Chapter 05: Degree sequences

(Combinatorial Graph Theory, SS 2019)

Gerhard Woeginger

SS 2019, RWTH
Announcements

Lecture times:
- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
https://algo.rwth-aachen.de/Lehre/SS19/kg/kg.py
A sequence $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ of non-negative integers is graphic, if there exists a simple, loopless graph $G = (V, E)$ with vertices $v_1, \ldots, v_n$ so that $\deg(v_i) = d_i$ for $1 \leq i \leq n$. 

- $\delta(G) = \text{minimum vertex degree in } G$
- $\Delta(G) = \text{maximum vertex degree in } G$
Theorem of Havel-Hakimi

Let $n \geq 2$ and let $D = \langle d_1, d_2, \ldots, d_n \rangle$ be a non-increasing sequence of non-negative integers. Let $\Delta = d_1$ be the largest number in $D$.

Then $D'$ is the sequence of lengths $n$ that results from $D$ by removing the first term $d_1$, subtracting 1 from the $\Delta$ next elements $d_2, \ldots, d_{\Delta+1}$, and bringing the resulting number sequence into non-increasing order.

Theorem (Václav Havel, 1955, and Seifollah Hakimi, 1963)

Sequence $D$ is graphic, if and only if sequence $D'$ is graphic.
Theorem (Pál Erdős, Tibor Gallai, 1960)

A sequence $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ of non-negative integers with even sum is graphic, if and only if it satisfies the Erdős-Gallai conditions:

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \quad \text{for } k = 1, \ldots, n.$$
Proof (Amitabha Tripathi, Sushmita Venugopalan, Douglas West, 2010):

- Grow a graph with vertices $v_1, \ldots, v_n$ from scratch
- There is a so-called critical vertex $v_r$ with $\deg(v_r) < d_r$
- For $1 \leq i \leq r - 1$: $\deg(v_i) = d_i$
- For $i \geq r + 1$: $\deg(v_i) \leq d_i$
- The vertices $v_{r+1}, \ldots, v_n$ form an independent set

Sub-goal: increase the degree of the critical vertex
Case 0: There exists some vertex $v_i$ with $\deg(v_i) < d_i$, that is not adjacent to the critical vertex $v_r$. 
Proof of Erdős-Gallai (2)

Case 0: There exists some vertex $v_i$ with $\deg(v_i) < d_i$, that is not adjacent to the critical vertex $v_r$.

Case 1: There exists some vertex $v_i$ with $i \leq r - 1$, that is not adjacent to the critical vertex $v_r$. 

Case 2: $v_1, \ldots, v_{r-1} \in \Gamma(v_r)$ and $\deg(v_k) \neq \min\{r, d_k\}$ for some $k \geq r + 1$.

Case 3: $v_1, \ldots, v_{r-1} \in \Gamma(v_r)$ and there are two non-adjacent vertices $v_i$ and $v_j$ with $i < j \leq r - 1$. 
Proof of Erdős-Gallai (2)

Case 0: There exists some vertex $v_i$ with $\deg(v_i) < d_i$, that is not adjacent to the critical vertex $v_r$

Case 1: There exists some vertex $v_i$ with $i \leq r - 1$, that is not adjacent to the critical vertex $v_r$

Case 2: $v_1, \ldots, v_{r-1} \in \Gamma(v_r)$ and $\deg(v_k) \neq \min\{r, d_k\}$ for some $k \geq r + 1$
Proof of Erdős-Gallai (2)

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Case 2: $v_1, \ldots, v_{r-1} \in \Gamma(v_r)$ and $\deg(v_k) \neq \min\{r, d_k\}$ for some $k \geq r + 1$

Case 3: $v_1, \ldots, v_{r-1} \in \Gamma(v_r)$ and there are two non-adjacent vertices $v_i$ and $v_j$ with $i < j \leq r - 1$
Consider the sequence $\langle n, n, n-1, n-1, \ldots, 3, 3, 2, 2, 1, 1 \rangle$ of length $2n$ that contains each integer in $\{1, \ldots, n\}$ exactly twice.

Prove that this sequence is graphic
(a) by applying the Havel-Hakimi algorithm;
(b) by showing that the Erdős-Gallai conditions are satisfied;
(c) by presenting an example graph.

Prove or disprove the following strengthening of the Erdős-Gallai theorem: An unordered sequence $d_1, d_2, \ldots, d_n$ of non-negative integers with even sum is graphic, if and only if it satisfies the Erdős-Gallai conditions.