Chapter 07: Connectivity

(Combinatorial Graph Theory, SS 2019)

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SS 2019, RWTH
Lecture times:
- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
https://algo.rwth-aachen.de/Lehre/SS19/KG/KG.py
Basic definitions (1)

**Definition**

Let $G = (V, E)$ be a graph.

- A subset $S \subseteq V$ is a **vertex-cut** (also called: cut-set, or separating set), if the graph $G - S$ is not connected.
- $G$ is **$k$-connected**, if every vertex-cut contains at least $k$ vertices.
- The **connectivity** $\kappa(G)$ of graph $G$ is the largest $k$ so that $G$ is $k$-connected.
- The connectivity of $K_n$ is $n - 1$.

- What is $\kappa(T)$ for a tree $T$?
- What is $\kappa(C_n)$?
- What is $\kappa(K_{s,t})$?
Definition

- Two paths are **independent**, if they share no common vertices, with the possible exception of their end-vertices.
- Two paths are **strictly independent**, if they share no common vertices.

Let $G = (V, E)$ be a graph, and let $X, Y \subseteq V$.

Definition

- An **X-Y-path** starts in $X$, traverses part of $V - (X \cup Y)$, and ends in $Y$.
- Sets $X$ and $Y$ are **linked** to each other, if $|X| = |Y|$ and there exist $|X|$ strictly independent $X$-$Y$-paths.
- A subset $Z \subseteq V$ **detaches** $X$ from $Y$, if $G - Z$ contains no $X$-$Y$-path.
The linking theorem (1)

Let $G = (V, E)$ be a graph, and let $X, Y \subseteq V$.
Let $Z \subseteq V$ with $|Z| = k \geq 1$ be a smallest set that detaches $X$ from $Y$.

Then there exist $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $|X_0| = |Y_0| = k$, so that $X_0$ is linked to $Y_0$.

Proof: by induction on the cardinality $|E|$.
In the first case, every $k$-element detaching set is $X$ or $Y$.

- Without loss of generality $|X| = k$ and $X \not\subseteq Y$.
- Pick $x_0 \in X - Y$. There exists an edge $e_0 = \{x_0, v_0\}$ with $v_0 \notin X$.
- Define $G_0 = G - e_0$. Let $Z_0 \subseteq V$ be a smallest detaching set in $G_0$.
Assume $|Z_0| = k - 1$.

- $Z_0 \cup \{x_0\}$ and $Z_0 \cup \{v_0\}$ are detaching sets in $G$.
  Then $Z_0 \cup \{x_0\} = X$, since $x_0 \in X - Y$.
  Then $Z_0 \cup \{v_0\} = Y$, since $v_0 \notin X$. 

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In the **second case**, there exists a $k$-element detaching set $Z \subseteq V$ with $Z \neq X$ and $Z \neq Y$.

- Let $G_1$ be subgraph induced by vertices on $X$-$Z$-paths.
  - Let $G_2$ be subgraph induced by vertices on $Y$-$Z$-paths.
- In $G_1$, one needs at least $k$ vertices to detach $X$ from $Z$.
  - By inductive hypothesis, there exists $k$-element $X_0 \subseteq X$ that is linked to $Z$ in $G_1$.
- In $G_2$, one needs at least $k$ vertices to detach $Y$ from $Z$.
  - By inductive hypothesis, there exists $k$-element $Y_0 \subseteq Y$ that is linked to $Z$ in $G_2$.
- Glue the paths in $G_1$ and in $G_2$ together.
Menger’s theorem

Theorem (Karl Menger, 1927)

Let $G = (V, E)$ be a graph, and let $x, y \in V$ with $\{x, y\} \notin E$. Let $k$ be the maximum number of independent $x$-$y$-paths in $G$. Let $\ell$ be the minimum number of vertices in a vertex-cut that isolates $x$ from $y$.

Then $k = \ell$.

Edge-version of Menger’s theorem:

- Let $k$ be the maximum number of edge-disjoint $x$-$y$-paths
- Let $\ell$ be the cardinality of smallest isolating edge-cut
Hall’s theorem (Marriage theorem)

**Theorem (Philip Hall, 1935)**

Let \( G = (X \cup Y, E) \) be a bipartite graph that satisfies the Hall condition

\[
\forall A \subseteq X : |\Gamma(A)| \geq |A|
\]

Then \( G \) contains a matching of size \(|X|\).
k-connected graphs

Auxiliary lemma

Let $G = (V, E)$ be a $k$-connected graph.
Let $H$ result from $G$ by adding a new vertex $y$ and connecting it to at least $k$ vertices.

Then $H$ is $k$-connected.

Theorem

Let $G = (V, E)$ be a graph with $|V| \geq k + 1$.
Then the following three statements are pairwise equivalent.

(a) $G$ is $k$-connected

(b) Between any two vertices, there are at least $k$ independent paths

(c) For every $k$-element subset $U \subseteq V$ and for every vertex $v \in V - U$, there exist $k$ independent $x-U$-paths
Two-connected graphs

Theorem

Let \( G = (V, E) \) be a graph with \(|V| \geq 3\).
Then the following five statements are pairwise equivalent.

(a) \( G \) is 2-connected
(b) \( G \) has no cut-vertex
(c) For any two vertices, there is a cycle containing them
(d) For any vertex and any edge, there is a cycle containing them
(e) For any two edges, there is a cycle containing them
Some exercises

Exercise
Prove or disprove:
A graph $G = (V, E)$ is 3-connected, if and only if for any three vertices $x, y, z \in V$ there exists a cycle containing them.

Exercise
Prove or disprove:
If the graph $G = (V, E)$ is $k$-connected, then for any $k$ vertices there exists a cycle containing them.