Exercise 1

Distributing Christmas presents was easy for Santa on the Cycle World. Every city with the name $v_i$ had only two outgoing roads to its nearest neighbors $v_{i+1}$ and $v_{i-1}$. The only strange thing was that the cities $v_{n-1}$ and $v_0$ were connected, although $n \neq 0$. But sometimes in life you have to accept, that $\mod n$ holds.

To make Santas job even more easy his elves wrote on every outgoing road the names of the cities which were reachable on the shortest way by this road. But because the elves were lazy, they wrote only two names on every street sign, the largest and smallest one. Thus, if Santa was in the city $v_i$ and he got a present, he just have to look at the two street names, which were $(v_i+1, v_i + \frac{n}{2} \mod n)$ and $(v_i + \frac{n}{2} \mod n, v_i-1)$. If the destination was part of the first interval, Santa knew that he had to go to the city $v_i+1$ and the other way around.

But Santas elves want to help Santa more. They construct shortcuts that go through the inner part of the cycle world, because no one knows what happens if you leave the outer barrier of the cycle world. To plan these routes, they used a piece of paper. Doing this, they avoided that the new shortcuts cross each other. Otherwise their planning would get too complicated.

But these new shortcuts can only be used if they create new labels for the roads and shortcuts.

(a) In great panic, the elves call you and ask if you can give them some algorithm that rewrites the signs on the old streets and the new shortcuts each time they add a new shortcut? They have Rename-o-Bots that can move from city to city via the streets, read street signs and rewrite them.

(b) One of the elves is very brave and wants to create a shortcut that leaves the border of the cycle world. You need to stop him! Talking about the risk and the danger is useless. Can you show him on the plan what such a shortcut would mean for the labels on the street? Maybe you can convince him by writing an explanation with a picture.

Exercise 2

The people of the isolated world are very unhappy. One half of the cities want to send presents to the other half. Their Santa is ready to send the packages if all the cities are connected by a grid. So, every city that wants to send a present has one distinct entrance to the present delivery grid. Every city that gets a present has one distinct exit from the present delivery grid on the other side of the grid. Santa makes a short calculation for the dimensions of the grid and realizes that one dimension of the grid should be $n$ if $n$ cities want to send one package at the same time to $n$ distinct cities on the other side of the grid.

Each of Santas elves, that carry the presents, start in one of the sending cities, walk equally fast and are not allowed to meet on the way, stop or walk backwards. Thus, in each time step every street and crossing in the present delivery grid can only be used by one elf. The elves know the direction of the streets in the grid, because each crossing is enumerated.
Santa asks you to design a simple route for every elf such that the unknown dimension of the grid is as small as possible.

A first suggestion was to build a $n \times n$ grid. If an elf gets a present in city $(\text{sender}, 0)$ that he should bring to city $(\text{receiver}, n)$, he goes $y$ steps downwards towards $(\text{sender}, y)$, then sideways until he reaches the crossing $(\text{receiver}, y)$. From there the elf moves to the destination of the package. Using this approach every elf makes a sideways movement in his own row on the grid and will never meet another elf.

Santa thinks that you can find a better approach to reduce the dimension of the grid.

**Exercise 3**

Celebrating Christmas is difficult in the De Bruijn world. Every city has the same name length of $d$ and can use only $k$ letters. Each city has $k$ outgoing and $k$ ingoing streets. A city with the name $ab$, where $|a| = 1$ and $|b| = d - 1$ has a street to every city with the name $bx$ where $x$ is one of the $k$ possible letters.

The people of the De Bruijn world believe that there exists only one Santa. The $k - 1$ Santas try to hold up this believe. Luckily each time around Christmas there are two cities $i$ and $j$ in the De Bruijn world, that are empty. The $k - 1$ Santas meet in $i$, plan their routes through the world and meet again in $j$. It is not allowed that two Santas visit the same city. But it is not necessary that every city gets visited (maybe some cities were evil over the year).

This year, something horrible happened. No one brought a well defined route. Can you help the $k - 1$ Santas and give them a route depending on the city names?