq z$ paths from $\mathcal{W}_v$. 

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- Therefore, there is at least one edge that is $t$-popular for $v$. 

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Therefore, there is at least one edge that is $t$-popular for $v$.

Given that $E'$ is non-empty, each node in $Q$ is incident to an edge in $E'$. 

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Show: \(|Q| \leq 2A_v(t)\)

- Let \(E'\) denote the set of edges that are \(t\)-popular for \(v\). To complete the proof of the lemma, we have to show \(|Q| \leq 2|E'| = 2A_v(t)\).

- At first, we observe that \(t \leq (n - 1)/\Delta\) implies that \(E' \neq \emptyset\). This is because
  - \(v\) has at most \(\Delta\) incident edges, and
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  such that at least one of the edges incident to \(v\) is contained in at least \((n - 1)/\Delta \geq z\) paths from \(\mathcal{W}_v\).

- Therefore, there is at least one edge that is \(t\)-popular for \(v\).

- Given that \(E'\) is non-empty, each node in \(Q\) is incident to an edge in \(E'\).

- Consequently, \(|Q| \leq 2|E'|\) as each of the edges in \(E'\) is incident to at most two nodes from \(Q\).
Proof of the lower bound by Borodin and Hopcroft

Show: \( \exists e^*: e^* \text{ is } t\text{-popular for } t \text{ different nodes, for } t = \Omega(\sqrt{n}/\Delta). \)

- Our next goal is to show that there exists an edge \( e^* \) that is \( t\)-popular for \( t \) nodes where \( t = \Omega(\sqrt{n}/\Delta) \).
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- We observe that

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\sum_{e \in E} A_e(t) = \sum_{e \in E} \sum_{v \in V} A_{e,v}(t) = \sum_{v \in V} \sum_{e \in E} A_{e,v}(t) = \sum_{v \in V} A_v(t) \geq \frac{n^2}{2\Delta t},
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where the inequality follows from the lemma.
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- Because of the “pigeonhole principle”, there has to exist an edge \( e^* \in E \) such that

\[
A_{e^*}(t) \geq \left\lfloor \frac{n^2}{|E| \cdot 2\Delta t} \right\rfloor \geq \left\lfloor \frac{n}{2\Delta^2 t} \right\rfloor,
\]

where the last step follows from \( |E| \leq \Delta n. \)
Proof of the lower bound by Borodin and Hopcroft

Show: \( \exists e^*: e^* \) is \( t \)-popular for \( t \) different nodes, for \( t = \Omega(\sqrt{n}/\Delta) \). We have \( A_{e^*}(t) \geq \lceil \frac{n}{2\Delta^2 t} \rceil \).

- Next we choose \( t \) such that \( t = \frac{n}{2\Delta^2 t} \), that is, we set \( t = \sqrt{n}/(\sqrt{2\Delta}) \).
Proof of the lower bound by Borodin and Hopcroft

Show: \( \exists e^*: e^* \text{ is } t\text{-popular for } t \text{ different nodes}, \text{ for } t = \Omega(\sqrt{n}/\Delta) \). We have \( A_{e^*}(t) \geq \left\lceil \frac{n}{2\Delta^2 t} \right\rceil \).

- Next we choose \( t \) such that \( t = \frac{n}{2\Delta^2 t} \), that is, we set \( t = \sqrt{n}/(\sqrt{2}\Delta) \).
- Observe that \( t = \sqrt{n}/(\sqrt{2}\Delta) \) implies \( t \leq (n - 1)/\Delta \), for any \( n \geq 2 \).
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- Next we choose \( t \) such that \( t = \frac{n}{2\Delta^2} \), that is, we set \( t = \sqrt{n}/(\sqrt{2}\Delta) \).
- Observe that \( t = \sqrt{n}/(\sqrt{2}\Delta) \) implies \( t \leq (n - 1)/\Delta \), for any \( n \geq 2 \),
- so that the assumption about \( t \) that we made in the lemma is satisfied.
Proof of the lower bound by Borodin and Hopcroft

Show: $\exists e^*: e^*$ is $t$-popular for $t$ different nodes, for $t = \Omega(\sqrt{n}/\Delta)$. We have $A_{e^*}(t) \geq \left\lceil \frac{n}{2\Delta^2 t} \right\rceil$.

- Next we choose $t$ such that $t = \frac{n}{2\Delta^2 t}$, that is, we set $t = \sqrt{n}/(\sqrt{2}\Delta)$.
- Observe that $t = \sqrt{n}/(\sqrt{2}\Delta)$ implies $t \leq (n-1)/\Delta$, for any $n \geq 2$,
- so that the assumption about $t$ that we made in the lemma is satisfied.

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  - \( A_{e^*}(t) \geq \lceil t \rceil \), that is,
  - \( e^* \) is \( \lceil t \rceil \)-popular for \( \lceil t \rceil \) nodes, where \( t = \sqrt{n}/(\sqrt{2}\Delta) \).
Proof of the lower bound by Borodin and Hopcroft

Construct a permutation $\pi$ such that $t$ of the paths selected by $\pi$ contain $e^*$. Finally, we construct a permutation $\pi$ such that $\lceil t \rceil$ of the paths selected by $\pi$ contain $e^*$:

- Let $V'$ denote a set of $\lceil t \rceil$ nodes for which $e^*$ is $\lceil t \rceil$-popular.
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- W.l.o.g., $V' = \{1, \ldots, \lceil t \rceil\}$.
- For every $\nu \in V'$, there exists a subset $U_\nu \subseteq V$ of cardinality $\lceil t \rceil$ such that, for every $u \in U_\nu$, the path $P_{\nu,u}$ contains $e^\ast$. 
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- For $v = 1$ to $\lceil t \rceil$, set $\pi(v) = u$ where $u$ is chosen arbitrarily from $U_v \setminus \{\pi(1), \ldots, \pi(v-1)\}$. 
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By our construction, $\pi$ and $e^*$ satisfy the properties described in the theorem.
Application to the hypercube and Goal

For the $d$-dimensional hypercube with $n = 2^d$ nodes, the lower bound of Borodin and Hopcroft implies a lower bound of $\Omega(\sqrt{n}/ \log n)$ for permutation routing.
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- Our goal is to devise a distributed permutation routing algorithm with time complexity $O(\log n)$. 
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- Consequently, when using bit-fixing paths the time complexity for permutation routing is $\Omega(\sqrt{n})$.

- Our goal is to devise a distributed permutation routing algorithm with time complexity $O(\log n)$.

- This will take some time.
Outline of the approach

- We build a dynamic system of storage devices supporting the addition and removal of storage devices using dynamic hashing:

1 independently, uniformly at random
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- We build a dynamic system of storage devices supporting the addition and removal of storage devices using dynamic hashing:
  - devices are mapped i.u.r.\(^1\) to the ring \([0, 1)\), that is, each device \(i\) gets assigned a random address \(a(i) \in [0, 1)\)

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- We assume an idealistic hash function, that is, the hash values are real numbers chosen i.u.r. from \([0, 1)\).

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Definition of successors

- Let \( V \) be the set of storage devices at some point of time, and let \( n = |V| \).
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- For address \( A \in [0, 1) \), define

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\text{succ}(A) = \begin{cases} 
\arg\min \{a(i) \geq A \mid i \in V\} & \text{if } \exists i \in V : a(i) \in [A, 1), \\
\arg\min \{a(i) \geq 0 \mid i \in V\} & \text{otherwise}.
\end{cases}
\]

\[
\text{pred}(A) = \begin{cases} 
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Object \( x \in U \) is mapped to device \( \text{succ}(h(x)) \).
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Quality of the load balancing

- The quality of the load balancing depends on the distribution of the sizes of the ring for which the storage devices are responsible.
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**Definition (weight of a device)**

For device $i \in V$, define the weight of device $i$ by

$$W_i = \begin{cases} a(i) - a(pred(a(i))) & \text{if } a(pred(a(i))) < a(i), \\ 1 - (a(pred(a(i))) - a(i)) & \text{otherwise.} \end{cases}$$

Let $W = \max_{i \in [n]} W_i$. 

The term "w.h.p." abbreviates "with high probability" and means with probability at least $1 - n^{-\alpha}$, for any constant $\alpha > 0$. 
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Let \( W = \max_{i \in [n]} W_i \).

- Ideally, we would have \( W = W_0 = W_1 = \ldots = W_{n-1} = \frac{1}{n} \).
- We will show that \( W = O\left(\frac{\log n}{n}\right) \), w.h.p.\(^2\)

---

\(^2\)The term "w.h.p." abbreviates "with high probability" and means with probability at least \( 1 - n^{-\alpha} \), for any constant \( \alpha > 0 \).
Quality of the load balancing

Lemma

Let $T \subseteq [0, 1)$ and $t = |T|$ the mass (length) of $T$. Suppose that $M$ points are chosen i.u.r. from $[0, 1)$. The probability that none of these points is from $T$ is at most $e^{-tM}$.

Proof:

$$\Pr[\text{no point in } T] = (1 - t)^M = ((1 - t)^{1/t})^{tM} \leq e^{-tM}$$

as, for every $x > 0$, it holds $(1 - \frac{1}{x})^x \leq \frac{1}{e}$. 
Quality of the load balancing

Theorem

\[ W = O\left(\frac{\log n}{n}\right), \text{ w.h.p.} \]

Proof:

- Fix \( j \in V \). Suppose \( j \)'s address \( a(j) \) is fixed arbitrarily.

\[ \Pr[\text{no point in } T] \leq e^{-tM}. \]
Quality of the load balancing

**Theorem**

\[ W = O\left(\frac{\log n}{n}\right), \text{ w.h.p.} \]

**Proof:**

- Fix \( j \in V \). Suppose \( j \)'s address \( a(j) \) is fixed arbitrarily.
- A necessary condition for the event \( W_j \geq t, t \in [0,1) \), is that no addresses of the other \( n-1 \) devices falls into the interval from \( a(j) - t \) to \( a(j) \).
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- for any \( \alpha > 0 \),

\[
\Pr\left[ W_j \geq 2(\alpha + 1) \frac{\ln n}{n} \right] \leq e^{-2(\alpha + 1) \frac{\ln n}{n} (n-1)} \leq e^{-(\alpha+1) \ln n} = n^{-(\alpha+1)}
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  and, hence,
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  \Pr \left[ W \geq 2(\alpha + 1) \frac{\ln n}{n} \right] \leq \sum_{j \in V} \Pr \left[ W_j \geq 2(\alpha + 1) \frac{\ln n}{n} \right] \leq n^{-\alpha}.
  \]

Pr [no point in \( T \)] \( \leq e^{-tM} \).
Improved quality of the load balancing

We have $W = O(\log \frac{n}{n})$, w.h.p.

- In order to improve the load balancing, we use $k$ virtual nodes for each device. Let $V'$ denote the set of $kn$ "virtual" nodes.
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- Let $W_i$ denote the weight of device $i$, i.e., the sum of the lengths of the intervals corresponding to $i$'s nodes, and $W = \max_{i \in [n]} W_i$. 
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**Theorem**

For any $k \geq 1$, $W = \frac{1}{n} \cdot O\left(1 + \frac{\log n}{k}\right)$, w.h.p.
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If $k \geq \log n$ then $W = O\left(\frac{1}{n}\right)$, w.h.p.
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**Proof of the Theorem:**

- Consider device \( j \) and suppose the address of the \( k \) nodes of this device are fixed arbitrarily.
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Proof of the Theorem:

- Consider device $j$ and suppose the address of the $k$ nodes of this device are fixed arbitrarily.
- For any $t \in [0, 1)$, we want to upper-bound $Pr[W_j \geq t]$. 
Improved quality of the load balancing

**Exact condition:**

The event $W_j \geq t$ happens if and only if there are $k$ intervals left of the $k$ addresses of $j$’s nodes so that

- these intervals have a total length of $t$, and
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In order to be able to enumerate all possibilities for choosing these $k$ intervals, we look at a slightly stronger necessary condition for the event $W_j \geq t$. 
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- none of the other $k(n-1)$ nodes have an address that falls into these intervals.

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If the event $W_j \geq t$ happens then there are $k$ intervals left of the $k$ addresses of $j$’s nodes so that

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- Now \( q + k = t'kn + k \leq (t - \frac{1}{n})kn + k = tkn \), so that this number is at most
  \[
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Once the intervals are fixed, the probability that these intervals with a total length of $t' \geq t - \frac{2}{n}$ are not hit by one other $k(n - 1)$ addresses is at most

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This gives $etn = e\beta$ and

$$\left(t - \frac{2}{n}\right)(n-1) \geq \left(t - \frac{2}{n}\right) \frac{n}{2} = \frac{\beta}{2} - 1$$

assuming $n \geq 2$. 
Consequently,

\[
\Pr \left[ W_i \geq \frac{\beta}{n} \right] \leq \left( e^{\beta} \cdot e^{-\beta/2+1} \right)^k
\]

\[
= \left( e^{\beta} \cdot e^{-\beta/2+1} \cdot \left( \frac{4}{3} \right)^\beta \right)^k \left( \frac{3}{4} \right)^{\beta^k}.
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- Now observe that \( e^\beta \cdot e^{-\beta/2 + 1} \cdot \left( \frac{4}{3} \right)^\beta \) decreases exponentially in \( \beta \) since \( e^{\frac{1}{2}} \geq \frac{4}{3} \). For \( \beta \geq 25 \), this term is less than 1. Consequently,
  \[
  \Pr\left[ W_i \geq \frac{\beta}{n} \right] \leq \left( \frac{3}{4} \right)^{\beta k} \leq \left( \frac{3}{4} \right)^{(\alpha + 1) \log_{4/3} n} = n^{-(\alpha + 1)},
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for \( \beta \geq \frac{(\alpha+1) \log_{4/3} n}{k} \).

- It follows \( \Pr \left[ W \geq \frac{\beta}{n} \right] \leq n^{-\alpha} \), for \( \beta = O(1 + \frac{\log n}{k}) \), which proves the theorem.
Overlay network

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The Chord network allows that devices enter and leave the system dynamically and supports the efficient search for data objects.

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- Observe that the set of links is finite. For $v \in V$, let $d(v)$ denote the smallest integer such that
  
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- The outdegree of $v$ is at most $d(v)$. Let $D = \max\{d(v)|v \in V\}$. 
Upper-bounding the outdegree

**Theorem**

\[ D = O(\log n), \text{ w.hp., where } n = |V|. \]

**Proof:**

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- All edges \( e(v, i) \) with \( i \geq d(v) \) point to \( \text{succ}(a(v)) \).
- In particular, it holds \( 2^{-d(v)} \leq \ell(v) \), which gives
  \[
  d(v) = \left\lceil \log \left( \frac{1}{\ell(v)} \right) \right\rceil.
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- If at least one of the other $n - 1$ nodes falls into the interval $[a(v), a(v) + \beta)$ which, for each of these nodes, happens with probability $\beta$. 

Now let $\alpha > 0$ be chosen arbitrarily. We obtain

\[
\Pr\left[d(v) \geq (\alpha + 3) \log n\right] \leq \Pr\left[\log \left(\frac{1}{\ell(v)}\right) \geq (\alpha + 3) \log n\right]
\leq \Pr\left[\log \left(\frac{1}{\ell(v)}\right) > (\alpha + 2) \log n\right]
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Hence, the probability that there exists a node $v \in V$ for which $d(v) \geq (\alpha + 3) \log n$ is at most $n^{-\alpha}$. \qed
Routing in Chord

- Suppose a node \( v \) (or the device corresponding to \( v \)) wants to access a data object \( x \).
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- The number of hops needed for finding an object is at most $D$ and, thus, $O(\log n)$, w.h.p., because the index of the outgoing links is increasing with every hop on the routing path.
Oblivious Randomized Routing

Definition

- One specifies a path system $\mathcal{W}$ containing a set of paths $W_{u,v}$ from $u$ to $v$. 

For each packet with source $u$ and destination $v$ one chooses a path $P \in W_{u,v}$ independently at random with probability $D_{u,v}(P)$ and forwards the packet along $P$ to its destination.
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Example:

- For any two nodes \( u, v \in V, u \neq v \), one specifies two alternative paths, that is, \( |W_{u,v}| = 2 \).
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Example:

- For any two nodes $u, v \in V, u \neq v$, one specifies two alternative paths, that is, $|W_{u,v}| = 2$.
- Let $D_{u,v}$ denote the uniform distribution on $W_{u,v}$.
- When sending a packet from $u$ to $v$ choose $P \in W_{u,v}$ with probability $D_{u,v}(P) = 1/2$. 
Packet Scheduling Problem and Scheduling Policies

Definition (Packet Scheduling Problem)

- Input: collection of paths $\mathcal{P}$, one for each packet.
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A scheduling policy is called greedy if a packet $p$ has to wait in a step $t$ before using the next edge $e$ on its path only because there is another packet $p'$ using $e$ in this step.
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**Definition (Packet Scheduling Problem)**

- **Input**: collection of paths $\mathcal{P}$, one for each packet.
- **Task**: One needs to specify which packet should be forwarded along which edge in which time step.

We will address the packet scheduling problem by describing a *scheduling policy*

- specifying which packet can go first and
- which packets have to wait if two or more packets are contending for the same edge.

**Examples:**

- **FCFS** (first-come-first-serve)
- **FTG** (Farthest-to-go)
- Random Rank (as defined later)

A scheduling policy is called **greedy** if a packet $p$ has to wait in a step $t$ before using the next edge $e$ on its path only because there is another packet $p'$ using $e$ in this step.

We say that $p$ is delayed by $p'$ at edge $e$ in time step $t$. 
Congestion and Dilation

**Definition (Dilation)**

The dilation $D$ of a path collection $\mathcal{P}$ is the length (number of edges) on the longest path in $\mathcal{P}$.

**Definition (Congestion)**

The congestion $C$ of a path collection $\mathcal{P}$ is the maximum number of paths from $\mathcal{P}$ that share the same edge (in the same direction).
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- In the following, we assume that every undirected edge is replaced by two edges in opposite direction.
- For a (directed) edge $e \in E$, $C(e)$ denotes the number of paths from $\mathcal{P}$ that contain $e$. 
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- In the following, we assume that every undirected edge is replaced by two edges in opposite direction.
- For a (directed) edge $e \in E$, $C(e)$ denotes the number of paths from $\mathcal{P}$ that contain $e$.
- The congestion is thus defined by $C = \max_{e \in E} C(e)$.
Observation (Lower Bound)

The routing time needed by any scheduling policy is at least
\[ \max\{C, D\} = \Omega(C + D) \]

because

- there is a packet which has path length \( D \) and thus needs at least \( D \) steps to reach its destination, and
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Observation (Upper Bound)

The routing time needed by any greedy scheduling policy is at most \(C \cdot D\) steps because each packet can be delayed at most for \(C - 1\) steps on each edge on its routing path.
Valiant’s trick

- We study permutation routing on the $d$-dimensional hypercube with $n = 2^d$ nodes.
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• For each packet $p$, with source node $s_p$ and destination $d_p$ we pick a node $v_p$ independently, uniformly at random from $V$. 
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- The node $v_p$ is thus used as intermediate destination.
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```markdown
- Valiant's trick: A method for simplifying the analysis of routing algorithms in parallel computing by dividing the problem into two phases: source-to-intermediate and intermediate-to-final.
- Source-to-intermediate phase: Routes packets from sources to intermediate destinations.
- Intermediate-to-final phase: Routes packets from intermediate destinations to their final destinations.
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This approach allows for a more manageable and tractable analysis of the routing problem, often leading to simpler algorithms and more efficient solutions.
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- Valiant’s trick reduces a "worst-case permutation routing problem" to two "random routing problems":
  - One with randomly picked destination nodes (phase 1) and
  - one with randomly picked source nodes (phase 2).

- Observe that Valiant’s trick follows the paradigm of randomized oblivious routing.
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- The same result can be shown for phase 2.

**Lemma**

*The congestion $C$ in phase 1 (phase 2) is $O(\log n / \log \log n)$, w.h.p.*
Proof of the lemma

- Let \( e \) be an edge of dimension \( i \), i.e., an edge that flips the \( i \)-th bit.
Proof of the lemma

- Let $e$ be an edge of dimension $i$, i.e., an edge that flips the $i$-th bit.
- Let $IN(e)$ be the set of nodes from which $e$ is reachable by a bit-fixing path. It holds $|IN(e)| = 2^{d-i-1}$. 
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- Fix any node in $IN(e)$. The path of the packet starting at $v$ contains $e$ if the packet’s intermediate destination is in $OUT(e)$.
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- Fix any node in $IN(e)$. The path of the packet starting at $v$ contains $e$ if the packet’s intermediate destination is in $OUT(e)$.
- As intermediate destinations are picked uniformly at random

$$\Pr[v's\ packet\ traverses\ e] = \frac{|OUT(e)|}{n} = \frac{2^i}{2^d} = 2^{i-d}.$$
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- For a subset $X \subseteq IN(e)$, let $A(X, e)$ denote the event that the paths of all packets starting from $X$ contain $e$.
- Let $C(e)$ be a random variable describing the congestion at edge $e$, i.e., $C(e)$ is the number of paths containing $e$.
- Let $k$ be any natural number.

\[
\Pr[C(e) \geq k] = \Pr[\exists X \subseteq IN(e), |X| = k : A(X, e)]
\]

(Union Bound)

\[
\leq \sum_{X \subseteq IN(e), |X| = k} \Pr[A(X, e)]
\]

\[
= \sum_{X \subseteq IN(e), |X| = k} \left(2^{i-d}\right)^k
\]

\[
= \binom{|IN(e)|}{k} \left(2^{i-d}\right)^k.
\]
Proof of lemma

- Binomial coefficients can be estimated by

\[ \left( \frac{a}{b} \right)^b \leq \binom{a}{b} \leq \left( \frac{e \cdot a}{b} \right)^b, \]

where \( e = 2.71 \ldots \) is the Eulerian number.
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  where \( e = 2.71 \ldots \) is the Eulerian number.

- This gives
  \[
  \Pr [C(e) \geq k] \leq \left( \frac{e|\ln(e)|}{k} \right)^k \left( 2^{i-d} \right)^k
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  = \left( \frac{e2^{d-i-1}}{k} \right)^k \left( 2^{i-d} \right)^k
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  = \left( \frac{e}{2k} \right)^k.
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Proof of lemma

- The congestion is defined to be $C = \max\{C(e) | e \in E\}$.

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\Pr[C \geq k] = \Pr[\exists e \in E : C(e) \geq k] \\
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The last bound follows from $|E| \leq dn \leq n^2$ and $\frac{e}{2k} \leq \frac{1}{2}$, where we assume $k \geq 3$. 
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- Now we choose $k$ such that $\Pr[C \geq k] \leq n^{-\alpha}$, for constant $\alpha > 0$.

- In particular, we set $k = \lceil(\alpha + 2)\log n\rceil \geq 3$ which gives

$$\Pr[C \geq k] \leq n^2 2^{-(\alpha+2)\log n} \leq n^2 n^{-(\alpha+2)} = n^{-\alpha},$$

which shows $C = O(\log n)$, w.h.p.
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- In order to show $C = O(\log n / \log \log n)$, w.h.p., we need to choose $k$ in a more clever way.
- We set

$$k = \max \left\{ \frac{e}{2} \sqrt{d}, 2(\alpha + 2) \frac{d}{\log d} \right\} = O \left( \frac{\log n}{\log \log n} \right)$$
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$$\Pr [C \geq k] \leq n^2 \left( \frac{e}{2k} \right)^k \leq n^2 \left( \frac{1}{\sqrt{d}} \right)^k \leq n^2 \left( \left( \frac{1}{\log d} \right)^{\frac{2}{2 \log d} (\alpha+2)d} \right) = n^2 \left( \frac{1}{2} \right)^{(\alpha+2)d} = n^2 \cdot n^{-(\alpha+2)} = n^{-\alpha}.$$
Definition (h-to-h-routing problem)

An \( h \)-relation is a routing problem in which every node is the source of \( h \) packets and the destination of \( h \) packets.

- Observe that a "1-relation" is a "permutation routing problem".
**Congestion of $h$-relations**

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**Lemma**

*Suppose we use Valiant’s trick for routing an arbitrary $h$-relation on the hypercube. The congestion $C$ is $O(\log n + h)$, w.h.p.*

Proof: Exercise
Scheduling on the hypercube

We study the problem of forwarding packets along prespecified paths on the $d$-dimensional hypercube.

**Theorem**

- *Suppose we are given a set of packets each of which coming with a bit-fixing path along which it should be sent from its source to its destination.*
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There is a distributed, randomized scheduling protocol that delivers all packets in time $O(C + \log n)$, w.h.p.
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There is a distributed, randomized scheduling protocol that delivers all packets in time $O(C + \log n)$, w.h.p.

Combining this result with Valiant's trick gives:

**Corollary**

There is a distributed algorithm that routes any $h$-relation in time $O(h + \log n)$, w.h.p., on the hypercube.
Randomized scheduling policy

The random rank protocol:

- Let $R$ denote a sufficiently large integer whose value will be specified later.
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- Besides every packet is assumed to have a unique integer ident number (id).
- If two or more packets contend for the same edge in a step, then the one with smallest rank is forwarded and the others have to wait.
- In case of equal ranks, packets with smaller ids are preferred.
Delay sequence analysis

Our analysis uses the following witness structure.

**Definition (delay sequence)**

A **delay sequence** $DS$ of length $s$ consists of

1. a **delay path** $P = (e(1), \ldots, e(L)), 1 \leq L \leq d$, with edges of increasing dimension (like a bit-fixing path in reverse order)
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4. $s + 1$ numbers $k_0, k_1, \ldots, k_s \in [R]$ with $k_0 \geq k_1 \geq \cdots \geq k_s$. 

Definition (active delay sequence)
$DS$ is called active if $r(p_i) = k_i$, for $0 \leq i \leq s$. 

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Lemma

If the random rank protocol needs $T > d$ steps, then there exists an active DS of length at least $T - d$.

Proof:

- Consider any packet packet arriving at its destination in step $T$. As $T > d$, this packet must have been delayed for at least one step. We call this packet $p_0$. 
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- Next we follow packet $p_2$ and so on until we reach a packet $p_s$, $s \geq 1$, that was not delayed before. We follow this packet back to its source.
Lemma

If the random rank protocol needs \( T > d \) steps, then there exists an active DS of length at least \( T - d \).

Proof:

- Consider any packet packet arriving at its destination in step \( T \). As \( T > d \), this packet must have been delayed for at least one step. We call this packet \( p_0 \).
- We follow the path of \( p_0 \) backwards from its destination until we reach an edge where it has been delayed by a packet that we call \( p_1 \).
- Now we follow \( p_1 \) backwards through time until we reach a time step where this packet has been delayed before by another packet that we call \( p_2 \) (possibly at the same edge).
- Next we follow packet \( p_2 \) and so on until we reach a packet \( p_s \), \( s \geq 1 \), that was not delayed before. We follow this packet back to its source.
- Our tour backward through time covers \( T \) steps and we observed \( s \) delays. Let \( L \) denote the number of edges on the recorded path.
Delay sequence analysis

- From this tour backwards through time, we can now construct an active DS as follows.
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5. For \( 0 \leq i \leq s \), we set \( k_i = r(p_i) \). Observe that this gives \( k_0 \geq k_1 \geq \cdots \geq k_s \) as packet \( p_{i-1} \) is delayed by packet \( p_i \) and the protocol prefers packets with smaller rank.
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This ends the proof of the lemma.
Delay sequence analysis

Now we bound the probability that there exists an active DS. Our analysis begins with counting delay sequences.

**Lemma**

The number of delay sequences of length $s$ is at most

$$n^2 \cdot \binom{L - 1 + s}{s} \cdot C^{s+1} \cdot \binom{R + s}{s + 1}.$$
Delay sequence analysis

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Proof:

1) Counting delay paths:
   The number of ways to choose a delay path is $n(n - 1) \leq n^2$ as this path corresponds to a bit-fixing path (in reverse order) that is determined by specifying the first and the last node on the path.
Delay sequence analysis

The number of delay sequences of length \( s \) is at most \( n^2 \cdot \binom{L-1+s}{s} \cdot C^{s+1} \cdot \binom{R+s}{s+1} \).

2) Counting the ways to choose the \( l_i \)'s and the \( k_i \)'s:
How many ways are there to choose the integers \( l_1, \ldots, l_s \) such that \( 1 \leq l_1 \leq l_2 \leq \cdots \leq l_s \leq d \)?
Delay sequence analysis

The number of delay sequences of length $s$ is at most $n^2 \cdot \binom{L}{s} \cdot C^{s+1} \cdot \binom{R}{s+1}$.

2) Counting the ways to choose the $\ell_i$'s and the $k'_i$'s:
How many ways are there to choose the integers $\ell_1, \ldots, \ell_s$ such that $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_s \leq d$?

- These integers can be encoded into a binary string as follows

$$0^{\ell_1-1} 10^{\ell_2-\ell_1} 10^{\ell_3-\ell_2} 1 \cdots 10^{\ell_s-\ell_{s-1}} 10^{d-\ell_s}.$$
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   These integers can be encoded into a binary string as follows
   
   $0^{\ell_1-1}10^{\ell_2-\ell_1}10^{\ell_3-\ell_2}1\ldots10^{\ell_s-\ell_{s-1}}10^{d-\ell_s}$.

   Observe that this string contains $s$ ones and the number of zeros in this string is
   
   $\ell_1 - 1 + \left( \sum_{i=2}^{s} (\ell_i - \ell_{i-1}) \right) + d - \ell_s = d - 1$. 
Delay sequence analysis

The number of delay sequences of length $s$ is at most $n^2 \cdot \binom{L-1+s}{s} \cdot C^{s+1} \cdot \binom{R+s}{s+1}$.

- Consequently, there is a one-to-one mapping between the $\ell_i$'s and the binary strings with $d-1$ zeros and $s$ ones. Hence, the number of ways to choose the $\ell_i$'s corresponds to the number of such strings which is

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Analogously the number of ways to choose $k_0, \ldots, k_s \in [R]$ such that $k_0 \geq k_1 \geq \cdots \geq k_s$ is equal to the number of binary strings consisting of $R-1$ zeroes and $s+1$ ones, which is

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The number of delay sequences of length $s$ is at most $n^2 \cdot \binom{L-1+s}{s} \cdot C^{s+1} \cdot \binom{R+s}{s+1}$.

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Now suppose that the delay path $P$ and the $\ell_i$'s are fixed.
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The number of delay sequences of length \( s \) is at most \( n^2 \cdot \left( \frac{L-1+s}{s} \right) \cdot C^{s+1} \cdot \left( \frac{R+s}{s+1} \right) \).

3) Counting the ways to choose delay packets:
Now suppose that the delay path \( P \) and the \( \ell_i \)'s are fixed.

Then, for each delay packet, we know an edge that is contained in its path: In particular, we know that packet \( p_i \), for \( 1 \leq i \leq s \), uses edge \( e(\ell_i) \) and packet \( p_0 \) uses edge \( e(\ell_1) \).
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How may possibilities are there to choose a packet whose path is leading through a known edge? – At most $C$ since each edge is contained in the paths of at most $C$ packets.
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How many possibilities are there to choose a packet whose path is leading through a known edge? – At most $C$ since each edge is contained in the paths of at most $C$ packets.

Hence, there are at most $C$ possibilities to choose $p_i$ and, hence, at most $C^{s+1}$ possibilities to choose all delay packets $p_0, \ldots, p_s$. 
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This ends the proof of the lemma.
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Lemma

The probability that a given DS of length $s$ is active is $R^{-(s+1)}$.

Proof:

- For every delay packet $p_i$, the probability that the packet’s rank is $k_i$ is $1/R$ because ranks are chosen uniformly at random from $[R]$. 
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- Thus, the probability that all $s+1$ delay packets have the prescribed rank is $R^{-(s+1)}$ because the ranks of different packets are chosen independently.
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- By the first Lemma, if the algorithm needs \( T \geq d + s \) steps, then there exists an active delay sequence of length at least \( s \).
Delay sequence analysis

- By the first Lemma, if the algorithm needs $T \geq d + s$ steps, then there exists an active delay sequence of length at least $s$.

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Delay sequence analysis

- By the first Lemma, if the algorithm needs $T \geq d + s$ steps, then there exists an active delay sequence of length at least $s$.
- Cutting this sequence after packet $p_s$ gives an active delay sequence of length exactly $s$.
- Let $\mathcal{DS}(s)$ denote the set of delay sequences of length $s$. It holds

\[
\Pr[T \geq d + s] \leq \Pr[\exists DS \in \mathcal{DS}(s) : DS \text{ is active}] \\
\leq \sum_{DS \in \mathcal{DS}(s)} \Pr[DS \text{ is active}] \\
(\text{third Lemma}) \Rightarrow \sum_{DS \in \mathcal{DS}(s)} R^{-(s+1)} \\
(\text{second Lemma}) \leq n^2 \cdot \binom{d - 1 + s}{s} \cdot C^{s+1} \cdot \binom{R + s}{s + 1} \cdot R^{-(s+1)}.
\]
Delay sequence analysis

We have so far:

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\Pr[T \geq d + s] \leq n^2 \cdot \binom{d - 1 + s}{s} \cdot C^{s+1} \cdot \binom{R + s}{s + 1} \cdot R^{-(s+1)}
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\[ \Pr [T \geq d + s] \leq n^2 \cdot \left( \frac{d - 1 + s}{s} \right) \cdot C^{s+1} \cdot \left( \frac{R + s}{s + 1} \right) \cdot R^{-(s+1)} \]

- Using \( \binom{a}{b} \leq 2^a \) and \( \binom{a}{b} \leq \left( \frac{ea}{b} \right)^b \) to upper-bound the binomial coefficients, we obtain

\[ \Pr [T \geq d + s] \leq n^2 \cdot 2^{d-1+s} \cdot C^{s+1} \cdot \left( \frac{e(R + s)}{s + 1} \right)^{s+1} \cdot R^{-(s+1)} \]

\[ \leq n^3 \cdot \left( \frac{2eC(R + s)}{(s + 1)R} \right)^{s+1} \]
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  \[
  \leq n^3 \cdot \left(\frac{2eC(R + s)}{(s + 1)R}\right)^{s+1}.
  \]

- Choosing \(R \geq s\) yields \(R + s \leq 2R\) and, hence,
  \[
  \Pr[T \geq d + s] \leq n^3 \cdot \left(\frac{4eC}{s + 1}\right)^{s+1}.
  \]
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- Hence, with probability at least \( 1 - n^{-\alpha} \), the random rank protocol needs at most \( d + s - 1 = O(C + \log n) \) steps.
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- Hence, with probability at least \( 1 - n^{-\alpha} \), the random rank protocol needs at most \( d + s - 1 = O(C + \log n) \) steps.
- This ends the proof of the theorem.
Consider any $n$-node network $G = (V, E)$. We study the problem of forwarding packets along arbitrary shortest paths in $G$.

**Theorem**

Suppose we are given a set of $N \geq n$ packets each of which coming with a shortest path in $G$ along which it should be sent from its source to its destination. Let $C$ and $D$ denote the congestion and the dilation of the paths, respectively. There is a distributed, randomized scheduling protocol that delivers all packets in time $O(C + D + \log N)$, w.h.p.

---

4Recall that edges are assumed to be directed. In order to represent an undirected network, one replaces each edge by two directed edges in opposite direction.
Randomized scheduling policy with increasing ranks

The growing rank protocol:

- Let $R$ denote a sufficiently large integer being a multiple of $D$. 
Randomized scheduling policy with increasing ranks

The growing rank protocol:

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Randomized scheduling policy with increasing ranks

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- Let $R$ denote a sufficiently large integer being a multiple of $D$.
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- Whenever the packet moves along an edge, its rank is increased by the value $R/D$. 
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- If two or more packets contend for the same edge in a step, then the one with smallest rank is forwarded and the others have to wait.
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- Besides every packet is assumed to have a unique integer ident number (id).
- If two or more packets contend for the same edge in a step, then the one with smallest rank is forwarded and the others have to wait.
- In case of equal ranks, packets with smaller ids are preferred.
Randomized scheduling policy

**Observation**

As the initial rank is at most $R - 1$ and the rank of a packet is increased at most $D$ times by $R/D$, the final rank of a packet is at most $2R - 1$.

Let $r_e(p) \in [2R]$ denote the rank of packet $p$ in those time steps in which $p$ contends for being forwarded along edge $e$. 
We adapt the definition of a delay sequence as follows.

**Definition (delay sequence)**

A delay sequence $DS$ of length $s$ consists of

1. a delay path $P = (e(1), \ldots, e(L))$, for $L \leq 2D$, with edges in reverse direction, that is, $(e(L), \ldots, e(1))$ is a path in $G$;
2. $s$ numbers $\ell_1, \ldots, \ell_s \in \{1, \ldots, L\}$ with $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_s$;
3. $s + 1$ distinct delay packets $p_0, p_1, \ldots, p_s$ such that, for $1 \leq i \leq s$, edge $e(\ell_i)$ is contained in the paths of packet $p_{i-1}$ and packet $p_i$;
4. $s + 1$ numbers $k_0, k_1, \ldots, k_s \in [2R]$ with $k_0 \geq k_1 \geq \cdots \geq k_s$.

**Definition (active delay sequence)**

$DS$ is active if $r_{e(\ell_i)}(p_i) = k_i$, for $1 \leq i \leq s$, and $r_{e(\ell_1)}(p_0) = k_0$. 
Delay sequence analysis

Lemma

If the growing rank protocol needs \( T \geq 2D \) steps, then there exists an active DS of length at least \( T - 2D \).

Proof:
Consider any packet packet arriving at its destination in step \( T \). As \( T \geq 2D \), this packet must have been delayed for at least one step. We call this packet \( p_0 \). We follow the path of \( p_0 \) backwards through time from its destination until we reach an edge where it has been delayed by a packet that we call \( p_1 \). Now we follow \( p_1 \) backwards through time until we reach a time step where this packet has been delayed before by another packet that we call \( p_2 \), and so on ... ... until we reach a packet \( p_s \), for some \( s \geq 1 \), that was not delayed before. We follow this packet back to its source.
Delay sequence analysis

From this tour backwards through time, we can now construct an active $DS$ as follows.

- The path that we have recorded by this process in reverse order gives us the delay path $P = (e(1), \ldots, e(L))$.

\footnote{This is the only part of the analysis where we need to assume that the paths of the packets are shortest paths in $G$.}
Delay sequence analysis

From this tour backwards through time, we can now construct an active DS as follows.

- The path that we have recorded by this process in reverse order gives us the delay path \( P = (e(1), \ldots, e(L)) \).
- The packets \( p_0, \ldots, p_s \) are defined to be the delay packets.

\[\text{Exercise: Show that the packets } p_0, \ldots, p_s \text{ are distinct, that is, no packet appears more than once in the delay sequence.}\]

\(^5\text{This is the only part of the analysis where we need to assume that the paths of the packets are shortest paths in } G.\]
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- For $1 \leq i \leq s$, we set $k_i = r_{e(\ell_i)}(p_i)$, and $k_0 = r_{e(\ell_1)}(p_0)$.

\[5\]This is the only part of the analysis where we need to assume that the paths of the packets are shortest paths in $G$. 
Delay sequence analysis

From this tour backwards through time, we can now construct an active DS as follows.

- The path that we have recorded by this process in reverse order gives us the delay path $P = (e(1), \ldots, e(L))$.
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**Exercise:** Show that the packets $p_0, \ldots, p_s$ are distinct, that is, no packet appears more than once in the delay sequence.\(^5\)

\(^5\)This is the only part of the analysis where we need to assume that the paths of the packets are shortest paths in $G$.\)
Delay sequence analysis

Observe that $k_0 \geq k_1 \geq \cdots \geq k_s$ as the ranks of the delay packets do not increase on our tour. More specifically:

- whenever we switch from packet $p_i$ to packet $p_{i+1}$ on our tour, the rank of $p_{i+1}$ is not larger than the rank of packet $p_i$ because $p_{i+1}$ delays $p_i$ and the protocol prefers packets with smaller rank, and
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- whenever we add an edge to the delay path and follow this edge, the rank of the currently observed packet is decreased (by $R/d$) as we follow the packet backwards in time.
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It only remains to prove $L \leq 2D$ and $s \geq T - 2D$.

- The final rank of $p_0$ is at most $2R - 1$. 
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- As ranks are non-negative, we obtain $(2R - 1) - L(R/D) \geq 0$ which gives $L \leq (2R - 1)/(R/D) \leq 2D$. 

Finally, $T = L + s$ implies $s = T - L \geq T - 2D$. This ends the proof of the lemma.
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Now we bound the probability that there exists an active DS. Our analysis begins with counting delay sequences.

**Lemma**

*The number of delay sequences of length s is at most*

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\binom{2D - 1 + s}{s} \binom{2R + s}{s + 1} N C^s.
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**Proof:**

Analogously to the analysis for the hypercube the number of ways to choose the $\ell_i$’s and the $k_i'$ can be bounded by

$$\binom{2D - 1 + s}{s} \binom{2R + s}{s + 1}.$$
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Now we assume that the $\ell_i$’s are fixed and we count the number of ways to choose the delay packets and the edges on the delay path:

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- This ends the proof of the lemma.
Delay sequence analysis

Lemma

*The probability that a DS of length $s$ is active is at most $R^{-(s+1)}$.***

Proof:

- Suppose $e(\ell_i)$ is the $j$th edge on the path of packet $p_i$. 
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- Suppose \( e(\ell_i) \) is the \( j \)th edge on the path of packet \( p_i \).

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- Consequently, the probability that all $s + 1$ delay packets have the prescribed rank is at most $R^{-(s+1)}$.
- This ends the proof of the lemma.
Delay sequence analysis

Now we proceed analogously to the analysis for the hypercube.

\[ \Pr [ T \geq 2D + s ] \leq \Pr [ \exists DS \in DS(s) : DS \text{ is active}] \]
\[ \leq \sum_{DS \in DS(s)} \Pr [ DS \text{ is active}] \]
\[ \leq \left( \frac{2D - 1 + s}{s} \right) \left( \frac{2R + s}{s + 1} \right) N C^{s} R^{-(s+1)} \]
\[ \leq 2^{2D-1+s} \left( \frac{e(2R + s)}{s + 1} \right)^{s+1} N C^{s} R^{-(s+1)} \]
\[ \leq 2^{2D} \left( \frac{6Ce}{s + 1} \right)^{s+1} N , \]

where the last equation assumes \( R \geq s \).
Delay sequence analysis

Finally, we set $s = \lceil \max\{12eC, (\alpha + 1) \log N + 2D\} \rceil - 1 = O(C + D + \log N)$. This gives

$$\Pr[T \geq 2D + s] \leq 2^{2D} N \left(\frac{1}{2}\right)^{s+1} \leq 2^{2D} N \left(\frac{1}{2}\right)^{(\alpha+1) \log N + 2D} \leq N^{-\alpha} \leq n^{-\alpha}$$

using $n \leq N$.

Hence, with probability at least $1 - n^{-\alpha}$, the growing rank protocol needs at most $2D + s - 1 = O(C + D + \log N)$ steps.
Literature