Load Balancing for Dynamic Spectrum Assignment with Local Information for Secondary Users

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***Abstract*—In this paper we study an idealized model of load balancing for dynamic spectrum allocation (DSA) for secondary users using only local information. In our model, each agent is assigned to a channel and may reassign its load in a round based fashion. We present a randomized protocol in which the actions of the agents depend purely on some cost measure (e. g., latency, inverse of the throughput, etc.) of the currently chosen channel. Since agents act concurrently, the system is prone to oscillations. We show how this can be avoided guaranteeing convergence towards a state in which every agent sustains at most a certain threshold cost (if such a state exists). We show that the system converges quickly by giving bounds on the convergence time towards approximately balanced states. Our analysis in the fluid limit (where the number of agents approaches infinity) holds for a large class of cost functions. We support our theoretical analysis by simulations to determine the dependence on the number of agents. It turns out that the number of agents affects the convergence time only in a logarithmic fashion. The work shows under quite general assumptions that even an extremely large number of users using several hundreds of (virtual) channels can work in a DSA fashion.**

I. Introduction

The radio spectrum is a finite and especially under 3 GHz scarce resource that needs to be used efficiently. Cognitive radio (CR) technology is emerging as an opportunity to enable dynamic and real-time spectrum access. The cognitive radio paradigm was originally introduced by Mitola [1]. Recently, efforts have been made to model opportunistic spectrum access and solve the channel allocation in a cognitive radio environment using a game theoretical approach, see [2], [3], [4] as examples. We believe that there are, in fact, two separate channel allocation problems related to cognitive radio networks. The first and obvious one is the much studied problem for the secondary users to detect spectrum opportunities that are opened by the non-transmitting primary users. In this case, the secondary users need to find out the spectrum opportunities and decide how to use them. Second, there is a problem which of the available channels secondary users can select. If DSA techniques provide an increasing amount of spectrum for secondary users there is still need to use these resources as efficiently as possible. The distributed load balancing problem thus arises.

We have noticed the secondary spectrum allocation and spectrum smoothing problem as a part of our practical implementation project, in which we are with the semi-realistic deployment of cognitive radio networks. Mähönen and Petrova [5] have noted that both the first and second problem may be open to flash crowd effects in the context of CRs. Some researchers have doubted if the dynamic spectrum allocation for secondary users is possible if those are using only local information, and the number of users and channels becomes extremely large. In this paper, we study this problem in an abstract model.

In principle, we want to find out if it is theoretically possible to design an algorithm that uses only local information to achieve dynamic *balanced* spectrum allocation for secondary users. Here, we expect simply that the problem of detecting primary users and blocking those channels from the allocation game is solved by some CR-technology. We want to pose this question in theoretically general form without relying on specific transmission technology or network topology models. Such details are left for the future work, which needs to rely on simulations instead of an analytical approach chosen for this work. We use cost function based optimization. Each cognitive radio, called *agent* in the following theoretical discussion, has a threshold T_i such that the agent is satisfied as far as the cost of communications is below that. The cost might be for example interference, delay, inverse of bit-rate or weighted combination of those. In this paper we consider a simplified case of $T_i = c$ for all users, in order to have analytically tractable solutions. Hence, we have in general a resource allocation game that must be solved under the described constraints. We will show that there is indeed a possibility to approach a state at equilibrium quickly, and the criticism against secondary user allocation does not hold in all cases. In order to keep the work analytical and general enough, we will not choose a single specific cost function but we will work with *a class of cost functions*. We will later make a simplified simulation model, where linear cost functions are used, and where the cost of the communication is related to number of cognitive radios on the same channel[1].

A. Dynamic Load Balancing

Our scenario can be modeled as a so-called *balls and bins* problem [6], [7] where n balls representing the agents are

[1]This is loosely justified simplification due to the fact that under perfect scheduling and equal traffic models for every agent each cognitive radio would introduce similar amount of traffic. Moreover each radio will produce in this toy-model an increased interference level, naturally the linearity does not hold exactly in realistic cases.

assigned to m bins representing the resources, i.e., channels of the radio spectrum. In this work, we assume that the number of balls n approaches infinity. Each ball strives to reduce its sustained cost determined by the number of balls sharing the same bin to a threshold of T or below. The balls have only local information, i.e., only the value of the sustained cost is available. Furthermore, the balls are memoryless so that they cannot remember previous states or decisions. No communication between the balls is necessary. Considering a parallel and round-based model, all balls are allowed to migrate concurrently in each round. A very natural way to obtain an allocation at which each ball sustains cost at most T, is to follow Protocol 1, which is described later in this article. We have studied this problem in the discrete case earlier in [8].

Our protocol can be described in a simple way. Let T denote a common threshold such that an agent is satisfied if its sustained cost is at most T. Agents are activated from time to time and perform the following step if their sustained cost exceeds T. Let the cost of the currently chosen resource be denoted by ℓ and let $p = (\ell - T)/\ell$ denote the relative factor by which the threshold is exceeded. Then, with probability p the agent migrates from its currently used resource to a new resource chosen uniformly at random. This simple protocol is executed by all agents in parallel.

The protocol has a number of appealing properties. First of all it is extremely simple. In particular, the protocol relies purely on local information. Its execution requires merely the measurement of the current cost which can be performed at various layers of the protocol stack. Furthermore, it does not require any coordination among the individual agents or interaction with some centralized authority[2]. In particular, it can be executed concurrently such that, sublinear convergence time becomes possible. The protocol is also stateless, i. e., it does not rely on any information about the past and is hence self stabilizing. As long as the system does not become overloaded, joining and leaving agents can be handled smoothly.

B. Our Results and Outline

Our theoretic analysis of the load balancing process assumes the so-called fluid limit model, in which we have an infinite number of agents. We first show that for any threshold T, our algorithm converges towards a state where all agents sustain a cost of no more than T (if such a state exists, i. e., T is chosen to be large enough). In such a state, the system is at Nash equilibrium. In order to guarantee this property, the agents must not be too greedy. In particular, if agents migrate with large probability already, when the threshold is exceeded by only a small amount, this will not lead to a stable behavior, but rather to overshooting and oscillation effects. How aggressively the agents may behave depends on the steepness of the cost functions. We introduce a damping factor α by which our protocol is slowed down in order to avoid such effects.

[2]We only assume (weak) timing synchronization so that agents can execute algorithms at the same time. This time synchronization might be provided by the system already due to communication protocol needs.

In order to guarantee convergence, we show that it is sufficient to set α to the maximum elasticity of the cost functions. For example, if the cost functions are linear in the load, the elasticity is one and the protocol does not have to be slowed down at all.

Subsequently, we give bounds on the time of convergence. Due to the fact that we consider the fluid limit, the load on resources with cost above T will decrease exponentially fast, but it will never actually reach zero. Therefore, we consider the time of convergence in terms of approximate equilibria. More precisely, we define δ-Nash equilibria as states in which no agent sustains cost of more than $(1+\delta)T$. Here, we present the following two upper bounds on the time to reach approximate equilibria in this sense, stated in Theorems 8 and 9.

As a first result we show that the time to reach such an approximately balanced state is bounded linearly in the number of resources m and logarithmically in the approximation parameter δ (and, of course, in the damping factor α). If m is large, this does not give a suitable convergence time, and our second bound may be more applicable. This second bound is particularly useful if the capacities of the individual resources do not differ too much, as is typically the case in many applications. This bound is proportional to the ratio between maximum and minimum capacity and the approximation parameter δ. This shows that the convergence time is almost independent of the system size.

The above bounds rely on the assumption that the cost of empty resources is 0. If this is not the case, we cannot guarantee that the cost of a resource actually decreases, eventually reaching a value below $(1+\delta)T$, when agents retreat from this resource. Hence, we introduce a second concept of approximate equilibria, in which we also allow an ϵ-fraction of the agents to deviate by more than a factor of $(1 + \delta)$ from T. For general cost functions we show that the time to reach an equilibrium of this type is bounded linearly in m, $1/\delta$, and $1/\epsilon$.

Finally, we underline our theoretical results in the fluid limit by some experiments for the case of finitely many agents. Our simulations indicate that the number of agents in the system affects the time of convergence only in a logarithmic fashion.

C. Related Work

Protocols similar to the one we are considering here have been studied in different models and under various assumptions. All of the protocols mentioned below operate by using more knowledge than just the cost of the currently chosen resource. They present analyses for discrete models with a finite number of users, but with restricted classes of cost functions.

Even-Dar and Mansour [9] consider concurrent protocols in a setting with linear cost functions. Their protocols require global knowledge in the sense that the users must be able to determine the set of underloaded and overloaded links. Given this knowledge, the convergence time is doubly logarithmic in the number of players. The protocol presented in [10] does not require this knowledge. The authors consider a distributed protocol for the case that the cost equals the load that does not rely on this knowledge. The protocol operates by sampling a target resource and migrating with a probability depending on the projected gain. Their bounds on the convergence time are also doubly logarithmic in the number of players but polynomial in the number of links. Furthermore, the analysis assumes that all resources have identical linear cost functions. These results can be generalized to the case of weighted jobs [11]. In the latter case, the convergence time is only pseudopolynomial, i. e., polynomial in the number of users, links, and in the maximum weight.

Finally, Fotakis *et al.* [12] consider a scenario with cost functions for every resource. Their protocol involves local coordination among the players sharing a resource. For the family of games in which the number of players asymptotically equals the number of resources they prove fast convergence to almost Nash equilibria. Intuitively, an almost Nash equilibrium is a state in which there are not too many agents that deviate considerably from the current average.

Game theory has been used extensively to model dynamic spectrum access in the context of CR, see for example [13], [2], [3], [4], [14], [15], [16], [17] just to mention few papers. There are also two recent excellent treatments studying games related to CR spectrum allocation by Etik *et al.* [18] and Suris *et al.* [19]. In the different context the semi-centralized or distributed channel allocation has also been studied in the context of ISM-band WLAN-systems [20], [21], [22], [23], [24], [25]. Useful background can be also found from [26], [27], [28], [29], [30]. The dynamic load balancing itself has been studied naturally also widely in the larger context of communications engineering. As discussed above our approach, which is using only pure local information, is a novel and extremely simple approach which does not expect any explicit cooperation. As some of the complexity is required for the first step to solve primary vs. secondary user allocation it is to be expected that a very simple load balancing algorithm for the secondary user load balancing would be useful. This has been our primary aim. Naturally our approach could also have applications beyond dynamic spectrum allocation.

II. Model

A. System Model

We consider a set of *resources* denoted by $[m]$. Each resource $i \in [m]$ is associated with a *cost function* $\ell_i : [0,1] \to \mathbb{R}_{\geq 0}$. In this work, we consider the *fluid limit* where the number of agents tends to infinity. The fraction of agents assigned to resource $i \in [m]$ is denoted by x_i, so $\sum_{i\in[m]} x_i = 1$. A *state* is a vector $(x_i)_{i\in[m]}$. The induced cost sustained by agents on link i is $\ell_i(x_i)$. When x is clear from the context we may also omit the argument writing ℓ_i for $\ell_i(x_i)$.

We consider cost functions that satisfy two properties: First, the cost of an unused resource is zero, and, second, the cost functions are not too steep in the sense that they have bounded *elasticity*. The elasticity of a function f at x is defined as $f'(x)\,x/f(x)$.

Definition 1: A function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ is *d-elasticity bounded* iff for all $x \in \mathbb{R}$ it holds that $f'(x)\,x/f(x) \leq d$. It has *zero offset* if $f(0) = 0$.

Throughout this paper, we assume that for all $i \in [m]$, ℓ_i is d-elasticity bounded. The zero offset property is necessary only for the first part of Theorem 8.

Let us illustrate these requirements by considering throughput as a utility measure and the reciprocal of the throughput as a cost measure. If the throughput is roughly proportional to $1/x_i$ and a constant depending on the SNR where x_i is the load of resource i, then the cost functions are roughly linear. The elasticity of a linear function is 1 and we have $d = 1$. For positive polynomial cost functions of degree p we have $d = p$.

For cost functions with bounded elasticity, the following fact holds:

Fact 1 (see, e. g., [31]): For all $\ell : [0,1] \to \mathbb{R}$ with elasticity at most d it holds that $\ell(y(1+\delta)) \leq (1 + 2\,d\,\delta)\ell(y)$ for $0 \leq \delta \leq \frac{1}{2\,d}$.

Furthermore we assume the existence of the threshold T with the property that an agent is satisfied if its cost is at most T. A state is then at a *Nash equilibrium* if $\ell_i(x_i) \leq T$ for all $i \in [m]$. We define the *capacity* of a resource $i \in [m]$ as the maximum load such that the cost is at most T, formally $S_i = \sup\{x \mid \ell_i(x) \leq T\}$. This can be seen as a threshold in terms of load rather than cost.

In the fluid limit, our protocol will not reach such a state since the probability to leave a resource decreases as its cost approaches T. Hence, in order to describe the convergence speed of our protocol, we will bound the time to reach an approximate equilibrium. A natural relaxation of Nash equilibria is the following:

Definition 2 (δ-Nash equilibrium): A state is at a *δ-Nash equilibrium* iff for all $i \in [m]$ it holds that $\ell_i(x_i) \leq (1+\delta)\,T$.

B. The Protocol

We can now introduce our protocol formally. The protocol is executed by all agents in parallel and is described as pseudocode in Algorithm 1.

Algorithm 1 The THRESHOLD-Protocol

```
loop
  for all agents in parallel do
    Let i denote the resource to which the agent is assigned.
    if ℓ_i(x) > T then
      With probability (ℓ_i(x) − T)/(α·ℓ_i(x)) do
      1. draw j ∈ [m] uniformly at random
      2. i ← j.
    end if
  end for
end loop
```

When all agents follow our protocol on an allocation x concurrently, we obtain a new allocation x'. This process is iterated indefinitely. We obtain a sequence of allocations denoted $x(0), x(1), \ldots$. Given a state x, the fraction of agents which leave bin i is denoted by $r_i(x) = \max\{0, x_i \cdot (\ell_i(x) - T)/(\alpha\,\ell_i(x))\}$.

In the following, we assume that the parameter α is set to $\alpha = d$, where d is an upper bound on the elasticity of the cost functions.

III. Analysis of the Convergence Time

A. Convergence to Equilibria

We first show that the protocol converges to a state where all agents sustain a cost at most T in the long run. To that end, we introduce a potential function which is zero only at the Nash equilibria and show that the potential decreases towards zero. We define the potential of a resource as the excess load

$$\phi_i(x) = \max\{x_i - S_i, 0\} \quad \text{and} \quad \phi(x) = \sum_{i\in[m]} \phi_i(x) \ .$$

Clearly, $\phi_i(x) = 0$ if and only if x is at Nash equilibrium. Furthermore, in the following, we assume that T is chosen such that $\sum_{i\in[m]} S_i \le 1$, i.e., the system is not overloaded and a Nash equilibrium in our sense exists.

Theorem 2: If $\alpha = d$, then for any initial state $x(0)$, our protocol converges towards a Nash equilibrium.

To see this, we show that our protocol does not move too many agents in any round. In the following, we will omit the formal statement of $\alpha = d$ from the formulation of the theorems.

Lemma 3: Consider any state x. Then, for any $i \in [m]$ with $\ell_i \ge T$, it holds that $r_i(x) \le \phi_i(x)$. If furthermore $\ell_i(0) = 0$ and ℓ_i is convex, it holds that $r_i(x) \ge \phi_i(x)/d$.

Proof: If $\ell_i(x) \ge T$, we have

$$r_i = x_i \cdot \frac{\ell_i - T}{d\,\ell_i} \ .$$

For the lower bound, note that, since ℓ_i is convex and $\ell_i(0) = 0$, we have

$$\begin{aligned}
\ell_i(x_i - d\,r_i) &\le \ell_i(x_i) - d\,r_i \cdot \frac{\ell_i(x_i)}{x_i} \\
&= \ell_i(x_i) - (\ell_i(x_i) - T) \\
&= T \ .
\end{aligned}$$

Consequently, $S_i \ge x_i - d\,r_i$ implying $\phi_i(x) = x_i - S_i \le d\,r_i$, our claim.

For the upper bound, consider the tangent

$$h(y) = y \cdot \frac{\ell_i(x_i) d}{x} - \ell_i(x_i)(d-1) \ .$$

and note that $\ell_i(y) \ge h(y)$ for all $y < x_i$. To see this, assume that for some $y_0 < x_i$, $\ell_i(y_0) < h(y_0)$. Then, using the bounded elasticity of ℓ_i as well as the above statement,

$$\begin{aligned}
&\ell_i(x_i) \\
=\ &\ell_i(y_0) + \int_{y_0}^{x_i} \ell_i'(u)\,du \\
\le\ &\ell_i(y_0) + d\int_{y_0}^{x_i} \frac{\ell_i(u)}{u}\,du \\
\le\ &\ell_i(y_0) + d\int_{y_0}^{x_i} \frac{h(u)}{u}\,du \\
=\ &\ell_i(y_0) + d\int_{y_0}^{x_i} \frac{1}{u}\left(u\,\frac{\ell_i(x_i)\,d}{x_i} - \ell_i(x_i)\,(d-1)\right)du
\end{aligned}$$

Now, using $u \le x_i$ and the assumption that $\ell_i(y_0) < h(y_0)$,

$$\begin{aligned}
&\ell_i(x_i) \\
\le\ &\ell_i(y_0) + d\int_{y_0}^{x_i} \frac{\ell_i(x_i)\,d - \ell_i(x_i)\,(d-1)}{x_i}\,du \\
=\ &\ell_i(y_0) + (x_i - y_0)\cdot\frac{d\,\ell_i(x_i)}{x_i} \\
<\ &h(y_0) + (x_i - y_0)\cdot\frac{d\,\ell_i(x_i)}{x_i} \\
=\ &\ell_i(x_i) \ ,
\end{aligned}$$

a contradiction. From $\ell_i(y) \ge h(y)$ it follows that

$$\begin{aligned}
&\ell_i(x_i - r_i) \\
\ge\ &h(x_i - r_i) \\
=\ &(x_i - r_i)\cdot\frac{d\,\ell_i(x_i)}{x_i} - (d-1)\,\ell_i(x_i) \\
=\ &\left(x_i - x_i\cdot\frac{\ell_i(x_i) - T}{d\,\ell_i(x_i)}\right)\cdot\frac{d\,\ell_i(x_i)}{x_i} - (d-1)\,\ell_i(x_i) \\
=\ &T \ ,
\end{aligned}$$

implying $S_i \le x_i - r_i$, our claim. ■

We can now prove the convergence theorem.

Proof of Theorem 2: We show that for any $\epsilon > 0$, we have $\phi(x(t)) \le \epsilon$ for all $t \ge t_0$ and some finite t_0.

Consider some state x not at Nash equilibrium and let x' denote the state generated from x by applying one round of our protocol. Let $r_{i,j}$ denote the fraction of agents moving from resource i to resource j. By Lemma 3, we know that for all $i \in [m]$, $\sum_{j\in m} r_{i,j} \le \phi_i(x)$. Hence, the entire fraction of migrating agents contributes to the potential in x. There exists at least one $j \in [m]$ with $x_j < S_j$ and up to $S_j - x_j$ agents migrating to j do not contribute any more to ϕ in x'. Hence, $\Delta\phi(x) = \phi(x') - \phi(x) < 0$.

Note that $\phi(x)$ and $\Delta\phi(x)$ are continuous in x. Consider the set $X_\epsilon = \{x \mid \phi(x) \ge \epsilon\}$. As shown above, $\Delta\phi(x) < 0$ for all $x \in X_\epsilon$. Since X_ϵ is closed, $\delta_\epsilon = \sup\{\Delta\phi(x) \mid x \in X_\epsilon\} < 0$. Hence, as long as $x(t) \in X_\epsilon$, the potential reduces by $\delta_\epsilon < 0$ in every round and reaches a value of at most ϵ in finite time. ■

B. Convergence Time for Zero-Offset Cost Functions

We have seen that our protocol approaches a stable state in the long run, however our analysis does not state how quickly such a state is approached. In this section, we consider the time to reach a state at δ-Nash equilibrium for cost functions with zero offset. Our proof of the convergence time will proceed by showing that the potential reduces in every round by at least a factor. The following shows how we can obtain a bound in the convergence time based on this factor.

Fact 4: For any constant $a \in (0,1)$, consider a sequence $y(0), y(1), \ldots$ such that for all $t \in \mathbb{N}$ it holds that $y(t+1) \leq (1-a)y(t)$. Let τ denote the smallest t such that $y(t) \leq \gamma$, then

$$\tau = \left\lceil \frac{1}{a} \cdot \log\left(\frac{y(0)}{\gamma}\right)\right\rceil$$

Proof: Substituting the above expression into $y(t) \leq y(0)\cdot(1-a)^t$ yields $y(\tau) \leq \gamma$. ∎

We will derive two different upper bounds, one in terms of the number of agents, and one in terms of the ratio between minimum and maximum capacity of the resources. Due to its simplicity, we start with the first bound.

Lemma 5: Assume that $\ell_i(0) = 0$ and ℓ_i is convex for all $i \in [m]$. For all initial allocations $x(0)$ it takes time at most

$$\mathcal{O}\left(m\,d \log\left(\frac{d\,\phi(x(0))}{\delta\, S_{\min}}\right)\right)$$

rounds to reach a state at δ-Nash equilibrium.

Proof: For $i \in [m]$ let $\delta_i(x) = \max\{0, S_i - x_i\}$ denote the gap in resource i. Note that, since the system is not overloaded, $\sum_{i\in[m]} \delta_i(x) \geq \phi(x)$. In particular, there exists some resource $i^* \in [m]$ with $\delta_i(x) \geq \phi(x)/m$. By Lemma 3, the fraction of agents migrating in one round is at least $\phi(x)/d$, and at least a fraction of $\phi(x)/(m\,d)$ migrates to resource i^*, effectively reducing the potential. Hence $\phi(x(t+1)) \leq \phi(x)\cdot(1-\frac{1}{dm})$. Thus, by Lemma 4, the time to reach a state with potential at most $\delta \cdot S_{\min}/(2\,d)$ is bounded by

$$\mathcal{O}\left(m\,d \log\left(\frac{d\,\phi(x(0))}{\delta\, S_{\min}}\right)\right) \quad .$$

Such a state is at a δ-Nash equilibrium since for all $i \in [m]$,

$$\begin{aligned} \ell_i(x_i) &\leq \ell_i(S_i + \phi_i(x)) \\ &\leq \ell_i(S_i + \phi(x)) \\ &\leq \ell_i\left(S_i \cdot \left(1 + \frac{\delta}{2\,d}\right)\right) \\ &\leq \ell_i(S_i)\cdot(1+\delta) \\ &= T\cdot(1+\delta) \quad , \end{aligned}$$

where the fourth inequality is due to the elasticity bound on ℓ_i (Fact 1). ∎

The proof of the second bound is more involved. It is divided into two phases. In the first phase, we reach a state with potential at most $\frac{m\,\delta\,S_{\min}}{4\,d^2}$. Based on this bound, we reach a δ-Nash equilibrium at the end of the second phase.

Lemma 6 (Phase 1): Assume that $\ell_i(0) = 0$ and ℓ_i is convex for all $i \in [m]$. For any initial state $x(0)$, the number of rounds to reach a state with $\phi(x(t)) \leq \frac{m\,\delta\,S_{\min}}{4\,d^2}$ is bounded by

$$\frac{8d^3 S_{\max}}{\delta\, S_{\min}} \log\left(\frac{4\,d^2}{m\,\delta\,S_{\min}}\right) + 1 \ .$$

Proof: First, we define two sets. Let $K(x) = \left\{i \mid x_i \leq S_i - \frac{\phi(x)}{m}\right\}$ be the set of all resources which have enough space to accommodate the fraction of agents migrating to each resource. Furthermore, let $L(x) = \left\{i \mid S_i - \frac{\phi(x)}{m} < x_i \leq S_i\right\}$ be the set of all resources that can only accommodate a part of the fraction of agents migrating to each resource. The fraction of agents resource i can accommodate is denoted by $\delta_i(x) = \max\{0, S_i - x_i\}$. Since we assumed, that the system is not overloaded it follows that $\phi(x) \leq \sum_{i\in[m]} \delta_i(x)$. Since the set of underloaded resources is precisely $L(x) \cup K(x)$, it holds that $\sum_{i\in[m]} \delta_i(x) = \sum_{i\in L(x)} \delta_i(x) + \sum_{i \in K(x)} \delta_i(x)$. We consider the following two cases.

1. Case: $\sum_{i\in L(x)} \delta_i(x) > \sum_{i\in K(x)} \delta_i(x)$.
   It follows that $\sum_{i\in L(x)} \delta_i(x) > \frac{\phi(x)}{2}$. Due to Lemma 3 each resource in $L(x)$ gets at least a fraction of $\frac{\phi(x)}{md}$ agents. Thus, after one round all resources in $L(x)$ have accommodated at least $\frac{1}{d}\delta_i(x)$ and the potential difference is $\Delta\phi(x) \geq \sum_{i\in L(x)} \frac{1}{d}\delta_i(x) > \frac{\phi(x)}{2\,d}$.
2. Case: $\sum_{i\in L(x)} \delta_i(x) \leq \sum_{i\in K(x)} \delta_i(x)$.
   It follows that $\sum_{i\in K(x)} \delta_i(x) \geq \frac{\phi(x)}{2}$. In each round we have a potential difference $\Delta\phi(x) \geq |K(x)| \cdot \frac{\phi(x)}{md} \geq |K(x)| \cdot \frac{m\,\delta\,S_{\min}}{4md^3} = |K(x)| \cdot \frac{\delta S_{\min}}{4\,d^3}$. The minimal number of resources in $K(x)$ is $|K(x)| \geq \frac{1}{S_{\max}} \sum_{i\in K} \delta_i(x) \geq \frac{\phi(x)}{2S_{\max}}$. Hence, $\Delta\phi(x) \geq \phi(x)\frac{\delta\,S_{\min}}{8d^3 S_{\max}}$.

Thus, in each round t we the potential is reduced by $\Delta\phi(x(t)) \geq \phi(x(t))\frac{\delta\,S_{\min}}{8d^3S_{\max}}$ and by Lemma 4 we need at most $\frac{8d^3S_{\max}}{\delta\,S_{\min}} \log\left(\frac{4\,d^2}{m\,\delta\,S_{\min}}\right) + 1$ rounds to decrease the potential from 1 down to $\frac{m\,\delta\,S_{\min}}{4\,d^2}$. ∎

Lemma 7 (Phase 2): Assume that $\ell_i(0) = 0$ and ℓ_i is convex for all $i \in [m]$. For all allocations $x(t)$ with $\phi(x(t)) \leq \frac{m\,\delta\,S_{\min}}{4\,d^2}$ and for all $\delta \leq 1$ it takes at most $2\,d\log(\frac{m}{2\,d}) + 1$ rounds to reach a δ-Nash equilibrium.

Proof: We consider an arbitrary resource $i \in [m]$. Due to Lemma 3 at least a fraction of $\frac{1}{d}\phi_i(x(t))$ agents leave resource i and at most a fraction of $\frac{1}{m}\phi(x(t)) \leq \frac{1}{m}\frac{m\,\delta S_{\min}}{4\,d^2}$ agents arrive at resource i. Thus,

$$\begin{aligned} \phi_i(x(t+1)) &\leq \phi_i(x(t)) - \frac{1}{d}\phi_i(x(t)) + \frac{\delta\,S_{\min}}{4\,d^2} \\ &= \phi_i(x(t))\left(1 - \frac{1}{d}\right) + \frac{\delta\,S_{\min}}{4\,d^2} \ . \end{aligned}$$

We proceed by showing that for resources with low cost (below $(1+\delta)T$) the cost remains low, and for resources with high cost (above $(1+\delta)T$), the potential decreases in a monotone fashion. Let us fix a resource i, and we have two cases:

1. Case: $\phi_i(x(t)) \leq \frac{\delta\,S_{\min}}{2\,d}$. Due to Fact 1, resource i has

a cost of at most $T(1+\delta)$. For this case it holds that

$$\begin{aligned}\phi_i(x(t+1)) &\le \frac{\delta\,S_{\min}}{2\,d}\left(1-\frac{1}{d}\right)+\frac{\delta\,S_{\min}}{4\,d^2}\\ &= \frac{\delta\,S_{\min}}{2\,d}\left(1-\frac{1}{d}+\frac{1}{2\,d}\right)\\ &< \frac{\delta\,S_{\min}}{2\,d}\ .\end{aligned}$$

Hence, after one round the cost of resource i is still below $T(1+\delta)$.

2. Case: $\phi_i(x(t)) > \frac{\delta\,S_{\min}}{2\,d}$. The cost on resource i is greater than $T(1+\delta)$. In this case it follows that

$$\begin{aligned}\phi_i(x(t+1)) &\le \phi_i(x(t))\left(1-\frac{1}{d}\right)+\frac{1}{2\,d}\phi_i(x(t))\\ &= \phi_i(x(t))\left(1-\frac{1}{2\,d}\right)\ .\end{aligned}$$

By Lemma 4 we know that for an overloaded resource i, it takes at most

$$\tau \le 2\,d\log\left(\frac{\frac{m\,\delta\,S_{\min}}{4\,d^2}}{\frac{\delta\,S_{\min}}{2\,d}}\right)+1 = 2\,d\log\left(\frac{m}{2\,d}\right)+1$$

rounds to reduce the potential from at most $\frac{m\,\delta S_{\min}}{4\,d^2}$ to $\frac{\delta\,S_{\min}}{2\,d}$, so that resource i has a cost of at most $T(1+\delta)$.

This completes our proof. ∎

Combining Lemmas 5, 6 and 7 yields the following theorem:

Theorem 8: Assume that $\ell_i(0)=0$ and ℓ_i is convex for all $i\in[m]$. For any initial state, the time to reach a state at δ-Nash equilibrium is bounded from above by

$$\mathcal{O}\left(\frac{d^3\,S_{\max}}{\delta\,S_{\min}}\ \log\left(\frac{d}{m\,\delta\,S_{\min}}\right)\right)$$

and by

$$\mathcal{O}\left(m\,d\log\left(\frac{d\,\phi(x(0))}{\delta\,S_{\min}}\right)\right)\ .$$

C. Convergence Time for General Cost Functions

For general cost functions (i. e. with non-zero offset), the above result does not hold. In particular, if there exist resources with constant but large cost functions, we cannot hope to reach a δ-Nash equilibrium since there will always remain an (exponentially fast decreasing) minority of agents utilizing this resource. Taking this into account, we modify our definition of approximate equilibria allowing a small fraction of agents to deviate by more than a factor $(1+\delta)$ from T.

Definition 3 (δ-ϵ-Nash equilibrium): A state x is at a *δ-ϵ-Nash equilibrium* iff

$$\sum_{i:\ell_i>(1+\delta)T} x_i \le \epsilon\ .$$

With this definition of approximate equilibria, we can show fast convergence again.

Theorem 9: For any initial state $x(0)$, the time to reach a state at δ-ϵ-Nash equilibrium is bounded by

$$\mathcal{O}\left(\frac{d\,m}{\epsilon\,\delta}\right)\ .$$

Proof: Consider a state not at δ-ϵ-Nash equilibrium. We consider only the contribution to the potential gain of the resources $i\in[m]$ with $\ell_i(x_i)>(1+\delta)\,T$. The fraction of agents leaving these resources is at least

$$\begin{aligned}V &= \sum_{i:\ell_i>(1+\delta)\,T} x_i\cdot\frac{\ell_i-T}{d\,\ell_i}\\ &\ge \sum_{i:\ell_i>(1+\delta)\,T} x_i\cdot\frac{\delta}{d\,(1+\delta)}\\ &\ge \epsilon\cdot\frac{\delta}{d\,(1+\delta)}\ .\end{aligned}$$

Also note that clearly $\phi(x)\ge\epsilon$ and by the pigeon hole principle, there exists a resource $i^*\in[m]$ with $S_{i^*}-x_{i^*}>\epsilon/m$. Among the agents in V, a fraction of V/m migrates to resource i^*. Thus, in any round t, the potential reduces by $\delta\,\epsilon/(d\,m\,(1+\delta))$. This can happen at most $\Phi(x(0))\cdot d\,m\,(1+\delta)/(\delta\,\epsilon)$ times. ∎

IV. Simulations

In the previous section we have analyzed our protocol in the fluid limit. The dependence of the convergence time on the number of users is invisible in this model since the number of users is assumed to approach infinity. In this section, we present simulation results to show that this dependence is very mild and show that the analytical results can be transferred into a model with finitely many agents.

We have studied scenarios where channels have linear cost functions $\ell_i(x)=a_i\cdot x$ and where the a_i were drawn randomly. We have used two distributions from which the a_i were drawn. In the first case the a_i were drawn uniformly at random from the interval $[0,1]$. In the second case the a_i were drawn from a Pareto distribution $\mathrm{Par}(k,z_{\min})$ with $k=3$ and $z_{\min}=0.01$. We considered 13, 100 and 1 000 resources and varied the number of agents from $m\cdot10$ to $m\cdot500$. The mean over 1 000 repetitions of each simulated combination of resources and agents are shown in the following figures. The convergence time data were fitted to a function of the form $t(n)=c_1\cdot\log^{c_2}(n)$ using minimum square regression.

We have simulated scenarios with $m=13$, 100 and 1 000 channels. In this paper we present results for $m=13$ and $m=100$. In Figure 1 we show the convergence time for the system which has uniform pdf for the a_i. The parameters found for the convergence time function are $c_1^{13}\approx1.37312$ and $c_2^{13}\approx1.8165$. It can be observed that the convergence time is sub-linear in the number of agents, which shows that the threshold protocol is well applicable also for large-scale systems.

When using a Pareto distribution for the a_i parameter of the cost functions, the resulting values of the a_i have a smaller variance than in the case of the uniform distribution. This is

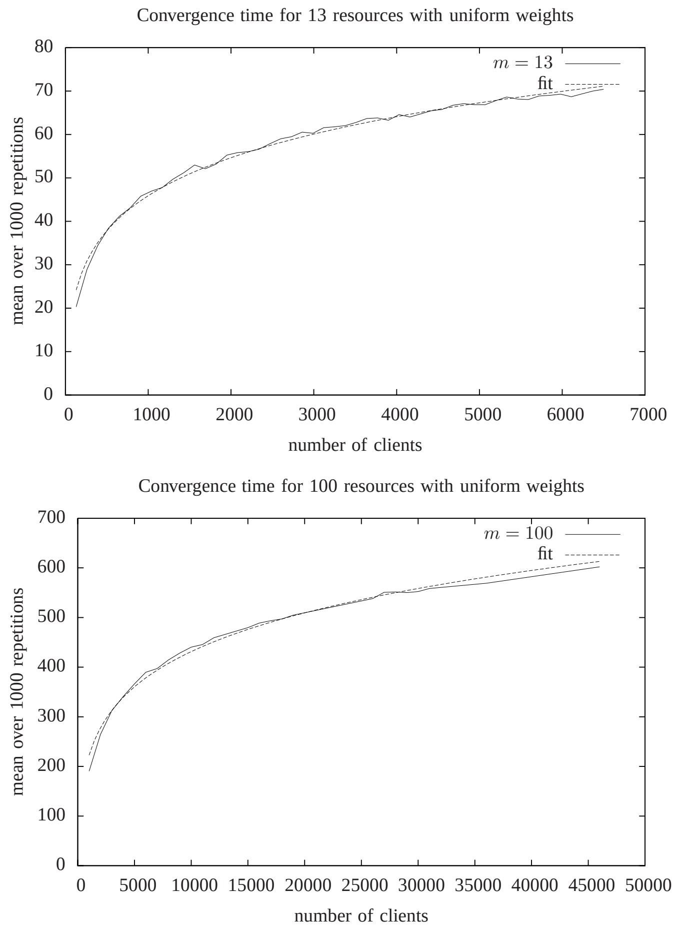

Fig. 1. Convergence time of the threshold protocol using uniformly distributed slopes for the cost functions. The upper panel is showing convergence time as a number of rounds for 13 channels. The lower panel shows the same for 100 channels.

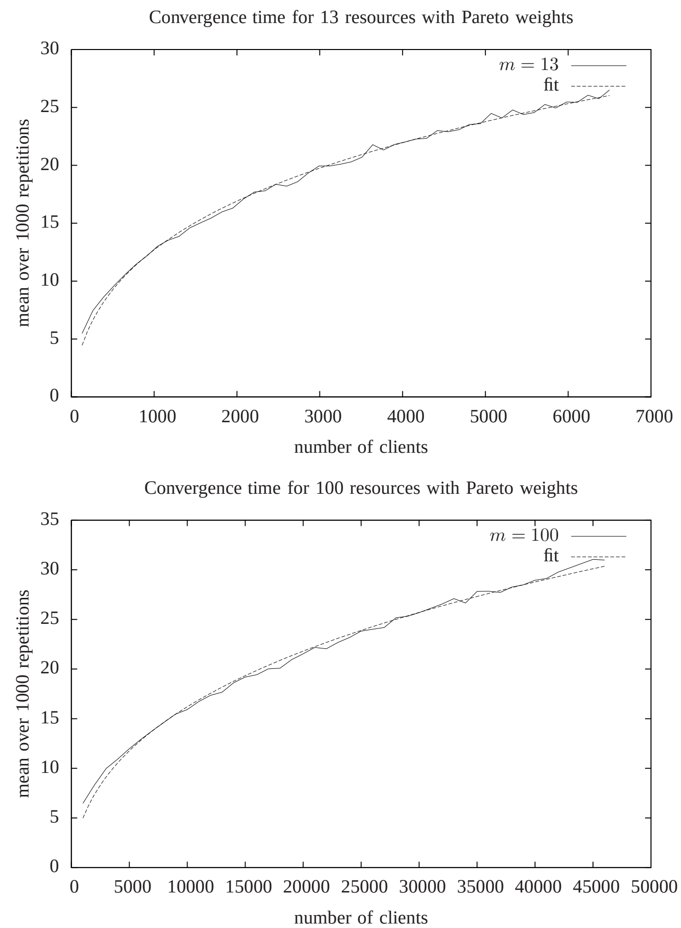

Fig. 2. Convergence time of the threshold protocol using Pareto-distributed slopes for the cost functions. The case for 13 and 100 channels is shown in the upper and lower panel, respectively.

often the case in the real world scenarios. Figure 3 shows the results for $m = 13$ and $m = 100$ channels. We observe a faster convergence time than in the case of uniformly drawn a_i. The simulations confirm that our fluid limit calculation is not limited to unrealistically high number of channels and users, but have also practical applicability. The used cost function is a simplification in order to show that the method is useful in general cases and it is not limited to some specific scenario. As the results are promising, we intend to make more extensive sets of simulations that will take into account true radio environment effects and topologies for this algorithm.

V. Summary and Further Directions

We have introduced a simple dynamic protocol for spectrum assignment with local information. Our analysis shows that the protocol converges quickly towards a state in which all agents sustain cost below a certain threshold parameter as long as such a state exists. Our analysis assumes the fluid limit model and therefore does not make a statement on the dependence of the number of users. Analysis in a discrete model indicate

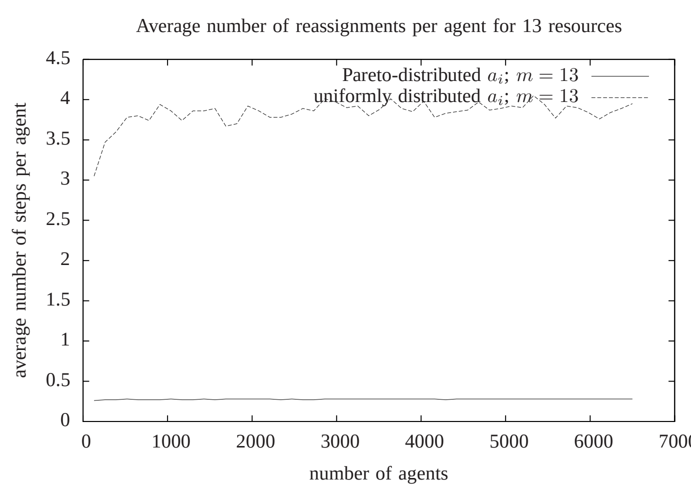

Fig. 3. Average number of reassignments per agent to reach a balanced state using 13 resources.

that the convergence time actually depends in a logarithmic fashion on the number of users [32].

Our model assumes that the cost sustained is equal for all users utilizing the same resource. We may also assume that the cost depends also on the user due to, e. g., fading and other parameters. In this case, the system can be described by a player-specific congestion game. The convergence analysis then becomes much more involved. Then, in general it is not even clear how many sequential steps are necessary to guarantee convergence towards a stable state.

An extension of our protocol would be to allow for the agents to have individual thresholds. In particular, agents should be able to find good values for T such that it is as small as possible while admitting a feasible solution. One way could be to adapt the threshold dynamically, increasing it if the system does not converge for some amount of time and decreasing it carefully otherwise.

The analytical solution and numerical simulation show that pure local information based stochastic allocation can work in principle. We are currently working towards the future work on implementing our algorithm to be part of our gnuRadio (USRP-boards) based test network. Further work is also needed to make detailed cost function models and provide simulation results using those in conjunction with wireless network simulators. We have done early feasibility tests with a simplified "radio model", where distribution of radios is random, the transmission power is fixed and the propagation loss exponent is 4 for all radios. The main contribution of this paper has been to make a position statement that the future CRs must also take in account the load balancing of secondary users to be true cognitive radios. We have also introduced the concept of seeing load balancing as a classical "balls and bins" type of congestion game. We have proved in the fluid limit that the congestion game can rapidly converge to feasible solutions (if those exists in the system at all), and that this can be reached by using simple algorithm that is based to only local information. These results pave way towards the future load balancing cognitive radio networks. Especially the need to find algorithms that do not need massive amount of information exchange between communicating nodes is important for the future CR technologies. One should note that our analysis formalism and algorithm is very general, and any reasonable cost function can be introduced into model without changing the fundamental properties. A change of a cost function is easy and through that one can analyze quickly different and more realistic radio cases quite readily. We hope that our results inspire more work on studying such algorithms.

ACKNOWLEDGMENT

The authors acknowledge the financial support of DFG through German UMIC-excellence cluster at the RWTH Aachen. One of us (PM) is also acknowledging a partial financial support from European Union through ARAGORN-project.

REFERENCES

[1] J. Mitola, *Cognitive Radio: An Integrated Agent Architecture for Software Defined Radio*. PhD Thesis, KTH, 2000.

[2] L. Berlemann and R. Huuhaa, "Spectrum load smoothing for cognitive medium access in open spectrum," *in Proc. IEEE PIMRC 2005*, vol. 3, pp. 1951–1956, 2005.

[3] A. Laufner and A. Leshem, "Distributed coordination of spectrum and the prisoner's dilemma," *in Proc. IEEE DySPAN*, vol. 1, pp. 94–100, 2005.

[4] S. H. Wong and I. J. Wassell, "Application of game theory for distributed dynamic channel allocation," *Proc. of IEEE Trans. on Vehicular Technology*, vol. 1, pp. 404–408, 2002.

[5] P. Mähönen and P. Petrova, "Flash crowds in cognitive radio environment: Beware an unpredictable mob," in *Proc. of 18th IEEE Annual International Symposium on Personal Indoor and Mobile Radio Communications (PIMRC 2007), Athens*, Sep. 2007.

[6] M. Raab and A. Steger, "Balls into bins — A simple and tight analysis," *Lecture Notes in Computer Science*, vol. 1518, p. 159, 1998.

[7] N. Johnson and S. Koty, *Urn Models and Their applications*. John Wiley and Sons, 1977.

[8] S. Fischer, M. Petrova, P. Mähönen, and B. Vöcking, "Distributed load balancing algorithm for adaptive channel allocation for Cognitive Radios," in *Proc. 2nd Conference on Cognitive Radio Oriented Wireless Networks and Communications (CrownCom)*, Orlando, FL, USA, Aug. 2007.

[9] E. Even-Dar and Y. Mansour, "Fast convergence of selfish rerouting," in *Proc. 16th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, 2005, pp. 772–781.

[10] P. Berenbrink, T. Friedetzky, L. A. Goldberg, P. Goldberg, Z. Hu, and R. Martin, "Distributed selfish load balancing," in *Proc. 17th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, 2006.

[11] P. Berenbrink, T. Friedetzky, I. Hajirasouliha, and Z. Hu, "Convergence to equilibria in distributed, selfish reallocation processes with weighted tasks," in *Proc. 15th Annual European Symposium on Algorithms (ESA)*, 2007, pp. 41–52.

[12] D. Fotakis, A. Kaporis, and P. Spirakis, "Atomic congestion games: Fast, myopic and concurrent," in *Proc. 1st Symposium on Algorithmic Game Theory (SAGT)*, 2008.

[13] S. Haykin, "Cognitive radio: brain-empowered wireless communications," *Selected Areas in Communications, IEEE Journal on*, vol. 23, no. 2, pp. 201–220, Feb. 2005.

[14] N. Nie and C. Comaniciu, "Adaptive channel allocation spectrum etiquette for cognitive radio networks," *Proc. IEEE DySPAN Conf., Baltimore, MD, USA*, pp. 269–278, Nob 2005.

[15] V. Srivastava, J. Neel, A. MacKenzie, J. Hicks, L. DaSilva, J. Reed, and G. Gilles, "Using game theory to analyze wireless ad hoc networks," *IEEE Communications Surveys and Tutorials*, Nob 2005.

[16] J. Hicks and A. MacKenzie, "A convergence result for potential games," in *11th Int. Symp. on Dynamic Games and Applications*, Dec 2004.

[17] M. Felegyhazi and J.-P. Hubaux, "Wireless operators in a shared spectrum," *INFOCOM 2006. 25th IEEE International Conference on Computer Communications. Proceedings*, pp. 1–11, April 2006.

[18] R. Etik, A. Parekh, and D. Tse, "Spectrum sharing for unlicensed bands," *IEEE JSAC*, vol. 25, pp. 517–527, 2007.

[19] J. Suris, L. DaSilva, Z. Han, and A. MacKenzie, "Cooperative game theory for distributed spectrum sharing," *Communications, 2007. ICC '07. IEEE International Conference on*, pp. 5282–5287, June 2007.

[20] J. Riihijärvi, M. Petrova, and P. Mähönen, "Frequency allocation for WLANs using graph colouring techniques," *Journal of Ad Hoc & Sensor Wireless Networks*, vol. 3, pp. 121–139, 2007.

[21] D. J. Leith and P. Clifford, "A self-managed distributed channel selection algorithm for wlans," *Proc. of RAWNET 2006, Boston, MA, USA*, 2006.

[22] A. Mishra, V. Shrivastava, D. Agarwal, S. Banerjee, and S. Ganguly, "Distributed channel management in uncoordinated wireless environments," *Proc. of Mobicom*, 2006.

[23] A. Mishra, V. Brik, S. Banerjee, A. Srinivasan, and W. Arbaugh, "A client-driven approach for channel management," *Proc. of INFOCOM 2006, Barcelona, Spain*, 2006.

[24] B. Leung and K. K. Kim, "Frequency assignment for IEEE 802.11 wireless networks," *Proc. of 58th IEEE Vehicular Technology Conference*, 2003.

[25] J. Riihijärvi, M. Petrova, and P. Mähönen, "Frequency allocation for WLANs using graph colouring technique," *Proc. of WONS 2005, St. Moritz, Switzerland*, January 2005.
[26] G. Cao and M. Singhal, "Distributed fault-tolerant channel allocation for mobile cellular network," *Proc. of IEEE INFOCOM 1999*, pp. 584–591, 1999.
[27] I. Katzela and M. Naghshineh, "Channel assignment schemes for cellular mobile telecommunication systems: A comprehensive survey," *IEEE Personal Comm.*, vol. 3, pp. 10–31, 1996.
[28] J. Jiang, T.-H. Lai, and N. Soundarajan, "On distributed dynamic channel allocation in mobile cellular networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 13, pp. 1024–1037, 2002.
[29] W. Hale, "Frequency assignment: theory and applications," *Proceedings of the IEEE*, vol. 68, pp. 1497–1514, 1980.
[30] K. I. Aardal, S. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino, and A. Sassano, "Models and solution techniques for frequency assignment problems," *4OR: Quarterly Journal of Operations Research*, vol. 1, pp. 261–317, 2003.
[31] S. Fischer, H. Räcke, and B. Vöcking, "Fast convergence to Wardrop equilibria by adaptive sampling methods," in *Proc. 38th Annual ACM Symposium on Theory of Computing (STOC)*. Seattle, WA, USA: ACM, May 2006, pp. 653–662.
[32] M. Schöngens, "Dynamic resource assignment for wireless devices," Master's thesis, RWTH Aachen, 2008.