Chapter 06: Girth

(Combinatorial Graph Theory, SS 2019)

Gerhard Woeginger

SS 2019, RWTH
Announcements

Lecture times:

- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
https://algo.rwth-aachen.de/Lehre/SS19/KG/KG.py
Definition

The girth $g(G)$ of a graph $G$ is the length of the shortest cycle in $G$. 
Large min-degree and high girth (1)

**Theorem**

For $d \geq 3$ and $\gamma \geq 3$, there exists a graph with $n \leq (2d)^\gamma$ vertices, with min-degree $\delta(G) \geq d$ and girth $g(G) \geq \gamma$.

- Define $N = (2d)^\gamma$ and $M = \frac{1}{2}N(N-1)$
- Family $\mathcal{F}$ consists of all the graphs with vertex set $\{1, \ldots, N\}$ and with exactly $dN$ edges.
- Number of cycles of length $\ell \geq 3$ is: $\frac{1}{2}(\ell - 1)!\binom{N}{\ell} < (N - 1)^\ell$
- Number of graphs in $\mathcal{F}$ that contain some fixed cycle $C$ of length $\ell$ is: $\binom{M-\ell}{dN-\ell}$
- Average number $A$ of cycles of length at most $\gamma - 1$ in graphs in $\mathcal{F}$ is smaller than $N$
Average number $A$ of cycles of length at most $\gamma - 1$ in graphs in $\mathcal{F}$ is strictly smaller than $N$.

Hence there exists a graph $G \in \mathcal{F}$ that contains at most $N - 1$ cycles of length at most $\gamma - 1$.

In $G$ remove one edge from each cycle of length at most $\gamma - 1$. Resulting graph $G'$ has $N$ vertices, more than $(d - 1)N$ edges, and girth at least $\gamma$.

Remove vertices of degree $\leq d - 1$ from $G'$. Repeat.

Auxiliary lemma

Let $n$ and $c$ be positive integer. Every graph with $n$ vertices and at least $c \cdot n$ edges contains a subgraph with min-degree at least $c + 1$. 
Moore graphs

Definition

Let $d \geq 2$ and $k \geq 2$.

A $(d, 2k + 1)$-Moore graph is a graph of girth $2k + 1$, minimum degree $d$, and with $1 + d \sum_{i=0}^{k-1} (d - 1)^i$ vertices.

Sloppy definition: A Moore graph is a regular graph that has as few vertices as possible for its girth.
Hoffman-Singleton theorem (1)

Theorem (Alan Hoffman, Robert Singleton, 1960)

If a \((d, 5)\)-Moore graph exists, then \(d \in \{2, 3, 7, 57\}\).

- Moore graph is \(d\)-regular.
  - Two non-adjacent vertices have one common neighbor.
  - Two adjacent vertices have no common neighbor.
- \(A^2 + A = (d - 1)I + J\)
- Eigenvalues: \(d\) with multiplicity 1; and \(\frac{1}{2}(-1 + \sqrt{4d - 3})\) with multiplicity \(m_1\); and \(\frac{1}{2}(-1 - \sqrt{4d - 3})\) with multiplicity \(m_2\)
- Trace of \(A\) is 0, and hence \(d = 2\) or \(4d - 3 = s^2\)
- \(16(m_1 - m_2)s = s^4 - 2s^2 - 15\)
Hoffman-Singleton theorem (2)

- $d = 2$: cycle $C_5$ (unique)
- $d = 3$: Peterson graph (unique)
- $d = 7$: Hoffman-Singleton graph with $n = 50$ vertices (unique)
- $d = 57$: existence of such a graph with $n = 3250$ vertices is open
Hoffman-Singleton graph (1)
Hoffman-Singleton graph (2)
Recapitulation: Eigenvalues and Eigenvectors
A number $\lambda$ is an **Eigenvalue** of an $n \times n$ real matrix $A = (a_{i,j})$, if there exists an $n$-dimensional vector $v \neq 0$ that satisfies the equation

$$Av = \lambda v.$$ 

The vector $v$ is called an **Eigenvector** corresponding to $\lambda$.

- The Eigenvalues are the roots of the polynomial $\det(A - \lambda I)$. In other words, they satisfy the equation $\det(A - \lambda I) = 0$, and they make the matrix $A - \lambda I$ singular.
- The Eigenvectors corresponding to $\lambda$ form a subspace $V_\lambda$. The dimension of $V_\lambda$ is called the **multiplicity** of $\lambda$; this coincides with the multiplicity of the root $\lambda$ for the polynomial $\det(A - \lambda I)$.
- $A$ has exactly $n$ Eigenvalues (when counted with multiplicities).
If matrix $A$ is symmetric, then all Eigenvalues are real and the Eigenvectors are pairwise orthogonal to each other.

The sum of the Eigenvalues equals the trace of matrix $A$:
$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{i,i}$$

The product of the Eigenvalues equals the determinant of matrix $A$:
$$\prod_{i=1}^{n} \lambda_i = \det(A)$$

The number of non-zero Eigenvalues (counted with multiplicities) is the rank of matrix $A$. 
The Eigenvalues of a graph $G$ (= Eigenvalues of the adjacency matrix of the graph) are real and add up to $0$.

If $G$ is $d$-regular, then $d$ is an Eigenvalue and the all-one vector $\vec{1} = (1, 1, \ldots, 1)$ is an Eigenvector.

The complete graph $K_n$ has Eigenvalues $n - 1$ (with multiplicity $1$) and $-1$ (with multiplicity $n - 1$).

The complete bipartite graph $K_{m,n}$ has
- Eigenvalues $0$ (with multiplicity $n + m - 2$), and
- Eigenvalues $\pm \sqrt{mn}$ (each with multiplicity $1$).

The Peterson graph has Eigenvalues $3; 1, 1, 1, 1, 1; -2, -2, -2, -2$. 