Chapter 00: The friendship theorem

(Combinatorial Graph Theory, SS 2019)

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SS 2019, RWTH
Announcements

Lecture times:
- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
https://algo.rwth-aachen.de/Lehre/SS19/KG/KG.py
The friendship theorem
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Theorem (Erdős, Rényi, Sós, 1966)

Let $G = (V, E)$ be an undirected graph, in which every two (distinct) vertices have exactly one neighbor in common. Then there exists a vertex that is adjacent to all other vertices.

- $G$ contains no $C_4$
- If $u, v \in V$ not adjacent, then $\deg(u) = \deg(v)$
- If $u, v \in V$ adjacent, then $\deg(u) = \deg(v)$
- $G$ is $k$-regular and $n = |V| = k^2 - k + 1$
- $k \geq 3$
- Adjacency matrix $A$ of $G$ satisfies $A^2 = (k - 1)I + J$
- $A^2$ has Eigenvalues $k - 1$ (multiplicity $n - 1$) and $k^2$ (multiplicity 1)
- $A$ has Eigenvalues $\pm k$ (multiplicity 1), $+\sqrt{k - 1}$ (multiplicity $r$); $-\sqrt{k - 1}$ (multiplicity $s$), where $r + s = n - 1$
Second proof by Craig Huneke (2002)

- $G$ is $k$-regular and $n = |V| = k^2 - k + 1$
- For $v \in V$ let $f_v(\ell)$ denote the number of walks $v = v_0, v_1, v_2, \ldots, v_{\ell-2}, v_{\ell-1}, v_\ell = v$
- $f_v(\ell) = (k - 1) \cdot f_v(\ell - 2) + k^{\ell-2}$

For a prime divisor $p$ of $k - 1$,
- let $N$ denote the number of closed walks of length $p$ in $G$
- $N$ is a multiple of $p$
- $f_v(\ell) \equiv 1 \mod p$ and $N \equiv 1 \mod p$
A graph $G = (V, E)$ is an $\ell$-friendship graph, if every pair of vertices is connected by precisely one (simple) path of length $\ell$.

**Conjecture (Anton Kotzig, 1977)**

For $\ell \geq 3$, there are no $\ell$-friendship graphs.

This conjecture has been proved for $3 \leq \ell \leq 33$. 
Exercise

In a group of $n$ persons, every pair is either friends or enemies. The group satisfies the following properties:

- Every person in the group has exactly three enemies.
- For every person, an enemy of a friend is automatically an enemy.

Find all possible values for $n$. 