Chapter 11: Coloring

(Combinatorial Graph Theory, SS 2019)

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SS 2019, RWTH
Announcements

Lecture times:
- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
https://algo.rwth-aachen.de/Lehre/SS19/KG/KG.py

- No lecture on Monday, June 17.
- Lecture on Friday, June 21.
Basic knowledge (1)

Definition

- A **proper vertex coloring** for a graph $G = (V, E)$ is a function $f : V \rightarrow \mathbb{N}$, so that $f(u) \neq f(v)$ for all edges $\{u, v\} \in E$.
- A graph $G = (V, E)$ is **k-colorable**, if there exists a proper vertex coloring $f : V \rightarrow \{1, \ldots, k\}$.
- The **chromatic number** $\chi(G)$ of a graph is the smallest integer $k$ so that $G$ is $k$-colorable.

What is $\chi(K_n)$?
What is $\chi(T)$ for a tree $T$?
What is $\chi(K_{s,t})$?
What is $\chi(C_{11})$?
Lemma

Every graph $G$ satisfies $\chi(G) \geq \omega(G)$.

Lemma

Every graph $G$ satisfies $\chi(G) \leq \Delta(G) + 1$. 
Theorem (Rowland Leonard Brooks, 1941)

Let $G$ be a connected graph with $\Delta(G) \geq 3$ and $G$ not complete. Then $\chi(G) \leq \Delta(G)$.

The exceptional cases:

- If $G = C_{2k+1}$, then $\Delta(G) = 2$ and $\chi(G) = 3$.
- If $G = K_k$, then $\Delta(G) = k - 1$ and $\chi(G) = k$.
- $G$ not connected, and one of the components is a clique or an odd cycle.
Proof for theorem of Brooks

- Consider smallest counter-example $G = (V, E)$. Let $x \in V$ with $\Gamma(x) = \{x_1, \ldots, x_d\}$, where $d \leq \Delta$. Remove $x$ from $G$ and consider remaining ($\Delta$-colorable) graph $H$.

- $d = \Delta$, and in every $\Delta$-coloring of $H$ the neighbors $x_1, \ldots, x_\Delta$ receive pairwise distinct colors.

- Without loss of generality: $x_i$ colored with color $i$. $H_{ij}$ is subgraph induced by color classes $i$ and $j$.

- For all $i < j$: vertices $x_i$ and $x_j$ are in same component $C_{ij}$ of $H_{ij}$.

- For all $i < j$: component $C_{ij}$ is path from $x_i$ to $x_j$.

- For all $j \neq k$: components $C_{ij}$ and $C_{ik}$ have only vertex $x_i$ in common.
As every planar graph contains a vertex of degree at most 5, we have

**Lemma**

Every planar graph $G$ satisfies $\chi(G) \leq 6$.

**Theorem (Alfred Bray Kempe, 1879)**

Every planar graph $G$ satisfies $\chi(G) \leq 5$.

- Kempe actually claimed $\chi(G) \leq 4$.
- In 1890, Percy Heawood found a mistake in Kempe’s argument and restructured the proof to yield the weaker bound.
- In 1977, Kenneth Ira Appel and Wolfgang Haken finally proved $\chi(G) \leq 4$.
- In 2005, Benjamin Werner and Georges Gonthier formulated a proof in the Coq proof assistant (an interactive theorem prover).
For a vertex \( v \in V \), a **k-list** \( L(v) \) is a \( k \)-element subset of \( \mathbb{N} \).

**Definition**

A graph \( G = (V, E) \) is **5-choosable**, if for every system of 5-lists \( L(v) \) for the vertices \( v \in V \), there exists a proper coloring \( f : V \to \mathbb{N} \) that satisfies \( f(v) \in L(v) \) for all \( v \in V \).

**Theorem (Carsten Thomassen, 1994)**

Every planar graph \( G \) is 5-choosable.
### Definition (Jan Mycielski 1955)
For a graph $G = (V, E)$, the corresponding Mycielskian graph $M(G)$ is defined as follows:

- The vertex set is $V \cup \{v' \mid v \in V\} \cup \{z\}$.
- The edge set contains all edges $\{u, v\} \in E$, together with all edges $\{z, v'\}$ with $v \in V$, together with all edges $\{u, v'\}$ with $\{u, v\} \in E$.

### Lemma 1
If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

### Lemma 2
If $g(G) \geq 4$, then $g(M(G)) \geq 4$. 

An immediate consequence of the preceding two lemmas:

**Theorem (Jan Mycielski 1955)**
For all $k \geq 2$, there exists a triangle-free graph $G$ with $\chi(G) > k$.

**Theorem (Pál Erdős, 1959)**
For all $k \geq 2$, there exists a graph $G$ with $\chi(G) > k$ and $g(G) > k$.

**Auxiliary lemma (Markov inequality)**
Let $X$ be a non-negative random variable, and let $a > 0$ be a real number. Then $\text{Prob}[X \geq a] \leq \mathbb{E}[X]/a$. 
Proof for the theorem of Erdős

- Let $n$ be a huge integer, and let $V = \{v_1, \ldots, v_n\}$.
  Let $p = n^{-k/(k+1)}$.

- Note that for $n$ sufficiently large $n^{1/(k+1)} > 6k \cdot \ln n$ and hence $p > 6k \cdot \ln n/n$.

- The probability space $\mathcal{G}(n, p)$ contains every graph over $V$. Every edge occurs with probability $p$.

We show for $n$ sufficiently large:

There exists a graph $G$ in $\mathcal{G}(n, p)$

1. with independence number $\alpha(G) \leq r := n/(2k)$
2. with at most $n/2$ cycles of length $k$ or shorter.