Lecture notes on

Combinatorial Graph Theory

held by
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Winter term 2016/2017
Disclaimer
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1 The Friendship theorem

For this section, we want to develop a solution to the following Friendship problem:

For groups of \( n \geq 3 \) people, where each pair of (distinct) persons has exactly one common friend. For which \( n \) can this condition be satisfied?

To get a grasp on this problem, let us start by testing the first few viable \( n \) (for this, let vertices take the roll of persons and edges model friendship):

\[
\begin{array}{c|c|c|c}
 n = 3 & n = 4 & n = 5 & \ldots \\
 \begin{tikzpicture}
  \node (v1) at (0,0) {$v$};
  \node (v2) at (1,1) {$v_2$};
  \node (v3) at (1,-1) {$v_3$};
  \node (v4) at (2,0) {$v_4$};
  \draw (v1) -- (v2);
  \draw (v1) -- (v3);
  \draw (v1) -- (v4);
\end{tikzpicture} & \text{Not possible} & \begin{tikzpicture}
  \node (v1) at (0,0) {$v$};
  \node (v2) at (1,1) {$v_2$};
  \node (v3) at (1,-1) {$v_3$};
  \node (v4) at (2,0) {$v_4$};
  \node (v5) at (3,0) {$v_5$};
  \draw (v1) -- (v2);
  \draw (v1) -- (v3);
  \draw (v1) -- (v4);
  \draw (v1) -- (v5);
\end{tikzpicture} & \ldots \\
\end{array}
\]

The solutions for \( n = 3 \) and \( n = 5 \) are found by experimenting just a bit. Trying to find a solution for \( n = 4 \) gives us that there is none. The solution to the general problem (with the above used graph-theoretic model) is given by the following theorem:

**Theorem 1** (Friendship theorem). Let \( G = (V,E) \) be a(n undirected) graph in which every pair of distinct vertices has exactly one common neighbor. Then there is a vertex which is connected to all other vertices.

**Proof (by Erdős, Rényi, Sós 1966).** This proof works indirectly, i.e. we assume that the theorem does not hold. We will proceed in steps:

1. \( G \) may not contain a cycle of length 4 (denoted as \( C_4 \)). This is an easy observation; if \( G \) contained a 4-cycle, then two non-adjacent vertices of such a cycle would share two neighbors (the two remaining vertices of the cycle).

2. For any two \( u,v \in V \) with \([u,v] \notin E\), we have \( \deg(u) = \deg(v) \). Without loss of generality, assume that \( u \) has the neighborhood \( \Gamma(u) = \{w_1, \ldots, w_k\} \). To satisfy the condition, \( v \) must be adjacent to some \( w \in \Gamma(u) \), for example (and again w.l.o.g.) \( w_1 \). By observation 1, \( v \) may not be connected to any \( w \in \Gamma(u) \setminus w_1 \) as this would give us a 4-cycle, but \( v \) still needs a common neighbors with the vertices in \( \Gamma(u) \). Our graph now may look something like the following:

\[
\begin{tikzpicture}
  \node (u) at (0,3) {$u$};
  \node (w1) at (1,2) {$w_1$};
  \node (w2) at (2,2) {$w_2$};
  \node (w3) at (3,2) {$w_3$};
  \node (vk) at (4,2) {$w_k$};
  \node (x1) at (0,0) {$x_1$};
  \node (x2) at (1,0) {$x_2$};
  \node (x3) at (2,0) {$x_3$};
  \node (xk) at (3,0) {$x_k$};
  \node (v) at (4,0) {$v$};
  \draw (u) -- (w1);
  \draw (u) -- (w2);
  \draw (u) -- (w3);
  \draw (u) -- (vk);
  \draw (w1) -- (v);
  \draw (w2) -- (v);
  \draw (w3) -- (v);
  \draw (vk) -- (v);
  \draw (x1) -- (v);
  \draw (x2) -- (v);
  \draw (x3) -- (v);
  \draw (xk) -- (v);
\end{tikzpicture}
\]

By the problem condition, \( v \) cannot be adjacent to any other \( w_i \) and for the same reason, there cannot be another vertex which is connected to more than one
element of $\Gamma(u)$. Hence, there must be at least different $k$ vertices $x_1, \ldots, x_k$ such that $v$ and $w_i$ have a common neighbor $x_i$. Note that there might be connected pairs in $\Gamma(u)$ as shown below, which means that $x_2$ could be $w_1$ (as shown in the visualization). This gives us $\deg(v) \geq \deg(u)$, and by symmetry, the claim holds.

3. For any two $u, v \in V$ with $[u, v] \in E$, we have $\deg(u) = \deg(v)$. We only consider the case where every other vertex is a neighbor of $u$ or $v$, as the other cases are easy – if there was a vertex $x$ not connected to $u$ or $v$, we can apply our previous findings to immediately get $\deg(u) = \deg(x) = \deg(v)$.

Let $\{v, w_2, \ldots, w_k\}$ be the neighborhood of $u$ and $\{u, z_2, \ldots, z_k\}$ the neighborhood of $v$. Now, by our assumption (from the beginning), $G$ does not contain a vertex which is adjacent to every other vertex and thus, w.l.o.g. $v$ is not connected to $w_2$ (since there must exist some $w_i$ which $v$ is not connected to). Furthermore we observe that $w_i$ cannot be connected to $z_j$, as this would give us a 4-cycle with $u$ and $v$. But this means that $w_i$ and $z_j$ share no neighbor and hence cannot be part of a valid solution graph. Hence, such $w_i$ (and $z_j$) cannot exist and we have $\deg(u) = \deg(v) = 1$ and thus, there cannot be any other vertices (than $u$ and $v$) in $G$ (which is also a contradiction to our assumption that $G$ does not contain vertices which are connected to all other vertices).

4. $G$ is $k$-regular and we have $n = k^2 - k + 1$. The $k$-regularity of $G$ is a direct corollary from the two previous results. Our graph now would look like this (again, let $w_1, \ldots, w_k$ be the neighbors of $u$):

```
    u
   /\  \
  /   \  /
 w_1-\-\-\-\-\-w_3-\-\-\-\-\-w_k
    \  \  /  \\
   /    / |  /  \\
  /    /  |  /  \\
 k-2  k-2  ... k-2  k-2
```

Counting the vertices gives us $n = 1 + k + k(k - 2) = k^2 - k + 1$.

5. Observe that $k \geq 3$. For $k = 1$ we would get a graph with one node (recall that we required at least 3 vertices in the problem) and for $k = 2$, the only solution is the complete graph $K_3$, which contains a vertex connected to all others.

6. The adjacency matrix $A$ of $G$ satisfies

$$A^2 = (k - 1)E_n + J_n$$

where $E_n$ denotes the identity matrix (of dimension $n \times n$; also sometimes denoted as $I$) and $J_n$ the $n \times n$-matrix of ones (i.e. every entry is 1).
This can be easily verified as the entries of \( A^2 \) represent the number of paths of length 2 from any vertex to another. By the desired property of our graph, there is exactly one path of length 2 from a vertex \( v_i \) to another vertex \( v_j \). Also, there are \( k \) paths (of length 2) from \( v_i \) to itself – we just go to any of its \( k \) neighbors and back. Since \( J_n \) already gives us ones on the main diagonal, we just have to add \((k - 1)E_n\).

7. The eigenvalues of \( A^2 \) are \( k - 1 \) (with multiplicity \( n - 1 \)) and \( k^2 \) (= \( n + k - 1 \) as seen before; with multiplicity 1). The matrix \( A^2 \) will look like this:

\[
A^2 = (k - 1)E_n + J_n = \begin{pmatrix}
k & 1 & \ldots & 1 & 1 \\
1 & k & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & k & 1 \\
1 & 1 & \ldots & 1 & k
\end{pmatrix}
\]

To determine the eigenvalues of \( A^2 \), we have to find the characteristic polynomial \( \chi_{A^2} = \det(XE_n - A^2) \). Since we can write \( A^2 \) as \((k - 1)E_n + J_n\), the eigenvalues of \( A^2 \) must be the eigenvalues of \( J_n \) shifted by \( k - 1 \) – just observe that

\[
A^2x = (k - 1)E_n x + J_n x = (k - 1)x + J_n x
\]

and hence, if \( x \) is an eigenvector of \( J_n \) (let us say \( J_n x = \lambda x \)), we have \( A^2x = (\lambda + k - 1)x \). Therefore, we only have to determine the eigenvalues of the matrix (of ones) \( J_n \). Following a simple rank argument, we know that \( J_n \) must have an eigenvalue of 0 (with multiplicity \( n - 1 \)), since it has rank 1) and furthermore, it is obvious that \( J_n \) has an eigenvalue of \( n \) (with multiplicity 1) where the corresponding eigenspace is generated by a column of \( J_n \).

Thus, by exhaustion we get that the eigenvalues of \( A^2 \) are \( k - 1 \) (with multiplicity \( n - 1 \)) and \( k^2 \) (= \( n + k - 1 \) as seen before; with multiplicity 1).

8. The eigenvalues of \( A \) are \( \sqrt{k - 1} \) (mult. \( r \)), \(-\sqrt{k - 1} \) (mult. \( s \)) and \( k \) (mult. 1) and additionally, \( r + s = n - 1 \). This is a direct corollary of the previous result as we have \( Ax = \pm \lambda x \) is equivalent to \( A^2x = A(Ax) = \lambda(\lambda x) = \lambda^2 x \), combined with the knowledge that \( G \) is \( k \)-regular (i.e. multiplying the vector of ones to \( A \) will net us a vector of \( k \) as there are exactly \( k \) vertices reachable from each vertex).

9. We know that \( k + (r - s)\sqrt{k - 1} = 0 \) from linear algebra: The sum of the eigenvalues of a matrix is equal to its trace (the sum of the entries on the main diagonal). Since we do not admit loops, the entries on the diagonal of \( A \) must be all zero and as such, we get \( \text{tr} A = 0 = k + (r - s)\sqrt{k - 1} \).

As a consequence, we get (observe that \( s - r \neq 0 \) for \( k \neq 0 \) – and we have \( k \geq 3 \))

\[
\frac{k}{s - r} = \sqrt{k - 1}
\]

which means, that \( k/(s - r) = t \) must be a perfect square (else, the left-hand-side is rational while the right-hand-side would be irrational) and we derive

\[
t = \frac{k}{s - r} = \frac{t^2 + 1}{s - r} \Rightarrow t^2 + 1 = t(s - r) \Rightarrow t(s - r - t) = 1
\]

\(^1\)Exchangably, one can use \( \det(A^2 - XE_n) \) as we only care for the roots of the polynomial, which are not changed by negating the polynomial.
and subsequently we have \( t = 1 \) and (using \( t = \sqrt{k-1} \)) \( k = 2 \), which is a contradiction. Hence, our assumption was false and the theorem must hold.

**Alternative proof (by Huneke 2002).** This proof also works indirectly and we can reuse the results from the steps 1 to 5 from the first proof, i.e. \( G \) is \( k \)-regular for some \( k \geq 3 \) and \( n = k^2 - k + 1 \) holds.

6. Let \( f(\ell) \) be the number of closed walks (not paths; we specifically allow the repetition of vertices and edges) of length \( \ell \) starting from some vertex \( v \in V \) which we denote as

\[
v = v_0, v_1, v_2, \ldots, v_{\ell-2}, v_{\ell-1}, v_\ell = v
\]

To get a useful form for \( f(\ell) \), we count the number of walks in two groups:

- The number of such walks where we have \( v_{\ell-2} = v \) is given by \( kf(\ell - 2) \), which is a simple corollary of the \( k \)-regularity of \( G \).
- The number of such walks where \( v_{\ell-2} \neq v \) is simply the number of all walks of length \( \ell - 2 \) (since then \( v_{\ell-2} \) and \( v \) only share one neighbor and thus, \( v_{\ell-1} \) is not choosable) minus the number walks not satisfying the condition (which is given by \( f(\ell - 2) \)), i.e. \( k^{\ell-2} - f(\ell - 2) \)

Hence, we end up with a recursive form for \( f(\ell) \) as the sum of the two complementary groups:

\[
f(\ell) = kf(\ell - 2) + k^{\ell-2} - f(\ell - 2) = (k - 1)f(\ell - 2) + k^{\ell-2}
\]

7. Let \( p \) be a prime divisor of \( (k - 1) \). Then, \( f(p) \equiv_p 1 \) holds (as \( (k - 1)f(\ell - 2) \) is divisible by \( p \)). Furthermore, the number of closed walks of length \( p \) in \( G \) is given by \( (k^2 - k + 1)f(p) \) (recall that \( n = k^2 - k + 1 \) due to \( p \) being prime and since \( k^2 - k + 1 \equiv_p 1 \), we have \( (k^2 - k + 1)f(p) \equiv_p 1 \).

8. Consider any fixed closed walk of length \( p \). This walk is counted \( p \) times (due to the \( p \) different starting vertices) in the total number of closed walks.

Now, this means the number of total closed walks of length \( p \) must be divisible by \( p \), but we already established that the total number of closed walks is not divisible by \( p \) (as we have \( (k^2 - k + 1)f(p) \equiv_p 1 \)), which gives us a contradiction. Thus, the theorem must hold. \( \square \)
2 Counting spanning trees

First recall some basic definitions:

- A tree is an acyclic, connected graph. Trees possess the property $|E| = |V| - 1$.
- A forest is an acyclic graph (i.e. all of its connected components are trees).
- A leaf is a vertex in a tree or forest with a degree of 1.

Example 2 (Counting spanning trees). Let $V = [1, 4]$. How many trees spanning $V$ are there?

There are 4 spanning trees with 3 leaves looking like this (up to renaming vertices):

```
  1
 /|
/  |
  2 3 4
```

The remaining spanning trees take the following form (up to renaming vertices):

```
  1
 /|
/  |
  2 3 4
```

If we also take symmetry into account (i.e. the graph given by 4-3-2-1 is just the graph 1-2-3-4 mirrored), we get that the number of such vertices is the number of 4-permutations halfed, which adds up to $4!/2 = 12$. Hence, there are 16 spanning trees over $[1, 4]$.

Theorem 3 (Cayley, 1889). There are $n^{n-2}$ trees over $[1, n]$.

Proof. Let $A \subseteq [1, n]$ with $|A| = k$. Denote by $T(n, k)$ the number of forests with $n$ vertices and $k$ connected components so that each vertex in $A$ is in a different connected component of the forest. We get a recursive formula for $T(n, k)$ as

$$T(n, k) = \sum_{i=0}^{n-k} \binom{n-k}{i} T(n-1, k+i-1)$$

which can be intuitively visualized as summing over the possibilities of creating a forest over $[1, n]$ with $k$ connected components by adding a new vertex to a $[1, n-1]$-forest with $k+i-1$ existing trees; connecting $i$ of those trees.

Further, we need to provide some base cases for $T(n, k)$: We say that $T(n, n) = 1$ (i.e. there is one graph over $[1, n]$ with no edges), $T(0, 0) = 0$ (there exists an empty graph with no vertices and no connected component) and lastly, we define $T(n, 0) = 0$ for any $n \geq 1$ (as there is no graph with $n \geq 1$ vertices and no connected component).

In terms of our counting function $T$, it remains to prove that $T(n, 1) = n^{n-2}$. We will actually prove a more general formula,

$$T(n, k) = kn^{n-k-1}$$
by induction on \( n \). For \( n = 1 \), this claim is easily verified using the base cases given above. Assuming the hypothesis for \( n - 1 \), we can deduce

\[
T(n, k) = \sum_{i=0}^{n-k} \binom{n-k}{i} T(n-1, k+i-1)
\]

\[
= \sum_{i=0}^{n-k} \binom{n-k}{i} (k+i-1)(n-1)^{n-k-i-1}
\]

Transforming the index from \( i \) to \( j = n - k - i \) (i.e. reversing the summation direction) and using symmetries in the binomial coefficient, we get

\[
= \sum_{j=0}^{n-k} \binom{n-k}{j} (n-j-1)(n-1)^{j-1}
\]

\[
= \sum_{j=0}^{n-k} \binom{n-k}{j} (n-1)(n-1)^{j-1} - \sum_{j=0}^{n-k} \binom{n-k}{j} j(n-1)^{j-1}
\]

Since the term in the second sum vanishes for \( j = 0 \), we can perform an index shift to get

\[
= \sum_{j=0}^{n-k} \binom{n-k}{j} (n-1)^{j} - \sum_{j=1}^{n-k} \binom{n-k}{j} j(n-1)^{j-1}
\]

\[
= n^{n-k} - (n-k)n^{n-k-1}
\]

\[
= kn^{n-k-1}
\]

which is what we wanted to get. Setting \( k = 1 \) for our special case concludes the proof of the theorem.

Alternative proof (Prüfer, 1918). This proof works by constructing a bijection between \([1, n]^{n-2}\) and the trees over \([1, n]\). For this, we will encode a tree over \([1, n]\) by its Prüfer code.

The Prüfer code of a \([1, n]\)-tree is constructed as follows:

**Algorithm 4** (Constructing the Prüfer code of a tree). For a \([1, n]\)-tree \( T \), repeat until one edge is left:

1. Remove the lexiographically smallest leaf \( x_i \).
2. Remove the edge \([x_i, y_i]\) incident to \( x_i \).
3. Store \( y_i \) in a tuple \( t \).
The Prüfer code of the tree is given by the tuple $t' = (y_1, \ldots, y_{n-2})$ (i.e. we delete the last entry as it is always $n$).

**Example 5 (Prüfer code).** Let $G$ be a tree over $[1,8]$ which is given by the following figure. We now want to construct the Prüfer code (of length 6) for $G$ using the above-given algorithm:

![Diagram of a tree with vertices labeled 1 to 8]

The steps of the algorithm give us the following values for $x$ and $y$ for the different $i \in [1,8]$.

<table>
<thead>
<tr>
<th>$y_i$</th>
<th>7 4 4 1 7 1 (8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>2 3 5 4 6 7 1</td>
</tr>
</tbody>
</table>

And thus, the Prüfer code of $G$ is given by $(7, 4, 4, 1, 7, 1)$.

We now want to use the Prüfer code to establish the bijection between $[1,n]^{n-2}$ and the trees over $[1,n]$.

**Lemma 6 (Injectivity of the Prüfer code mapping).** Distinct trees have distinct Prüfer codes.

**Proof.** We start by making two observations:

1. A vertex $v$ occurs $\text{deg}(v) - 1$ times in the Prüfer code. To see this, recall that $v$ occurs in the Prüfer code when one of its leaf children get removed and that in a tree, each inner node (spare the root) has $\text{deg}(v) - 1$ children.

2. Once the first leaf $x_1$ and its connecting edge $y_1$ have been removed, the Prüfer code of the remaining tree is $(y_2, \ldots, y_{n-2})$. This is easily provable using induction on $n$.

Now, we consider two trees $T_1 \neq T_2$ over $[1,n]$ and distinguish three cases:

1. The smallest leaf of $T_1$ and $T_2$ differ. Without loss of generality, let $x$ and $y$ with $x < y$ be the smallest leaves of $T_1$ and $T_2$ respectively. Then, $x$ cannot be a leaf in $T_2$ as it would be a smaller leaf then the smallest leaf $y$ and thus, it must be an inner node of $T_2$. Hence, $x$ does not appear in the Prüfer code of $T_1$ while it appears in the Prüfer code of $T_2$ by the first observation and thus, the Prüfer codes of $T_1$ and $T_2$ must differ.

2. If the smallest leaves of $T_1$ and $T_2$ are the same but they have different neighbors (which make up the first entry of the Prüfer code), the Prüfer codes of $T_1$ and $T_2$ have different first entries and thus differ.
3. If $T_1$ and $T_2$ both have $x$ as the smallest leaf and in both trees $y$ is the parent of $x$ (i.e. the first column of the $y_i/x_i$-table is identical for both trees), then $T_1 \setminus \{x\} \neq T_2 \setminus \{x\}$. Using the second observation and induction, it is easy to show that the codes of $T_1$ and $T_2$ must differ.

In conclusion, the Prüfer code mapping is injective.

Lemma 7 (Surjectivity of the Prüfer code mapping). Every string $[1, n]^{n-2}$ is a Prüfer code of some tree over $[1, n]$.

Proof. We construct a $[1, n]$-tree $T$ from a given Prüfer code $t = (y_1, \ldots, y_{n-2})$ (with $y_{n-1} = n$ being implicitly given by the construction of the Prüfer code).

The construction works as follows:

Algorithm 8 (Constructing a tree with a given Prüfer code). Let $t = (t_1, \ldots, t_{n-1})$ as given above. Repeat for $k \in [1, n-1]$:

1. Set $x_k$ to be the smallest number from $[1, n-1]$ not contained in $\{x_1, \ldots, x_{k-1}\} \cup \{y_k, \ldots, y_{n-1}\}$.

2. Add the edge $[x_k, y_k]$ to our work-in-progress tree $T$.

Now, we want to argue that this construction indeed gives us a tree $T$ with Prüfer code $t$ (we only prove the first part as the second one is clear from the construction). We start by making some observations and steps:

1. The set of numbers $\{x_1, \ldots, x_{n-1}\}$ is a permutation of $[1, n]$.

2. Assume the construction gives us a graph $G$. We now produce an oriented graph $G'$ by orienting the edges from $x_k$ to $y_k$ (i.e. upwards in the tree). Observe that every vertex in $[1, n-1]$ has exactly one outgoing edge.

Now, suppose there is a cycle in $G$, then there is a directed cycle $c$ in $G'$ (to see this, recall the construction of $G$ and $G'$). This directed cycle now contains a node $x_k$ from the construction with maximal $k$, and this node $x_k$ has only the one outgoing edge $[x_k, y_k]$ and $y_k$ cannot occur in $\{x_1, \ldots, x_k\}$ if our algorithm was used. Hence, $y_k$ cannot have an outgoing edge to any of the other nodes in $c$ and thus, we have a contradiction and $T$ must be a tree.

Putting the pieces together, we have established that the Prüfer code mapping is indeed a bijection and thus, the number of spanning trees over $[1, n]$ is $n^{n-2}$. 

Our next goal is to count the spanning trees of some graph $G$ (instead of $[1, n]$). An example:

Example 9 (Counting spanning trees of a graph). Let $G$ be a graph over $[1, 4]$ given by

By counting, $G$ has 8 spanning trees (2 by connecting either 2 or 4 with all other vertices, 4 by using the 1-2-3-4 cycle and leaving out one edge and 2 more by using the edge in the middle to get 1-2-4-3 and 1-4-2-3).
For this, we need an auxiliary lemma (without proof):

Lemma 10 (Cauchy-Binet formula). Let $A$ be a $(p \times m)$-matrix and $B$ be a $(m \times p)$-matrix with $p \leq m$. Then,

$$\det(AB) = \sum_{S \in \binom{[1,m]}{p}} \det(A_S) \det(B_S)$$

where $\binom{[1,m]}{p}$ denotes the set of $p$-element subsets of $[1,m]$ and $A_S$ denotes the submatrix generated by selecting the columns of $A$ indexed by the elements of $S$ while $B_S$ denotes the submatrix generated by selecting the rows of $B$ indexed by the elements of $S$.

We can now formulate and prove the following theorem:

Theorem 11 (Matrix-tree theorem, inspired by Kirchhoff). Let $A$ be the adjacency matrix of a $[1,n]$-graph $G$ and $D$ the diagonal matrix of the degrees of the vertices in $G$. Let $Q^*$ be the matrix of dimension $(n - 1) \times (n - 1)$ produced by deleting the last row and the last column from $D - A$ (the matrix $D - A$ is also called the Laplace matrix of $G$).

Then, $G$ has $\det(Q^*)$ spanning trees.

Proof. First, we handle the case where $G$ is not connected, i.e. $G$ has no spanning tree. Then $G$ possesses the connected components $c_1, c_2, \ldots, c_k$ and we can relabel the vertices so, that $D - A$ takes the form

$$D - A = \begin{pmatrix} c_1 & 0 \\ 0 & c_2, \ldots, c_k \end{pmatrix}$$

as there are no connections between any vertex of $c_1$ and any vertex of the other connected components. The Laplace matrix of a connected graph (or here, connected component viewed as a graph) has determinant 0 as the $i$-th column (and row) of the adjacency matrix has exactly $\deg(i)$ ones and thus, the sum of all columns of the Laplace matrix is the zero vector. Hence, the determinant of the Laplace matrix of a connected graph is 0.

Returning to our case, we see that the Laplace matrix of a non-connected graph is a block matrix and subsequently, its determinant is given by the product of the determinants of the Laplace matrix of $c_1$ and the Laplace matrix of $\{c_2, \ldots, c_k\}$. Since we already argued that the determinant of the Laplace matrix of $c_1$ is zero, the determinant of $Q^*$ is also zero (since we get $Q^*$ from the Laplace matrix and we can relabel vertices such that no column of the Laplace matrix of $c_1$ gets deleted), which was the desired result.

From now on, we require $G$ to be connected. We will prove the theorem for this case using multiple steps:

1. Choose an arbitrary orientation $G'$ of $G$ and let $M$ be the incidence matrix of $M$ (i.e. $M_{v,e} = \pm 1$ if $e$ is an outgoing/incoming edge of $v$). Then, we get $D - A = MM^t$. To see this, consider at first the diagonal entries where we have

$$(MM^t)_{i,i} = (M_{i,-})(M^t_{-,i}) = \sum_{j=1}^{E} M_{i,j}^2$$
which is exactly the number of edges incident to $i$ (alternatively formulated: the scalar product of a vector $v$ with entries from $\{-1, 0, 1\}$ with itself gives the number of nonzero entries of $v$).

For the non-diagonal entries, we have

$$(MM^tr)_{i,j} = (M_{i,-})(M^tr_{j,-}) = \sum_{p=1}^{|E|} M_{i,p}M^tr_{p,j} = \sum_{p=1}^{|E|} M_{i,p}M_{j,p}$$

Now, by the definition of the incidence matrix $M$, for any nonzero entry in a row of $M$ there exists exactly one other nonzero entry at the same place in another row of $M$ (as every column has exactly two nonzero entries, those being $-1$ for the start vertex and 1 for the target vertex) and their product is $-1$. So, iff $i$ and $j$ are adjacent, there exists exactly one edge between them and hence, $(MM^tr)_{i,j} = -1$ which is what we wanted.

2. Let $B$ be a submatrix of dimension $(n-1) \times (n-1)$ of $M$. We make two observations:

(a) If the underlying edges of the submatrix $B$ contain a cycle (or induce a non-connected graph), then $\det(B) = 0$.

To see this, add the corresponding columns of $M$ with a factor of -1 or 1 (if an edge is oriented clockwise/anticlockwise) to get a nontrivial linear combination that gives the zero vector, and thus the determinant of $M$ is zero (and by the same argument, the determinant of $B$).

(b) If the underlying edges of the submatrix $B$ form a tree, then $\det(B) = \pm 1$.

This is provable using the Laplace expansion and induction (informally: the idea is that for a tree $T$ and a subtree $S$, Laplace expansion gives $\det(T) = \pm 1 \cdot \det(S)$).

3. Let $M^*$ result from $M$ by deleting its last row. Then, $Q^* = M^*(M^*)^tr$ and by the Cauchy-Binet formula (and the fact that for any matrix $A$ we get $\det(A) = \det(A^tr)$),

$$\det(Q^*) = \sum_{S \subseteq \binom{[1,|E|]}{n-1}} (\det(M^*_S))^2$$

holds.

As we have seen before, every spanning tree (using $n-1$ edges) corresponds to a submatrix of $M$ with determinant $-1$ or 1 (i.e. their square is 1) while every other submatrix has determinant 0 (induced by a cycle). Thus the summation over all edge-sets of size $n-1$ and adding up the squared determinants of the corresponding submatrix counts the number of spanning trees of $G$ which is equal to $\det(Q^*)$ and the proof is concluded.
3 Turán’s theorem and extremal graph theory

Again, we start by recalling some basic definitions:

- A *clique* of size $n$ is a complete subgraph (i.e. $K_n$ as a subgraph). By $\omega(G)$ we denote the size of the largest clique in $G$.

- An *independent set* (or *stable set*) of size $n$ is a subgraph without any edges. By $\alpha(G)$ we denote the size of the largest independent set in $G$.

Note the dualism between cliques in a graph and independent sets in its complement graph (and vice versa).

We now want to answer the question of how many edges are needed in a graph $G$ over $[1,n]$ to guarantee the existence of a triangle ($K_3$ subgraph) in $G$. An example:

**Example 12** (Minimal number to guarantee a $K_3$ subgraph). Consider $n = 4$. We can build a graph with 4 edges and no triangle, but no graph with 5 (note that the graphs are unique up to renumbering vertices):

Thus, a triangle-free graph with 4 vertices has at most 4 edges – the pictured complete bipartite graph $K_{2,2}$ has 4 edges and no triangle, while the pictured graph with 5 edges has triangles (and cannot be repaired).

Similarly, the graph with the most edges and no triangle over $[1,6]$ is given by the complete bipartite graph $K_{3,3}$. It has 9 edges:

**Theorem 13** (Mantel’s theorem, 1906). Let $G$ be a graph over $[1,n]$. If $G$ is triangle-free, then $|E| \leq n^2/4$.

**Proof.** Observe: For $[u,v] \in E$, we have $\deg(u) + \deg(v) \leq n$ (else $u$ and $v$ would have a common neighbor and thus $G$ would contain a triangle). Thus,

$$n|E| \geq \sum_{[u,v] \in E} \deg(u) + \deg(v)$$
Since every vertex \( v \) has \( \deg(v) \) incident edges, we count the degree of \( v \) exactly \( \deg(v) \) times, giving us

\[
\sum_{v \in V} \deg^2(v)
\]

Using the arithmetic-quadratic mean inequality, we obtain

\[
\geq \frac{1}{n} \left( \sum_{v \in V} \deg(v) \right)^2 = \frac{4|E|^2}{n}
\]

And hence,

\[
n|E| \geq \frac{4|E|^2}{n} \iff |E| \leq \frac{n^2}{4}
\]

Note that this bound cannot be improved, as the complete bipartite graph \( K_{\frac{n}{2}, \frac{n}{2}} \) contains no triangle and has \( \frac{n^2}{4} \) edges. Next up is the generalization of this first result:

**Theorem 14** (Turán’s theorem, 1941). Let \( G \) be a \( K_p \)-free graph on \( n \) vertices. Then,

\[
|E| \leq \frac{p - 2}{p - 1} \cdot \frac{n^2}{2}
\]

**Proof.** For this proof, we will consider the following continuous optimization problem:

Maximize \( \sum_{[i,j] \in E} x_i x_j \) subject to \( \sum_{i \in V} x_i = 1 \) and \( 0 \leq x_i \) for all \( i \in V \).

We will now argue that there exists an optimal solution \( x^* = (x_1^*, \ldots, x_n^*) \) for this problem and try to quantify it. Some observations:

1. Suppose \( x_i^* > 0 \) and \( x_j^* \) and \( [i, j] \notin E \). Let \( W_i^* = \sum_{k \in \Gamma(i)} x_k^* \) (define \( W_j^* \) similarly).

Now, without loss of generality \( W_i^* \geq W_j^* \). Then, replacing \( x_i^* \) by \( x_i^* + x_j^* \) and replacing \( x_j \) by 0, the objective function changes by \( x_j^*(W_i^* - W_j^*) \geq 0 \). Also, this gives us that the set \( \{ i \in V \mid x_i^* > 0 \} \) forms a clique (else we can use the above procedure to find another solution by transferring weight from a vertex outside the clique). Hence, from now on we consider only feasible solutions where the above-mentioned set indeed forms a clique.

2. Suppose that \( x_i^* > 0 \) and \( x_j^* > 0 \) with \( x_i^* \neq x_j^* \).

Replacing both \( x_i^* \) and \( x_j^* \) by their mean \( \frac{1}{2}(x_i^* + x_j^*) \) changes the objective function (note that this replacement does not change any values besides \( x_i^* x_j^* \) in our summation due to symmetry) by

\[
\frac{1}{4}(x_i^* + x_j^*)^2 - x_i^* x_j^*
\]

which we can multiply by 4 to give us

\[
(x_i^*)^2 - 2x_i^* x_j^* + (x_j^*)^2 = (x_i^* - x_j^*)^2 \geq 0
\]
Hence, the optimal solution to the optimization problem for a graph $G$ whose largest clique $K$ has size $k$ is given by

$$x_i^* = \begin{cases} \frac{1}{k} & \text{if } i \in K \\ 0 & \text{else} \end{cases}$$

and the optimal objective value is (recall that $G$ is $K_p$-free, thus $k \leq p - 1$)

$$\sum_{i \in \binom{V}{2}} \frac{1}{k^2} = \binom{k}{2} \cdot \frac{1}{k^2} = \frac{k(k-1)}{2} \cdot \frac{1}{k^2} = \frac{1}{2} \left( 1 - \frac{1}{k} \right) \leq \frac{1}{2} \left( 1 - \frac{1}{p-1} \right) = \frac{1}{2} \cdot \frac{p-2}{p-1}$$

3. Now, consider the feasible solution $x^+ = (x_1^+, \ldots, x_n^+)$ where we set $x_i^+ = 1/n$ for all $i \in V$. Its objective function value is

$$\sum_{(i,j) \in E} \frac{1}{n^2} = \frac{|E|}{n^2}$$

Putting the pieces together, we know that

$$\frac{|E|}{n^2} \leq \frac{1}{2} \cdot \frac{p-2}{p-1}$$

which is equivalent to

$$|E| \leq \frac{p-2}{p-1} \cdot \frac{n^2}{2}$$

and thus, the proof is concluded. \hfill \Box

We will now look at a few concepts that yield an alternative proof of Turán’s theorem.

**Definition 15** (Graph domination). Let $G, H$ be graphs over $n$ vertices and let

$$g_1 \geq g_2 \geq \ldots \geq g_n \quad \text{and} \quad h_1 \geq h_2 \geq \ldots \geq h_n$$

be the degree sequences (i.e. the degrees a graph’s vertices descendingly sorted) of $G$ and $H$ respectively.

We say that $G$ is **dominated by** $H$ (write $G \preceq H$) if $g_i \leq h_i$ for all $i$.

**Theorem 16** (Erdős, 1967). Let $G$ be a $K_{p+1}$-free graph. Then, there exists a complete $p$-partite graph $H$ with $G \preceq H$.

**Proof.** We conduct induction on $p$. The base case ($p = 1$) is easy: A $K_2$-free graph does not contain any edges as does a 1-partite graph (giving us that both graphs are independent sets). This leaves the inductive step: Consider a $K_{p+1}$-free graph $G$ and let $u \in V$ have maximal degree $\Delta$. We define $V_1 = \Gamma(u)$ and $V_2 = V \setminus (V_1 \cup \{u\})$ denote the remaining vertices of $G$. Our $G$ might be visualized as follows:
Now, observe that the subgraph of $G$ induced by $V_1$ is $K_p$-free as every vertex in $V_1$ is connected to $u$ and furthermore, every vertex in $V_2$ has a degree less or equal to $\Delta$. By the assumption, there exists complete $(p - 1)$-partite graph $H_1$ with $G[V_1] \preceq H_1$. So, we can now construct a graph $H$ from $G$ by replacing $V_1$ with $H_1$ and deleting all edges in $V_2$ and adding edges from every vertex in $V_2$ (and from $u$) to every vertex in $H_1$, giving us something like this:

Thus, every vertex in $V_2$ now has degree $\Delta$ and thus, $G \preceq H$ (recall that $V_1 \preceq H_1$). Since $H_1$ is complete $(p - 1)$-partite and $u$ is not connected to any vertex in $V_2$ (and $V_2$ has no edges in $H$), $H$ is $p$-partite and complete as we added the required edges between $H_1$ and $\{u\} \cup V_2$. This means that $G \preceq H$ and $H$ is complete $p$-partite, which is completes the inductive step. 

Note that this yields a new proof for Turán’s theorem as we now only have to consider complete $p$-partite graphs on $n$ vertices. To maximize the number of edges in such a graph, we have to minimize the edges missing due to the elements of a partition class being not connected. The number of edges missing in a partition class of size $n$ is precisely the number of edges in the complete graph $K_n$ on $n$ vertices, which grows quadratically in $n$. One easily verifies that an uniform (as uniform as possible, at least) distribution of vertices onto the partition classes minimizes the number of missing edges and thus maximizes the number of edges present in our graph and further, the number of edges in such a graph satisfies (note that we formulated the above theorem with respect to $K_{p+1}$-free graphs opposed to $K_p$-free graphs in our formulation of Turán’s theorem)

$$|E| \leq \frac{p-1}{p} \cdot \frac{n^2}{2}$$

which is exactly what Turán’s theorem states.

After considering graphs which do not contain certain cliques, we now want to disallow the presence of cycles, starting with the $C_4$:

**Theorem 17.** Let $G$ be a graph satisfying

$$|E| > \frac{n}{2} \left( 1 + \sqrt{n - 1} \right)$$
3 Turán’s theorem and extremal graph theory

Then $G$ contains a cycle of length 4.

**Proof.** This proof works indirectly, i.e. we assume that $G$ has no cycle of length 4, but sufficiently many edges. We now want to count the number of $K_{1,2}$-subgraphs (also called cherries), where $K_{1,2}$ denotes the complete bipartite graph with partition classes of size 1 and 2, looking like this:

![Diagram of a cherry](image)

We now double-count those cherries:

1. By a midpoint argument, there are exactly

$$c = \sum_{v \in V} \left( \frac{\deg(v)}{2} \right)$$

cherries.

2. By counting possible endpoints, we get

$$c \leq \binom{n}{2}$$

Hence, we have

$$\binom{n}{2} = \frac{n(n-1)}{2} \geq c$$

$$= \sum_{v \in V} \left( \frac{\deg(v)}{2} \right)$$

$$= \frac{1}{2} \sum_{v \in V} \deg^2(v) - \frac{1}{2} \sum_{v \in V} \deg(v)$$

Using the arithmetic-quadratic mean inequality (cf. Mendel’s theorem), we get

$$\geq \frac{1}{2n} \left( \sum_{v \in V} \deg(v) \right)^2 - \frac{1}{2} \sum_{v \in V} \deg(v)$$

$$= \frac{1}{2n} (2|E|)^2 - |E|$$

$$\geq \frac{n}{2} \left( \frac{2|E|}{n} - 1 \right)^2$$

This means we have

$$\frac{n(n-1)}{2} \geq \frac{n}{2} \left( \frac{2|E|}{n} - 1 \right)^2$$

which one can easily rewrite as

$$|E| \leq \frac{n}{2} \left( 1 + \sqrt{n-1} \right)$$

giving us a contradiction to our assumption that $G$ has more edges. \qed
One now naturally wonders if this bound is tight, which is actually an open problem, but we know that it is asymptotically tight:

**Definition 18.** A *finite projective plane* of order $q$ is a set system $S_1, \ldots, S_k$ with $k = q^2 + q + 1$ over some $k$-element ground set $X$ such that

1. Each set $S_i$ contains exactly $q + 1$ elements.
2. Every element of $X$ occurs in exactly $q + 1$ sets.
3. Two sets $S_i$ and $S_j$ share exactly one element.

**Example 19 (Finite projective planes of order 1 and 2).** One can visualize finite projective planes of order $q$ as a collection of $k = q^2 + q + 1$ points and $k$ curves such that

1. Each curve passes through $q + 1$ points.
2. Every point lies on $q + 1$ curves.
3. Two curves share exactly one point.

Using this, the finite projective planes of order 1 ($q = 1, k = 3$) and 2 ($q = 2, k = 7$) can be visualized by the following points and curves:

```
\begin{center}
\begin{tikzpicture}
  \vertex[fill] (x1) at (0,0) {x_1};
  \vertex[fill] (x2) at (-1,-1) {x_2};
  \vertex[fill] (x3) at (1,-1) {x_3};
  \draw (x1) -- (x2) -- (x3) -- (x1);
\end{tikzpicture}
\end{center}
```

\begin{center}
\begin{tikzpicture}
  \vertex[fill] (x1) at (0,0) {x_1};
  \vertex[fill] (x2) at (-1,-1) {x_2};
  \vertex[fill] (x3) at (1,-1) {x_3};
  \vertex[fill] (x4) at (-1,1) {x_4};
  \vertex[fill] (x5) at (1,1) {x_5};
  \vertex[fill] (x6) at (0,2) {x_6};
  \vertex[fill] (x7) at (0,-2) {x_7};
  \draw (x1) -- (x2) -- (x3) -- (x1);
  \draw (x4) -- (x5) -- (x6) -- (x7) -- (x4);
  \draw (x2) -- (x4) -- (x7) -- (x5) -- (x3);
\end{tikzpicture}
\end{center}

**Theorem 20.** For every prime power $q = p^f$, there exists a finite projective plane of order $q$. (There is a proof that there exists no finite projective plane of order $q = 6$.)

With those auxiliary results, we get our asymptotic tightness result:

**Theorem 21.** There exists a graph over $n$ vertices with

$$|E| \geq \frac{1}{2\sqrt{2}} n^{3/2}$$

which has no circle of length 4.

**Proof outline.** We build a graph using the finite projective plane of order $q$ to build a bipartite graph on $2k = 2(q^2 + q + 1)$ vertices.

For this, let $V = X \cup \{S_1, \ldots, S_k\}$ and put an edge between $x_i$ and $s_j$ iff $x_i \in S_j$. Note that such a graph cannot contain cycles of length 4 or else there would be two sets which both share at least two elements, violating the properties of the finite projective plane. Our graph contains $k(q + 1) \in O(q^3)$ edges (as there are $k$ sets and every set has exactly $q + 1$ elements) and $O(q^2)$ vertices (in terms of $k$, we get the desired $O(k^{3/2})$ edges). ✷
Finally, we want to give a very interesting result (without proof):

**Theorem 22** (Erdős, Simonovits 1966). Let $H$ be a graph and let $\text{ex}(n, H)$ denote the most edges a graph which does not contain $H$ can have. Then,

$$\text{ex}(n, H) = \left( 1 - \frac{1}{\chi(H) - 1} \right) \binom{n}{2} + o(n^2)$$

where $\chi(H)$ is the chromatic number of $H$ (cf. Chapter 9).
4 Hamiltonian cycles

We start by recalling some basic concepts:

- A *Eulerian cycle* is a cycle which contains each edge of a graph exactly once (a connected graph is Eulerian iff every vertex has even degree).

- A *Hamiltonian cycle* is a cycle which contains each vertex of a graph exactly once. Determining if a graph is Hamiltonian is $\mathbf{NP}$-complete.

**Example 23** (Hamilton cycles). The *dodecahedron* is Hamiltonian as it possesses a Hamiltonian cycle:

![Dodecahedron](image)

The *Petersen graph* is not Hamiltonian:

![Petersen graph](image)

**Theorem 24** (Dirac, 1952). Let $\delta(G)$ be the minimal degree of a vertex in $G$. If $\delta(G) \geq n/2$, then $G$ is Hamiltonian.

*Proof.* Assume the theorem does not hold for some value of $n$, i.e. there exists a non-Hamiltonian graph on $n$ vertices such that all vertices have degree at least $n/2$. Without loss of generality we can assume that $G$ is edge-maximal, i.e. for all $[u, w] \notin E$, the graph $G + [u, w]$ is Hamiltonian.

Now, since $G + [u, w]$ is Hamiltonian, there must be a Hamiltonian path $(v_1, \ldots, v_n)$ from $u = v_1$ to $w = v_n$ in $G$ itself. We define $S = \{i \mid [u, v_i] \in E\}$ and $T = \{i \mid [w, v_i] \in E\}$ and recall that $|S| + |T| = |S \cap T| + |S \cup T|$. Furthermore, we know that $|S \cup T| \leq n - 1$ as $n \notin S \cup T$. 
From our assumption we get
\[ n \leq \deg(u) + \deg(w) \]
\[ = |S| + |T| \]
\[ = |S \cap T| + |S \cup T| \]
and using \(|S \cup T| \leq n - 1\) we get \(|S \cap T| \geq 1\). But this means that \(G\) is Hamiltonian:

If \(i\) is in \(S \cap T\), this means \(u\) is connected to \(v_{i+1}\) and \(w\) is connected to \(v_i\) and thus,
\[(u, v_2, \ldots, v_i, w, v_{n-1}, \ldots, v_{i+1}, u)\]
is a Hamiltonian cycle in \(G\), which means our assumption was not true. Hence, the theorem must hold.

**Theorem 25** (Ore, 1960). Let \(G = (V, E)\) with vertices \(u, w \in V\) and \([u, w] \notin E\) and \(\deg(u) + \deg(w) \geq n\). Then \(G\) is Hamiltonian if and only if \(G + [u, w]\) is Hamiltonian.

**Proof idea.** The direction from \(G\) to \(G + [u, w]\) is trivial as a Hamiltonian cycle in \(G\) is also a Hamiltonian cycle in \(G + [u, w]\).

The other direction can be proved in complete analogy to Dirac’s theorem.

**Definition 26** (Hamiltonian closure). The Hamiltonian closure \(HC(G)\) results from \(G\) by consecutively adding missing edges between vertices \(v, w \in V\) with \(\deg(v) + \deg(w) \geq n\) until no such non-edge remains.

Note that the Hamiltonian closure is well-defined as we do not make any (relevant) choice; every edge that could be added will get added at some point and the construction terminates (at latest when we have the complete graph).

**Lemma 27.** A graph \(G\) is Hamiltonian iff its Hamiltonian closure \(HC(G)\) is Hamiltonian.

**Proof.** By Ore’s theorem.

**Theorem 28** (Chvátal, 1972). Let \(G\) be a graph with the (ascending) degree sequence \(d\)
\[ d_1 = \deg(v_1) \leq \ldots \leq \deg(v_n) = d_n \]
If \(d_i > i\) or \(d_{n-i} \geq n - i\) for all \(i \leq n/2\), then \(G\) is Hamiltonian.

**Proof.** Suppose the theorem is false, i.e. there exists a Hamiltonian graph \(G\) which does not satisfy the conditions. Without loss of generality we may assume that \(G = HC(G)\).

Consider two vertices \(u, w \in V\) with \([u, w] \notin E\) such that \(\deg(u) + \deg(w)\) is maximal. Since \(G = HC\), we have \(\deg(u) + \deg(w) < n\) and w.l.o.g. we assume that \(\deg(u) \leq \deg(w)\) (and thus \(\deg(u) < n/2\)).

Now pick \(i = \deg(u) < n/2\). There are \(n - \deg(w) - 1 \geq \deg(u)\) vertices with a degree not larger than \(i = \deg(u)\) by the maximality of \(\deg(u) + \deg(w)\) and thus, we have \(d_i \leq i - \) the first condition is violated and we have to check the second one.

There are \(n - \deg(u) = n - i\) vertices satisfying with a degree not bigger than \(\deg(w) < n - 1\) and hence, \(d_{n-i} < n - i\) and the second condition is violated.
**Lemma 29.** Let $G$ be Hamiltonian. Then for every $S \subseteq V$, the graph $G \setminus S$ has at most $|S|$ components.

**Proof.** Since we only remove $|S|$ vertices from a Hamiltonian cycle, the cycle splits into at most $|S|$ paths which connect the components of $G \setminus S$. \hfill \Box

**Example 30 (Chvátal’s theorem).** Consider $G = (E_i \cup K_{n-2i}) \oplus K_i$, an independent set of size $i$ and a clique of size $n - 2i$ both completely connected to a clique of size $i$. The (ascending) degree sequence $d$ of $G$ is given by

$$d = i, \ldots, i, n-i-1, \ldots, n, n-1, \ldots, n-1$$

and thus, the conditions of Chvátal’s theorem are violated for $i$ as $d_i = i$. Hence, $G$ has no Hamiltonian cycle.

We now try to characterize Hamiltonian graphs via a new graph property:

**Definition 31 (Vertex cuts, Toughness).** Let $G$ be a graph and let $c(G)$ denote the number of components of $G$.

- A set $S \subseteq V$ is a vertex cut if $G - S$ is not connected.
- $G$ is called t-tough if all vertex cuts satisfy $|S| \geq t \cdot c(G - S)$.

Observe that every Hamiltonian graph must be 1-tough (as removing $p$ vertices leaves at most $p$ connected components in a Hamiltonian graph). What about the converse statement?

We want to argue that this is unlikely using an argument based on computational complexity theory. The problem of deciding if a graph is Hamiltonian is known to be NP-complete (we can easily encode a Hamiltonian cycle as a certificate), while deciding if a graph is 1-tough is coNP-complete (as the problem of deciding if a graph is not 1-tough is NP-complete). Hence we get that if 1-toughness implies Hamiltonicity, $NP = coNP$ would hold. This is regarded as very unlikely by complexity theorists.

**Example 32 (Hamiltonicity and 1-toughness).** Consider the following non-Hamiltonian graph which is 1-tough:

![Graph](image)

**Theorem 33 (Chrátal, 1973).** For every $t < 3/2$ there exist non-Hamiltonian, $t$-tough graphs.

**Proof.** We start by picking a large $n$ such that $3n/(2n+1) > t$. Now, build a suitable $G$ from 3 sets:
A clique $U$ of size $n$ connected to everything.

- A clique $R$ of size $2n + 1$.
- An independent set $T$ of size $2n + 1$ perfectly matched with $R$.

First, see that $G$ is not Hamiltonian.

Now, $G$ is $3n/(2n + 1)$-tough. To get more components, we need to delete the whole of $U$ (as $U$ is connected to every vertex). Once we are done with this, observe that deleting vertices of $T$ does not produce any new components while deleting vertices of $R$ creates a new component (an isolated vertex of $T$). Therefore, the number of components $k$ of $G - S$ where $U \subseteq S$ and $T \cap S = \emptyset$ is maximal and $k + 1 = |R \cap S|$.

Distinguish two cases:

- If $k \leq 2n$, then
  \[
  \frac{|S|}{c(G - S)} = \frac{n + k}{k + 1} \geq \frac{3n}{2n + 1}
  \]

- If $k = 2n + 1$, we lose the $R$-based component and get
  \[
  \frac{|S|}{c(G - S)} = \frac{3n + 1}{2n + 1} \geq \frac{3n}{2n + 1}
  \]

It is conjectured that there exists some real $t^*$ such that every $t^*$-tough graph is Hamiltonian (and we know that $t^* \geq 2.25$).

Our next topic is Hamiltonicity of planar graphs (the problem is still NP-complete). Tait conjectured in 1880 that every planar, 3-regular cubic graph is Hamiltonian (which implies the four color theorem which in turn was proven in 1976). This conjecture is false and a counterexample on 46 vertices was given by Tutte in 1946.

**Theorem 34** (Grinberg, 1968). Let $G$ be planar and Hamiltonian. Let $\phi'_i$ ($\phi''_i$ respectively) denote the number of faces of degree $i$ (the degree of a face is the number of edges forming its border) inside (outside resp.) the Hamiltonian cycle. Then,

\[
\sum_{i=3}^{n} (\phi'_i - \phi''_i)(i - 2) = 0
\]

**Proof.** Let $E'$ be the set of edges inside the cycle and let $m' = |E'|$. There are $m' + 1$ faces inside the cycle (as every edge splits exactly one existing face) and thus,

\[
\sum_{i=3}^{n} \phi'_i = m' + 1
\]

and hence, we have

\[
\sum_{i=3}^{n} i \cdot \phi'_i = n + 2m'
\]

as every inner edge is counted twice and each edge of the Hamiltonian cycle (of which there are $n$) is counted once. Now, using the above sums we get

\[
\sum_{i=3}^{n} (i - 2)\phi'_i = n - 2
\]
The same argument (view inner edges as outer edges and vice versa) gives us the same results for the outside ($\phi''_i$):

$$\sum_{i=3}^{n} (i - 2)\phi''_i = n - 2$$

and thus we get

$$\sum_{i=3}^{n} (i - 2)\phi'_i - \sum_{i=3}^{n} (i - 2)\phi''_i = \sum_{i=3}^{n} (i - 2)(\phi'_i - \phi''_i) = 0$$

Example 35 (Using Grinberg’s theorem). Adding this later, maybe. It’s somewhat big.
5 Connectivity

Let us recall some definitions:

- A (vertex) set \( S \subseteq V \) is a vertex cut (or cut set, separating set) if the subgraph induced by \( V \setminus S \) is disconnected.
- A graph \( G \) is \( k \)-connected if every vertex cut \( S \) has size at least \( k \).
- The connectivity \( \kappa(G) \) is the largest \( k \) such that \( G \) is \( k \)-connected.

**Example 36** (Connectivity of selected graphs). Consider the following graphs (or rather, graph classes):

- For a tree \( T \) (on at least 3 vertices) we have \( \kappa(T) = 1 \), since deleting any inner vertex disconnects the graph.
- For a cycle \( C_n \) (on at least 3 vertices) we have \( \kappa(C_n) = 2 \), since deleting just one vertex does not disconnect the graph, but deleting two non-adjacent ones does.
- It is \( \kappa(K_{s,t}) = \min\{s,t\} \) since we have to delete at least one of the partition classes (i.e. the smaller one) to disconnect the graph.
- For \( K_n \), we set \( \kappa(K_n) = n - 1 \). This is a definition, as no vertex cut will disconnect \( K_n \).

**Definition 37** ((Strictly) independent paths). We call two paths independent if they share no inner vertices (i.e. we allow the paths to have the same endpoints) and strictly independent if they share no vertices.

**Definition 38** (\( X-Y \)-paths, linkage, detaching sets). Let \( X, Y \subseteq V \).

- An \( X-Y \)-path is a path from an \( x \in X \) to an \( y \in Y \) using no other vertices from \( X, Y \).
- We call \( X \) and \( Y \) linked if \( |X| = |Y| \) and there exist \(|X|\) strictly independent \( X-Y \)-paths.
- A set \( Z \subseteq V \) detaches \( X \) from \( Y \) if the subgraph induced by \( V \setminus Z \) has no \( X-Y \)-path.

**Lemma 39.** Let \( X, Y \subseteq V \) for some graph. We see:

- \( X \) detaches \( X \) from \( Y \).
- If \( X \cap Y \neq \emptyset \), then \( X \cap Y \) is a subset of every detaching set of \( X \) and \( Y \).
- If \( Z \) is a detaching set of \( X \) and \( Y \) and \( X \cap Z \neq \emptyset \) and \( Y \cap Z \neq \emptyset \), then \( Z \) is a vertex cut.

**Theorem 40** (Technical theorem). Let \( Z \) of size \( k \geq 1 \) be a detaching set of \( X \) and \( Y \) in \( G \) of smallest size. Then there exist \( X_0 \subseteq X, Y_0 \subseteq Y \) with \( |X_0| = |Y_0| = k \) and \( X_0 \) is linked to \( Y_0 \).
Proof. By induction on $|E|$. The base case for $|E| = 1$ is easy. For the inductive step, we do a case distinction:

For the first case, we assume that the only detaching sets of size $k$ are $X$ and (or) $Y$. Without loss of generality we may assume that $|X| = k$ and $Y \not\subseteq X$ (else, the statement trivially holds by using paths of length 0). Now, let $x_0 \in X \setminus Y$. Then, if $X \setminus \{x_0\}$ does not detach $X$ from $Y$, there must be some edge $[x_0, v_0]$ with $v_0 \not\in X$ (as there must still remain a path from $x_0$ to $Y$). Define $G_0 = G - [x_0, v_0]$ and let $Z_0$ be the smallest detaching set for $X$ and $Y$ in $G_0$.

If $|Z_0| = k$, we can use the inductive hypothesis to get the desired statement. If not, we must have $|Z_0| = k - 1$ (since we assumed that the detaching set was minimal, and we only deleted one edge). Now, we know that $Z \cup \{x_0\}$ and $Z \cup \{v_0\}$ are detaching sets in $G$ (for $X$ and $Y$; recall that we are performing induction on $|E|$). But now, we have that $Z \cup \{x_0\} = X$ (as $x_0 \in X \setminus Y$) and $Z \cup \{v_0\} = Y$ (as $v_0 \not\in X$) by the fact that $X$ and (or) $Y$ are the only detaching sets of size $k$ in $G$. We now see that there are $k - 1$ trivial paths in $Z_0$ and together with the $X$-$Y$-path given by $[x_0, v_0]$, we get that $X = X_0$ and $Y = Y_0$ are linked.

For the second case, we assume that there exists some detaching set $Z$ of size $k$ with $X \neq Z \neq Y$. Let $G_1$ and $G_2$ respectively be the subgraph of $G$ induced by the vertices $X \setminus Y$ (and $Z \setminus Y$ respectively). There must be at least $k$ vertices in $G_1$ to detach $X$ from $Z$ and by the inductive hypothesis there exists some $X_0 \subseteq X$ linked to $Z$ in $G_1$. We get a similar result for $G_2$ and thus, we get that there exist $X_0, Y_0$ of size $k$ that are linked via $Z$.

**Theorem 41** (Menger, 1927). Let $x \neq y$ be vertices of a graph with $[x, y] \notin E$ and let $k$ be the smallest size of a vertex cut separating $x$ from $y$. Furthermore, let $l$ be the maximum number of independent $x$-$y$-paths. Then, $k = l$.

**Proof.** Use the technical theorem with $X = \Gamma(x)$ and $Y = \Gamma(y)$. \qed

**Theorem 42** (Marriage theorem, Hall, 1935). Let $G$ be a bipartite graph with the partition classes $X$ and $Y$. If $|\Gamma(A)| \geq |A|$ holds for all $A \subseteq X$, then $G$ has a matching of size $|X|$.

**Proof.** Consider the smallest set $Z$ detaching $X$ from $\Gamma(X)$ and $Z_1 \subseteq X, Z_2 \subseteq \Gamma(X)$. Now, if we have $|Z| < |X|$, then $\Gamma(X \setminus Z_1) \subseteq Z_2$ and thus $|\Gamma(X \setminus Z_1)| \leq |Z_2|$. Furthermore, we have $|X \setminus Z_1| = |X| - |Z_1| > |Z| - |Z_1| = |Z_2|$ and thus, $A = X \setminus Z_1$ violates the condition of the theorem.

Hence, the smallest detaching set of $X$ and $\Gamma(X)$ has at least $|X|$ elements. Using the technical theorem completes the proof (since the paths between $X$ and $\Gamma(X)$ are edges). \qed

**Theorem 43** (Menger, edge version). Let $x \neq y$ be vertices of a graph with $[x, y] \notin E$ and let $k$ be the smallest size of an edge cut separating $x$ from $y$. Furthermore, let $l$ be the maximum number of edge-disjoint $x$-$y$-paths. Then, $k = l$.

**Proof.** Define the line graph $L(G)$ of $G$ to be the graph resulting from creating an vertex for every edge of $G$ and connecting two of these vertices if the corresponding edges share an endpoint.
Apply the technical theorem to the line \( L(G) \) of \( G \) and set \( X \) and \( Y \) to the vertex sets representing edges incident to \( x \) and \( y \) respectively.

**Lemma 44.** Let \( G \) be \( k \)-connected and let \( H \) result from \( G \) by adding a new vertex \( y \) and connecting it to \( k \) vertices in \( G \). Then \( H \) is \( k \)-connected.

**Proof.** We distinguish two cases. For the first one, we assume that \( H = K_{k+1} \) and the definition of connectivity gives \( \kappa(K_{k+1}) = k \).

For the second case, suppose \( S \) is a vertex cut of size \( k - 1 \) in \( H \). We again get two cases:

1. We have \( y \in S \). But then, \( S \setminus \{y\} \) is a vertex cut of size \( k - 2 \) in \( G \), which is a contradiction.

2. We have \( y \notin S \). Then \( S \) cannot disconnect \( y \) as it has \( k \) neighbors and thus, \( S \) would have to be a vertex cut for \( G \), which is a contradiction (\( G \) is \( k \)-connected).

**Theorem 45.** A graph \( G \) is \( k \)-connected if and only if for every \( k \)-element subset \( U \subseteq V \) and for every \( x \in V \setminus U \) there exist \( k \) independent \( x-U \)-paths only intersecting at \( x \).

**Proof.** To see that the first statement implies the second one, create a new vertex \( y \) connected to each vertex in \( U \) and thus, Menger’s theorem and the previous lemma show the claim. The other direction follows by the fact that there are \( k \) independent paths between \( x \) and \( \Gamma(y) \).

**Theorem 46.** Let \( G \) be a graph with at least 3 vertices. The following statements are equivalent:

1. \( G \) is 2-connected.
2. \( G \) has no cut vertex (a vertex cut of size 1).
3. Any two vertices of \( G \) lie on a common cycle.
4. Any vertex and edge lie on a common cycle.
5. Any two edges lie on a common cycle.

We will give no proof for this theorem, but at this point you should be able to very easily do it yourself.

**Theorem 47.** If \( G \) is \( k \)-connected, then any \( k \) vertices lie on a common cycle.

**Proof idea.** By induction on \( k \). The base case for \( k = 2 \) is already established by the previous theorem.

For the inductive step, we use the previous lemma. Let \( G \) be \( k \)-connected and \( S \subseteq V \) with \( |S| = k \) and \( x \in S \). Then the \( k - 1 \) vertices in \( S \setminus \{x\} \) lie on a common cycle and there exist \( k \) independent \( x-(S \setminus \{x\}) \)-paths. But since \( S \setminus \{x\} \) has only \( k - 1 \) elements, there must be 2 paths from \( x \) to \( S \setminus \{x\} \) whose first intersection with the common cycle of \( S \setminus \{x\} \) lie in the same segment of the cycle. Now, we get the common cycle for \( S \) by breaking the cycle for \( S \setminus \{x\} \) at those intersection points and adjoining the paths to \( x \).

Note that this theorem only states that \( k \)-connectivity implies the existence of such a cycle and the converse statement which holds for \( k = 2 \) is not true for \( k \geq 3 \). For a counterexample, consider \( C_k \) which is only 2-connected, but all its elements lie on a common cycle.
6 Planarity and Kuratowski’s theorem

We start by going over some definitions:

**Definition 48** (Planar graph). A graph $G$ is called planar if it can be embedded in a plane without any two edges intersecting.

Let us recall some facts:

**Lemma 49** (Euler’s formula). Let $G$ be a planar graph with $n$ vertices, $m$ edges and $f$ faces (areas enclosed by at least 3 edges plus the remaining plane, the so-called infinite face). Then:

- If $G$ is connected, $n + f = m + 2$ holds. In general, we have $n + f = m + c + 1$ where $c$ is the number of components of $G$.
- Any face corresponds to at least 3 edges (which can be used twice) and thus $2m ≥ 3f$.
- In general, $m ≤ 3n − 6$ holds (previous bounds) and $m ≤ 2n − 4$ holds if $G$ is bipartite (as bipartite graphs contain no triangles, we have $2m ≥ 4f$).

**Example 50** (Non-planar graphs). The graphs $K_5$ ($n = 5$, $m = 10$) and $K_{3,3}$ ($n = 6$, $m = 9$) are not planar by the above inequalities,

![Graphs K5 and K3,3](image)

while the graphs resulting from taking one of these two graphs and removing one edge are planar.

Note that every graph obtained by subdividing edges of a non-planar graph is in turn non-planar, and of course a graph containing a non-planar subgraph is not planar either. This hereditary property gives rise to a theorem characterizing planarity via forbidden subgraphs (namely subdivisions of the examples above) which takes up the remainder of this chapter.

**Theorem 51** (Kuratowski’s theorem, Kuratowski, 1930). A graph $G$ is planar if and only if it neither contains a subdivision of $K_5$ nor one of $K_{3,3}$ as a (not necessarily induced) subgraph.

The proof of Kuratowski’s theorem is given using seven auxiliary lemmata:

**Lemma 52** (Auxiliary lemma 1). Let $G$ be planar and let $F$ denote the set of edges on the boundary of some face in some planar embedding of $G$.

Then there exists a planar embedding of $G$ so that $F$ is the boundary of the infinite face.

*Proof idea.* Take some cylindrical cloth representing the graph (one of the holes represents the infinite face, the other one the face bounded by $F$) and turn it inside out.
**Lemma 53** (Auxilliary lemma 2). Let $G$ be minimally non-planar, i.e. removing any edge yields a planar graph. Then $G$ is 2-connected.

*Proof.* Without loss of generality, $G$ is connected (else we get the same contradiction for its components). Suppose that $G$ is not 2-connected, i.e. it has a cut-vertex $v$ and let $G_1, \ldots, G_k$ be the components of $G \setminus v$.

Now, all $G_i$ must be planar (as they are induced proper subgraphs of $G$ which was minimally non-planar) and we can embed $G_i \cup \{v\}$ planarily such that $v$ lies on the infinite face. Combining the embeddings of $G_1, \ldots, G_k$ and identifying their copies of $v$ gives a planar embedding of $G$, which is a contradiction. Thus, $G$ cannot have a cut vertex and must be 2-connected. \[\square\]

**Lemma 54** (Auxilliary lemma 3). Let $S = \{x, y\}$ be a vertex cut of $G$ and let $G_1, G_2$ be subgraphs of $G$ satisfying $G_1 \cap G_2 = S$ and $G_1 \cup G_2 = G$. Further, let $H_i = G_i \cup [x, y]$ for $i \in [1, 2]$.

If $G$ is non-planar, then at least one of $H_1$ and $H_2$ is non-planar.

*Proof.* If $H_1$ and $H_2$ can be planarly embedded, we can again get a planar embedding of $G$ choosing embeddings of $H_1$ and $H_2$ where the edge $[x, y]$ lies on the infinite face. Identifying $x$ and $y$ (and removing $[x, y]$ if needed) we can obtain a planar embedding of $G$, contradicting the non-planarity of $G$. \[\square\]

From now on, we refer to subdivisions of $K_5$ and subdivisions of $K_{3,3}$ as *Kuratowski subgraphs*.

**Lemma 55** (Auxilliary lemma 4). Suppose $G$ is non-planar and has no Kuratowski subgraph and is edge-minimal (i.e. the graph with the least edges exhibiting the desired properties). Then $G$ is 3-connected.

*Proof.* Suppose $G$ has vertex cut $x, y$. By the previous lemma (cf. Lemma 54) there exist subgraphs $H_1, H_2$ of $G$ and without loss of generality, $H_1$ is non-planar, but contains a Kuratowski subgraph.

Now, the edge $[x, y]$ is either not in the Kuratowski subgraph (which instantly yields a contradiction as then the same subgraph is a Kuratowski subgraph of $G$) or else, we can find some $x$-$y$-path in $H_2$ which corresponds to a subdivision of $[x, y]$ in $G$ and thus giving us a Kuratowski subgraph in $G$. Hence, $G$ cannot have a vertex cut of size 2 and is therefore 3-connected. \[\square\]

Note that at this point we have established that if there exists a counterexample $G$ to Kuratowski’s theorem, then $G$ must be 3-connected.

**Lemma 56** (Auxilliary lemma 5, Thomassen 1980). Every 3-connected graph $G$ with at least 5 vertices contains at least one edge $e$ such that the contraction $G/e$ (the graph resulting by combining the endpoints of $e$ into one vertex) is 3-connected.

*Proof.* Suppose not, i.e. for no edge $e$ the contraction $G/e$ is 3-connected. Then $G/e$ has a vertex cut of size 2 and one vertex of the cut is the contraction vertex of $e = [x, y]$ (recall that $G$ is 3-connected) while the other vertex is the so-called *companion vertex*.

Pick $x, y$ and the companion $z$ so that $G \setminus \{x, y, z\}$ has a component $H$ of largest possible size. Let $u$ be a neighbor of $z$ which is not in $H$ and $v$ a companion of $[u, z]$ (note
that $\{u, v, z\}$ is a vertex cut as well – else there is no such $v$ to form a vertex cut with the contraction of $[u, z]$ and we are done).

We distinguish two cases:

1. We have $v \notin H$: Then there exists a larger component $H'$ in $G \setminus \{u, v, z\}$ containing $H$ and at least one of $x$ or $y$.

2. We have $v \in H$: Then $(H \cup \{x, y\}) \setminus \{v\}$ is connected (otherwise $\{u, z\}$ is a vertex cut in $G$, which is 3-connected).

So in both cases we get a contradiction by the maximality of $H$ and thus, there must be some edge $e$ for which $G/e$ is 3-connected.

\textbf{Lemma 57} (Auxilliary lemma 6). If a contraction $G/e$ of a graph $G$ and one of its edges $e$ contains a Kuratowski subgraph, then so does $G$ itself.

\textbf{Proof.} Let $H$ be a Kuratowski subgraph in $G/e$ and let $z$ be the vertex resulting from contracting $e$ in $G$. Without loss of generality we may assume that $z$ is in $H$ (or else $H$ already exists in $G$ and we are done).

We distinguish three cases:

1. We have $\deg_H(z) \leq 2$. Then we can use $[x, y]$ to emulate $z$ in $G$ (in this case, $z$ must be on a path in the Kuratowski subgraph and replacing it with $[x, y]$ produces the same result as subdividing one of the edges of $z$).

2. We have $\deg_H(z) \geq 3$ and at most one edge incident to $z$ in $H$ is incident to $x$ in $G$. Again, we emulate $z$ by replacing it with $y$ in $G$ (and if needed, $x$ and $[x, y]$ as in the first case).

3. For the remaining cases, there are at least two edges incident to both $x$ and $y$ which are incident to $z$ in $H$. Thus, $H$ must be a subdivision of $K_5$ (as it has a vertex of degree 4, which a $K_{3,3}$-subdivision does not contain) and replacing $z$ by $[x, y]$ gives us a $K_{3,3}$-subdivision in $G$,

\begin{center}
\begin{tikzpicture}
  \node[vertex, fill=black] (z) at (0,0) {$z$};
  \node[vertex, fill=black] (x) at (0,1) {$x$};
  \node[vertex, fill=black] (y) at (0,-1) {$y$};
  \node[vertex, fill=black] (a) at (-1,0) {};\node[vertex, fill=black] (b) at (1,0) {};
  \node[vertex, fill=black] (c) at (-2,0) {};\node[vertex, fill=black] (d) at (2,0) {};
  \node[vertex, fill=black] (e) at (-3,0) {};\node[vertex, fill=black] (f) at (3,0) {};
  \draw (z) -- (x) -- (y);
  \draw (z) -- (a) -- (b);
  \draw (z) -- (c) -- (d);
  \draw (z) -- (e) -- (f);
  \node[vertex, fill=white, text=black] at (1.5,0) {Replace $z$ by $[x, y]$};
\end{tikzpicture}
\end{center}

where the bipartitions of the corners of the $K_{3,3}$-subgraph are given by the squares and circles.

\textbf{Lemma 58} (Auxilliary lemma 7). Every 3-connected graph without Kuratowski subgraph is planar.
Proof. We use induction on $n$. For $n = 4$, the only 3-connected graph is the $K_4$, which is planar.

So let $G$ be a 3-connected graph on $n \geq 5$ vertices without a Kuratowski subgraph. By the fifth auxiliary lemma (cf. Lemma 56), $G$ has some edge $e = [x, y]$ so that $G/e$ is still 3-connected. Let $z$ denote the vertex resulting from the contraction of $e$.

Then by the sixth auxiliary lemma (cf. Lemma 57), $G/e$ has no Kuratowski subgraph and is thus planar by the inductive hypothesis. We consider a planar embedding of $G/e$ and remove $z$ from the drawing and put $[x, y]$ back in. Let $C$ denote the boundary cycle of the resulting face and let $x_1, \ldots, x_k$ be the neighbors of $x$ on $C$ in a clockwise order. This gives us the following situation after reconnecting $x$ with its neighbors:

We distinguish four cases:

1. All neighbors of $y$ belong to the same segment of $C$, that is every neighbor $y_i$ of $y$ is contained between two consecutive neighbors of $x$ in the clockwise enumeration of the neighbors of $x$.

Then $G$ is obviously planar as we can just connect $y$ with straight edges to its neighbors to obtain a planar embedding of $G$.

2. There exist two neighbors $y_1, y_2$ of $y$ which are not in the same segment of $C$.

Then $G$ contains a Kuratowski subgraph, namely a $K_{3,3}$-subdivision,

which is a contradiction to the assumption that $G$ has no Kuratowski subgraph.

3. There are at least 3 shared neighbors of $x$ and $y$.

Then $x$ and $y$ are connected to each other and to all of the three shared neighbors (by edges which are not in $C$) which form a subdivided triangle with the edges in
C. Thus, $x, y$ and the three shared neighbors form a $K_5$-subdivision which again is a contradiction to the assumption that there is no Kuratowski subgraph in $G$.

4. There are at most 2 shared neighbors of $x$ and $y$. If the situation is not covered by the first case, we can find a $K_{3,3}$-subdivision similarly to the construction in the second case.

Proof of Kuratowski’s theorem (cf. Theorem 51). By the auxiliary lemmata 1 to 4 we know that if there exists a counterexample to Kuratowski’s theorem, then it must be 3-connected. Using the auxiliary lemmata 5 and 6, we can prove auxiliary lemma 7 which states that every 3-connected graph containing no Kuratowski subgraph is planar.

Hence no such counterexample to Kuratowski’s theorem can exist, and the proof is concluded.
7 Drawings and crossings

In this section, we want to study drawings of (planar and non-planar) graphs and crossings of edges in such drawings.

Formally, a drawing of a graph can be understood as a 2-tuple of functions \((f_V, f_E)\) where \(f_V : V \rightarrow \mathbb{R}^2\) maps the vertices to the points in the euclidean plane where they are drawn and \(f_E : E \rightarrow \text{Map}([0, 1], \mathbb{R}^2)\) maps the edges onto rectifiable curves in the euclidean plane (with the restriction of \((f_E([x, y]))(0) = f_V(x)\) and \((f_E([x, y])(1)) = f_V(y)\)).

**Theorem 59** (Fáry’s theorem, Fáry 1948, Wagner 1936). Every planar graph has a planar drawing such that all edges are straight line segments.

To prove Fáry’s theorem, we use two auxiliary lemmata.

**Lemma 60** (Auxilliary lemma 1). Every planar graph on at least 4 vertices contains 4 vertices of degree at most 5.

**Proof.** Without loss of generality we may assume that \(G\) is triangulated (i.e. we add as many edges as possible without making \(G\) non-planar), thus any vertex will have degree at least 3.

Since \(G\) is planar, it satisfies the inequality \(m \leq 3n - 6\) and we have

\[
2m = 6n - 12 = \sum_{v \in V} \deg(v) \geq 3k + 6(n - k)
\]

if we assume that \(G\) has \(k\) vertices of degree at most 5. Solving for \(k\) yields \(4 \leq k\).

**Lemma 61** (Art gallery theorem (special case for \(n = 5\)), Chvátal 1975, Auxilliary lemma 2). Every simple pentagon contains a point that sees all 5 corners.

**Proof.** Every simple pentagon can be triangulated into three simple triangles and there must exist some vertex which is adjacent to all three triangles and thus sees all 5 corners.

**Proof of Fáry’s theorem (cf. Theorem [59]).** Without loss of generality we may assume that \(G\) is triangulated. We show the stronger statement that for any face \(uvw\) there exists a straight line drawing that makes \(uvw\) the infinite face by induction on \(n\) (the case \(n = 4\) is easy; consider a top-down-projection of a tetrahedron and remove edges if needed):

We start by picking some vertex \(x \notin u, v, w\) with degree at most 5 (which is known to exist by the first auxiliary lemma) and remove it and retriangulate the resulting face. By the inductive hypothesis, there exists a straight line drawing for the resulting graph.

Now, we delete the added edges from the drawing and re-add \(x\). Since \(x\) has at most 5 neighbors, the face that we place \(x\) in is a pentagon (or a degenerated pentagon) and we can use the second auxiliary lemma to embed \(x\) so that no two edges of \(x\) cross.

**Definition 62** (Crossing number). Consider a (not necessarily planar) drawing of a graph. In said drawing, two edges cross if they intersect at any non-vertex point in \(\mathbb{R}^2\).

The crossing number \(\text{cr}(G)\) of a graph \(G\) is defined as the minimal number of crossings in a drawing of \(G\).
Example 63 (Crossing number).

1. Obviously, we have \( cr(G) = 0 \) if and only if \( G \) is planar.

2. As mentioned previously, deleting any edge from the \( K_5 \) or the \( K_{3,3} \) makes them planar and as such, \( cr(K_5) = cr(K_{3,3}) = 1 \).

Next up is a normalization result for crossing-minimal drawings.

**Definition 64** (Simple drawing). A drawing of \( G \) is simple if

1. Any two edges cross at most once.
2. If two edges are incident to the same vertex they do not cross.
3. No three edges cross in a common point.

**Theorem 65.** Every graph \( G \) has a simple, crossing-minimal drawing (i.e. a simple drawing with \( cr(G) \) crossings).

*Proof idea.* Starting from any crossing-minimal drawing of \( G \), iteratively construct a drawing satisfying the above definition. Indeed, a crossing-minimal drawing of \( G \) cannot violate the properties 1 and 2 (as one could easily remove crossings then) and property 3 can be satisfied by replacing a \( n \)-multicrossing by \( \binom{n}{2} \) pairwise crossings. \( \Box \)

This theorem allows us to use the term “drawing” instead of “simple drawing” without any loss of generality.

Our next result gives an (tight in the sense of the \( \Theta \)-notation) asymptotic for the crossing number of the class of complete graphs, but first let us study an example from said class:

**Example 66** (Crossing number of \( K_6 \)). We conjecture that \( cr(K_6) \) is equal to 3.

If we had \( cr(K_6) \leq 2 \), then we could produce a planar graph from \( K_6 \) by removing two edges, but the resulting graph obviously has a \( K_{3,3} \)-subgraph (by choosing the bipartitions in such a manner that the deleted edges run between elements of the same bipartition). Another way to see that this cannot be the case is by using the edge bound \( m \leq 3n - 6 \) for planar graphs, since a \( K_6 \) with two deleted edges has \( \binom{6}{2} - 2 = 13 \) edges, which is more than \( 3n - 6 = 12 \).

On the other hand, we can show that \( cr(K_6) \leq 3 \) by giving a drawing of \( K_6 \) with three crossings:

![Diagram of K6 with three crossings](image)

And thus, we have \( cr(K_6) = 3 \).
A conjecture about the asymptotic behaviour of $K_n$ is easily formulated as “for large $n$, almost all edges will cross, giving $\text{cr}(K_n) \in \Theta\left(\left(\frac{|E|}{2}\right)\right) = \Theta(n^4)” and indeed, this simple idea turns out to be correct:

**Theorem 67** (Guy 1972). The crossing number for the class of complete graphs satisfies $\text{cr}(K_n) \in \Theta(n^4)$.

**Proof.** It is easy to see that $\Theta(n^4)$ is an upper bound by the fact that we can place all $n$ vertices on a cycle with the edges running through the middle. Such a drawing produces exactly one crossing for every 4-combination of vertices, of which there are $\binom{n}{4} \in \Theta(n^4)$.

To show that $\Theta(n^4)$ also constitutes a lower bound, note that any drawing of the $K_n$ contains $n$ drawings of the $K_{n-1}$. Further, every crossing of the drawing of $K_n$ occurs in exactly $n - 4$ of the $K_{n-1}$-drawings (i.e. in all such drawings which contain all four vertices which the two crossing edges are incident to) which gives us the recurrence relation

$$\text{cr}(K_n) \geq \frac{n}{n-4} \text{cr}(K_{n-1})$$

This in turn gives us (recall $\text{cr}(K_5) = 1$)

$$\begin{align*}
\text{cr}(K_n) &\geq \frac{n}{n-4} \cdot \frac{n-1}{n-5} \cdot \frac{6}{2} \text{cr}(K_5) \\
&= \frac{1}{5} \cdot \frac{n}{n!} \cdot 4!(n-4)! \\
&= \frac{1}{5} \cdot \left(\frac{n}{4}\right) \in \Theta(n^4)
\end{align*}$$

It is known that

$$\text{cr}(K_n) \leq \frac{1}{4} \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor$$

and it is conjectured to be tight for all $n$ (but only proven for $n \leq 12$).

Note that Fáry’s theorem (cf. Theorem 59) does not make a statement for non-planar graphs (i.e. graphs with crossing number not equal to zero) and in fact, it cannot be generalized in the sense that not all graphs have a straight-line drawing with an optimal number of crossings (w.r.t. all possible drawings). For example, it is known that the crossing number of the $K_8$ is $\text{cr}(K_8) = 18$, but its straight-line crossing number is $\text{cr}(K_8) = 19$.

Our next result attempts to give a closed form for the crossing number of complete bipartite graphs (called the Turán brick factory problem):

**Theorem 68** (Zarankiewicz 1954). The crossing number for the class of complete bipartite graphs satisfies

$$\text{cr}(K_{s,t}) \leq Z(s,t) = \left\lfloor \frac{s}{2} \right\rfloor \cdot \left\lfloor \frac{s-1}{2} \right\rfloor \cdot \left\lfloor \frac{t}{2} \right\rfloor \cdot \left\lfloor \frac{t-1}{2} \right\rfloor$$

**Proof.** We only prove the case where $s$ and $t$ are even (the others are similar), i.e. we have $s = 2a$ and $t = 2b$ for some $a, b$. 

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Then we have
\[
Z(s, t) = Z(2a, 2b) = a \cdot (a - 1) \cdot b \cdot (b - 1) = 4 \binom{a}{2} \binom{b}{2}
\]
and a simple drawing using this many crossings can be constructed by placing the 2a
s-vertices on the y-axis (a above the origin, a below) and the 2b t-vertices on the x-axis
(again b left of the origin, b right) and connecting them.

Thus there are exactly \( \binom{a}{2} \binom{b}{2} \) crossings in each quadrant of the euclidean plane, totaling
\[
Z(s, t) = 4 \binom{a}{2} \binom{b}{2}
\]
crossings.

For some cases, we are actually able to prove equality:

**Example 69** (Crossing number of \( K_{2,n} \)). We have \( cr(K_{2,n}) = 0 \) by the previous theorem
(cf. Theorem 68), and indeed any such graph has a planar drawing:

![Diagram of a planar drawing of \( K_{2,n} \)]

**Theorem 70.** For all \( n \) we have \( cr(K_{3,n}) = Z(3, n) = \lfloor n/2 \rfloor \lfloor (n - 1)/2 \rfloor \).

**Proof.** Consider some simple drawing of \( K_{3,n} \) with \( cr(K_{3,n}) \) crossings and let \( A \) and \( B \)
be its bipartition with \( |A| = 3 \) and \( B = n \). Define the auxiliary graph \( H \) with vertex
set \( B \) and put an edge between \( x \) and \( y \) if and only if the \( K_{2,3} \) induced by \( A \cup \{x,y\} \)
is drawn crossing-free in our drawing.

It is easy to see that \( H \) must have at least \( \binom{n}{2} - cr(K_{3,n}) \) edges (as no crossing in our
drawing can eliminate more than one edge from \( H \)). Further, \( H \) must be triangle-free
or there would exist three vertices \( a, b \) and \( c \) such that \( A \cup \{a, b, c\} \) form a \( K_{3,3} \) (induced
from \( K_{3,n} \)) which is drawn crossing-free. By Mantel’s theorem (cf. Theorem 13), \( H \) has
at most \( \lfloor n^2/4 \rfloor \) edges and we get (after solving for \( cr(K_{3,n}) \))

\[
\begin{align*}
cr(K_{3,n}) & \geq \binom{n}{2} - \lfloor n^2/4 \rfloor \\
& \geq \binom{n}{2} - \frac{n^2}{4} \\
& = \frac{n^2}{4} - \frac{n}{2} \\
& = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor
\end{align*}
\]

It is known that the Zarankiewicz bound (cf. Theorem 68) is tight for \( K_{s,t} \) for \( s \leq 6 \)
and arbitrary \( t \) and for \( (s, t) = (7, 9) \).
**Observation 71.** For all \( n \), we have \( \text{cr}(K_{4,n}) \geq 2 \text{cr}(K_{3,n}) \).

*Proof.* Any drawing of the \( K_{4,n} \) contains four drawings of \( K_{3,n} \) and every crossing occurs in exactly two of those four copies, yielding the desired inequality.

**Lemma 72.** For any graph \( G \) on \( n \) vertices and \( m \) edges, we have

\[
\text{cr}(G) \geq m - 3n + 6
\]

*Proof.* We use induction on \( f(G) = m - 3n + 6 \). For \( f(G) \leq 0 \), the inequality trivially holds (recall that \( m \leq 3n - 6 \) for planar graphs).

So, let \( f(G) \geq 1 \) and pick a crossing-minimal drawing of \( G \). Let \( e \) be an edge involved in a crossing to get \( G' \), which satisfies \( f(G') = f(G) - 1 \) and \( \text{cr}(G') \leq \text{cr}(G) - 1 \). But then we have

\[
\begin{align*}
\text{cr}(G) & \geq \text{cr}(G') + 1 \\
& \geq f(G') + 1 \\
& = f(G)
\end{align*}
\]

This lemma is intuitively clear after considering Euler’s formula – adding an edge to an edge-maximal planar graph produces a crossing, and so does the addition of further edges by the same idea.

**Theorem 73** (The Crossing Lemma, Ajtai, Chvátal, Newborn, Szemerédi 1982). Every graph \( G \) with \( m \geq 4n \) satisfies \( \text{cr}(G) \geq m^3 / 64n^2 \)

*Proof (by Alon 1990).* Fix a simple drawing of \( G \) with \( \text{cr}(G) \) crossings. Consider a random subset \( V' \subseteq V \) which contains every vertex with probability \( p \). Then the resulting induced subgraph \( G[V'] \) satisfies

1. \( \mathbb{E}[|V'|] = np \), and
2. \( \mathbb{E}[|E'|] = mp^2 \).

Now, the expected number of crossings in \( G[V'] \) is

\[
\mathbb{E}[\text{cr}(G[V'])] = p^4 \text{cr}(G)
\]

by the fact that a crossing of \( G \) is present in \( G' \) if and only if the four end vertices of the crossing edges are in \( V' \).

We know by the previous lemma (cf. Lemma 72) that

\[
p^4 \text{cr}(G) \geq \mathbb{E}[\text{cr}(G[V'])] \\
\geq mp^2 - 3np
\]

holds. If we now pick \( p = 4n/m \leq 1 \) we get the desired inequality by plugging in \( p \) (since there exists at least one such subset \( V' \) which has the expected number of crossings or more).

**Theorem 74** (Szemerédi, Trotter 1983). Let \( P \) be a set of \( n \) points in the euclidean plane \( \mathbb{R}^2 \) and \( L \) be a set of \( m \) lines in \( \mathbb{R}^2 \). Then the number \( t \) of point-line-incidences satisfies

\[
t \in O(n^{2/3}m^{2/3} + n + m)
\]
Proof (by Székely 1996). Define a graph $G$ on the vertex set $P$. Put an edge between two points $p_1, p_2 \in P$ if and only if $p_1$ and $p_2$ lie consecutively along some line in $L$.

We have $|E| \geq t - m$ and trivially $cr(G) \leq \binom{m}{2} \leq m^2/2$. We distinguish two cases:

1. We have $|E| < 4|V|$. Then $t - m < 4n$ (by $|E| > t - m$) and thus, $t < m + 4n \in O(n^{2/3}m^{2/3}n + m)$.

2. We have $|E| \geq 4|V|$. Thus by the crossing lemma (cf. Theorem 73) we get

$$\frac{m^2}{2} \geq cr(G) \geq \frac{(t - m)^3}{64n^2}$$

and hence we have $32m^2n^2 \geq (t - m)^3$, giving us $t \leq 32^{1/3}m^{2/3}n^{2/3} + m \in O(n^{2/3}m^{2/3}n + m)$.

Finally, we look at an application of the theory we developed.

**Example 75.** Consider two subsets $A$ and $B$ of $\mathbb{R}$ of size $s$.

1. How many elements does $A + B = \{a + b \mid a \in A, b \in B\}$ contain?

Some number between $cs$ (for some constant $c$, consider $A = B$ the integers 1 to $s$) and $s^2$ (consider $A$ as the integers 1 to $s$ and $B$ the fractions $1/2$ to $1/(s + 1)$).

2. What about $A \cdot B = \{ab \mid a \in A, b \in B\}$?

Again, as in the first case (consider $A = B$ the set of the first $s$ powers of 2 for the lower bound and two disjoint sets of primes for the second one).

What about $A \cdot B + C$? The upper bound is again easy (at $s^3$), but the lower bound requires a more subtle analysis:

**Theorem 76** (Elekes 1997). Let $A, B, C \subseteq \mathbb{R}$ with $|A| = |B| = |C| = s$ and let $R = \{ab + c \mid a \in A, b \in B, c \in C\}$.

Then $r = |R| \in \Omega(s^{3/2})$.

**Proof.** Let $P = \{(a, q) \mid a \in A, q \in R\}$ and $L = \{y \mid y = bx + c$ with $b \in B, c \in C\}$. For the sizes of these sets, we get $|L| = |B \times C| = s^2$ and $|P| = |A \times R| = rs$.

Now, every line in $L$ contains $s$ points of $P$ (of the form $a_i b + c = q$), so the number of total point-line-incidences is $s^3$. Using the previous theorem (cf. Theorem 74), we get that there exists some constant $c$ such that

$$s^3 \leq c(m^{2/3}n^{2/3} + m + n) = c(s^{4/3}r^{2/3} + s^2 + rs)$$

which implies that $s \leq cr^{2/3}$ and thus $r \in \Omega(s^{3/2})$. 

\[\square\]
8 Sperner’s lemma and bandwidth

Definition 77 (Simplicial subdivision). A simplicial subdivision of a triangle $T$ is a partition of $T$ into triangular cells so that the intersection of two cells is a common vertex, a common edge or empty.

Definition 78 (Proper labelling). A proper labelling of a simplicial subdivision assigns every vertex a label $\ell \in \{0, 1, 2\}$ such that each corner of the triangle has a different label and the label of a corner does not show up on the edge opposing it.

Example 79. A properly labelled simplicial subdivision might look something like this:

![Diagram of a properly labelled simplicial subdivision]

The above example contains three cells whose vertices are labelled 0-1-2, and indeed this is not a coincidence. In fact, any properly labelled simplicial subdivision has at least one such cell:

Lemma 80 (Sperner’s lemma, Sperner, 1928). Every properly labelled simplicial subdivision $T$ contains a cell labelled 0-1-2.

Proof. We start by constructing an auxiliary graph (based on the dual graph of $T$) $G$, which has a vertex for every face of $T$ (including the infinite face) and putting an edge between two vertices if and only if their associated cells share an edge which is labelled 0-1.

Now, the vertices of cells with no 0-1-edge have degree 0 in $G$ and vertices of cells labelled 0-0-1 or 0-1-1 have degree 2, while cells labelled 0-1-2 have degree 1. Further, the infinite face has odd degree. To see this, it obviously suffices to consider outer border path of $T$ connecting 0 and 1 (which can only contain vertices labelled 0 or 1). Now, adding any vertex to this path either adds no or two edges to the vertex of the infinite face in $G$, which one can prove by considering the two neighbors of the added vertex $v$:

1. Both neighbors are labelled 1 (or 0 by symmetry). Then we either add 0 edges (if $v$ is labelled 1) or 2 (if $v$ is labelled 0).

2. One neighbor is labelled 0, the other one is labelled 1. Then we remove one edge and add one back, as the new path has the form $0 - 0 - 1$ or $0 - 1 - 1$ and the total number of edges is unchanged.

Since there exists at least one vertex of odd degree, there must exist an odd number of 0-1-2-labelled cells (as they are the only other cells to produce vertices of odd degree in $G$) by the handshaking lemma.
Sperner’s lemma has a number of (maybe surprising) applications all over mathematics. Its topological counterpart is *Brouwer’s fixed-point theorem*, which can be proven using Sperner’s lemma:

**Theorem 81** (Brouwer’s fixed-point theorem, Hadamard, Brouwer 1910). Let $T$ be a simply connected region in $\mathbb{R}^2$. Then every continuous function $f : T \to T$ has at least one fixed point.

**Proof.** By homotopy, it suffices to consider triangular regions, so without loss of generality let $T$ be a triangular region in $\mathbb{R}^2$. We embed $T$ in $\mathbb{R}^3$ so that its corners have the coordinates $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$ (barycentric coordinates) and its edges are the direct connections of these points. Further, we want $T$ to be embedded on the plane described by $x + y + z = 1$, i.e. the plane spanned by the edges of $T$.

Now, define the subsets $S_0, S_1$ and $S_2$ of $T$ with $S_0 \cup S_1 \cup S_2 = T$ to have the following property: Let $x \in T$ with $f(x) = y$, then $x \in S_i$ if and only if $y_i \leq x_i$ (note that $x_i < y_i$ cannot hold for all $i$ since the coordinates of $x$ and $y$ both sum to 1 and thus every $x$ is in at least one set $S_i$).

By definition, $x$ is a fixed point of $f$ if and only if it is contained in all sets $S_i$, i.e. $x \in S_0 \cap S_1 \cap S_2$. Also, notice that the sets $S_i$ are closed as it contains its limit points (as we defined it using $\leq$ as opposed to $<$).

We now define a sequence of subdivisions of $T$ by dividing $T$ using $2^k - 1$ equidistant lines parallel to each side:

As $k \to \infty$, the diameter of the cells approaches 0. We label each inner intersection point $x$ on the grid by $i \in 0, 1, 2$ where $i$ is the smallest number so that $x \in S_i$ and choose a labelling for the outer intersection points to obtain a proper labelling.

Now, Sperner’s lemma yields a 0-1-2-labelled triangular cell (say $A_kB_kC_k$) in each subdivision and there exists some point $X^*$ such that there exist subsequences of $(A_k)$,
(\(B_k\)) and (\(C_k\)) which converge to \(x^*\) as the diameter of the cells converges to 0. As \(A_k B_k C_k\) is labelled 0-1-2, the limit point \(x^*\) carries all three labels and hence we have \(x^* \in S_0 \cap S_1 \cap S_2\) and \(x^*\) is a fixed point of \(f\) in \(T\).

Another application of Sperner’s lemma is in cake cutting:

**Definition 82** (Envy-free cake cutting). We model a cake as the interval \([0, 1]_{\mathbb{R}}\) and the desirability of certain parts of that cake for three players \(A, B\) and \(C\) as measures \(\mu_A, \mu_B\) and \(\mu_C\) which are non-negative, additive and continuous (and thus especially non-atomic) on \([0, 1]_{\mathbb{R}}\).

An *envy-free allocation* is a partition of \([0, 1]_{\mathbb{R}}\) into intervals \(A, B, C\) such that every player is satisfied with his part of the cake (according to his measure), i.e. the choice of \(A, B\) and \(C\) satisfies

\[
\int_A \mu_A(x) \, dx \geq \int_B \mu_A(x) \, dx \quad \text{and} \quad \int_A \mu_A(x) \, dx \geq \int_C \mu_A(x) \, dx
\]

and the analogue criterions for \(B\) and \(C\).

**Theorem 83.** There always exists an envy-free allocation of a cake.

*Proof.* We encode the three pieces of the unit interval by their euclidean length (note that it makes no difference which player recieves the exact cutting point). An envy-free allocation might look like this:

![Diagram of a cake cut into three parts with measures \(\mu_A, \mu_B, \mu_C\) and lengths \(|A| = 1/3\), \(|B| = 5/12\), \(|C| = 1/4\).]

Thus, we have \(|A| + |B| + |C| = 1\) and we reuse the concept of modelling the space of possible allocations in barycentric coordinates (in the \((0, 0, 1)-(0, 1, 0)-(1, 0, 0)\)-triangle) from the proof of Brouwer’s theorem (cf. Theorem 81). Every intersection point in our grid is now owned by one of the players \(A, B\) or \(C\) so that every cell has one vertex for each player which is owned by said player.

We now label every intersection point with the index of the most-preferred piece of owner of the vertex (i.e. ask the owner of the vertex which of the pieces is the largest according to his measure). This gives us a proper labelling (on the edges of the triangle we always have one empty piece and two empty pieces on its corners), and we can apply Sperner’s lemma like in the proof of Brouwer’s theorem to obtain a limit point which carries all three labels, i.e. we get a partition of the cake where every player prefers a different piece. \(\square\)
Next, we study the bandwidth of graphs.

**Definition 84** (Dilation, Bandwidth). Let $G$ be a graph on $n$ vertices.

1. For a bijection $f : V \to [1, n]$, the *dilation* of $G$ under $f$ is defined as
   \[ \text{dil}(G, f) = \max_{[u, v] \in E} |f(u) - f(v)| \]

2. The *bandwidth* of $G$ is
   \[ \text{bw}(G) = \min_f \text{dil}(G, f) \]

**Example 85** (Bandwidth of paths, $K_n$ and $C_n$). Let $n \in \mathbb{N}$

1. The bandwidth of the path on $n$ vertices is 1 (by choosing a linear embedding).

2. The bandwidth of the complete graph $K_n$ is $n - 1$ (as there will always be an edge from the vertex mapped to 1 to the vertex mapped to $n$).

3. The bandwidth of the cyclic graph $C_n$ is 2 (starting from some vertex, embed the vertices left/right of it alternatingly):

   \[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

**Definition 86** ((Vertex) boundary). For $S \subseteq V$, the *vertex boundary* $\text{bd}(S)$ of $S$ is the set of vertices of $S$ with some neighbors outside of $S$.

**Theorem 87** (Harper bound, Harper 1966). For $k \geq 1$,

\[ \text{bw}(G) \geq \min_{S \subseteq B, |S| = k} |\text{bd}(S)| \]

*Proof.* Fix one dilation-minimizing embedding of $G$ and consider the vertices numbered 1 through $k$ as $S$. Then there must exist some edge of length at least $(k + 1) - (k - \text{bd}(S) + 1) = \text{bd}(S)$. \qed

Determining the bandwidth is $\text{NP}$-complete (when stated as a decision problem), and even for well-studied graph classes it is quite nontrivial to prove closed forms for their bandwidth:

**Theorem 88** (Bandwidth of $G_{n \times n}$). The bandwidth of the $n \times n$-grid $G_{n \times n} = P_n \times P_n$ is $n$ (for $n \geq 2$).

*Proof.* To see that $n$ constitutes an upper bound, number the vertices row-by-row. This numbering has a dilation of $n$ (the inter-row edges have length $n$ as the vertex numbered 1 is connected to the vertex numbered $n + 1$ and so on).

For the lower bound, let $q$ be the smallest integer so that the set $S$ of vertices numbered 1 through $q$ contains a full row or a full column in some fixed dilation-minimal embedding of $G_{n \times n}$. This gives us two cases:
1. $S$ contains a full row (column) but no full column (row). Then for each column one vertex in $S$ is on the boundary and applying the Harper bound (cf. Theorem 87) gives $\text{bw}(G_{n\times n}) \geq n$.

2. $S$ contains a full row and a full column. Then $|\text{bd}(S)| \geq 2(n - 1)$ as the vertex labelled $q$ must complete a cross in $G_{n\times n}$ (or else we would have the first case) and as such, there must be at least one vertex missing in every column and row apart from the cross and as such there exist vertices for every such row and column which is on the boundary. Since we already know that $n$ is an upper bound for the dilation of the embedding, we get that our embedding is not optimal which contradicts our assumption.

Hence, we know that $\text{dil}(G_{n\times n}) \geq n$ holds and together with the upper bound of $n$, we obtain the desired result.

Next, we study the bandwidth of triangular grids:

**Definition 89** (Triangular grid $T_n$). For $n \geq 2$ let $T_n$ denote the triangular grid graph on $n(n+1)/2$ vertices.

**Definition 90** (Connectors (of $T_n$)). A connector is a connected subset of vertices that contains at least one vertex from every outer edge of $T_n$.

**Example 91** (Connector of triangular grid $T_4$). A connector of $T_4$ might look like this (connector vertices marked as squares, the induced edges are dashed):

![Diagram of a connector in a triangular grid]

Before concerning ourselves with the bandwidth of $T_n$, we prove two auxiliary lemmata.

**Lemma 92** (Auxilliary lemma 1). In any red-blue-labelling of the vertices of $T_n$, exactly one color contains a connector of $T_n$.

**Proof.** To see that at least one of the colors contains a connector, suppose that neither color does. Now label every vertex with the smallest triangle side that cannot be reached from within its color class. This yields a proper labelling of the vertices of $T_n$ and by Sperner’s lemma (cf. Lemma 80) there exists a triangle whose corners are labelled 0-1-2.

In this triangle, there must be two connected vertices which are in the same color class but are labelled differently, which is a contradiction. Thus, at least one color must contain a connector of $T_n$.

Further, at most one color can contain a connector of $T_n$, or else we can find a planar embedding of the $K_{3,3}$ like this:
Using the sets \{1, 2, 3\} and \{a, b, c\} as the bipartition sets gives a \(K_{3,3}\) subdivision, which is planarly drawn if the \(T_n\) is (which is trivially planar). The connections from \{1, 2, 3\} to \(a\) and \(b\) are built using the color classes (red/circles for \(a\) and blue/squares for \(b\)) which are connectors for \(T_n\). Notice that it is not a problem if \(a\) or \(b\) happen to coincide with the corner points of the \(T_n\) (draw it yourself and trace the paths).

**Lemma 93** (Auxilliary lemma 2). Every connector of \(T_n\) contains at least \(n\) vertices.

**Proof.** We use induction on \(n\). The cases for \(n = 1\) and \(n = 2\) are clear.

So, let \(n \geq 3\) and let \(C\) be a connector of \(T_n\). We distinguish two cases:

1. \(C\) contains at least two corners of \(T_n\) (w.l.o.g. the top corner and one of the bottom corners). Then \(C\) contains at least one vertex of every layer, of which there are \(n\), giving us \(|C| \geq n\).

2. \(C\) contains at most one corner of \(T_n\). Then we remove one side-layer (which \(C\) is not completely contained in as it only contains one corner) to obtain a \(T_{n-1}\), for which the remainder of \(C\) is a connector. By the inductive hypothesis, the remainder of \(C\) has at least \(n-1\) vertices and thus, \(C\) must have at least \(n\) vertices (since we deleted at least one by removing the side-layer).

**Theorem 94** (Hochberg, McDiarmid, Saks 1995). The bandwidth of \(T_n\) is \(n\).

**Proof.** Again, we get \(n\) as an upper bound by a row-by-row-enumeration of the graph.

For the lower bound, we fix some dilation-minimal embedding of \(T_n\) and consider the largest integer \(q\) so that the vertices numbered 1 through \(q\) do not contain a connector. Let \(R\) denote the set of preimages of \([1, q]\) under the numbering and let \(S\) be the set of
vertices adjacent to some vertex in $R$ (but not in $R$ itself). Further, let $T$ denote the
vertices of $T_n$ which are not in $R$ nor in $S$.

Now, let $v$ be the vertex numbered $q+1$ is in $S$, i.e. $v$ is connected to some vertex in $R$ as
$R \cup \{v\}$ contains a connector. This implies that $T$ contains no connector (cf. Lemma 92)
and also, $R \cup T$ has no connector (as neither of them contains one and the sets are not
interconnected). Thus, $S$ must contain a connector and we have $S = bd(S \cup T)$. By
the second auxilliary lemma (cf. Lemma 93) we have $|S| \geq n$ and hence a lower bound
on the bandwidth of $T_n$ by the Harper bound (cf. Theorem 87).

**Theorem 95** (Local density bound). For some subset $H$ of $V$ let $\text{diam}(H)$ be the
maximum over all $v, w \in H$ of the minimum distance between $v$ and $w$. Then

$$\text{bw}(G) \geq \max_{H \subseteq V} \frac{|H| - 1}{\text{diam}(H)} = \text{ldb}(G)$$

*Proof.* Fix some dilation-minimal embedding of $G$ and consider the first vertex $x \in H$ and
the last vertex $y \in H$ (w.r.t. the embedding). Then one of the edges on the
shortest $x-y$-path (which has length at least $\text{diam}(H)$) has dilation at least $(|H| - 1)/\text{diam}(H)$. \qed

**Definition 96** (Caterpillar). A *caterpillar* is a tree which has a path of length $p + 1$
(for some $p$) incident to all edges like this

Denote by $L_k$ (for $k \in [1, p]$) be the set of leaves attached to $v_k$ on the path of the
caterpillar which are *not* part of the path.

**Lemma 97.** Let $C$ be a caterpillar with a path of length $p + 1$ and $1 \leq \alpha \leq \beta \leq p$ and
$B = \lceil \text{ldb}(C) \rceil$. Then

$$\sum_{k \in [\alpha, \beta]} |L_k| \leq (B - 1)(\beta - \alpha + 2)$$

*Proof.* Consider the subgraph $H$ induced by $v_\alpha, \ldots, v_\beta$ and their neighbors. For the
number of vertices in $H$ we have

$$|V_H| = 2 + \sum_{k \in [\alpha, \beta]} (|L_k| + 1)$$

and the diameter of $H$ is $\beta - \alpha + 2$ (the distance between some outer neighbors of $v_\alpha$
and $v_\beta$).

The local density bound (cf. Theorem 95) gives

$$\frac{|V_H| - 1}{\text{diam}(H)} = \frac{1 + \sum_{k \in [\alpha, \beta]} (|L_k| + 1)}{\beta - \alpha + 2} \leq B$$
and thus (note that $|\alpha, \beta| = \beta - \alpha + 1$) we get

$$\sum_{k \in [\alpha, \beta]} |L_k| \leq (B - 1)(\beta - \alpha + 2)$$

which is the desired result. \hfill \Box

**Theorem 98.** For every caterpillar $C$ we have $\text{bw}(G) = \lceil \lfloor \text{db}(C) \rfloor \rceil$.

**Proof.** Construct a bandwidth-numbering $f$ with $f(v_i) < f(v_j)$ for $i < j$ (recall that we named the vertices on the path $v_i$). Between $v_{i-1}$ and $v_i$ (for $1 \leq i \leq p + 1$) we embed a set of leaves so that

1. $L_i \subseteq S_i \cup S_{i+1}$, and
2. $|S_i| \leq B - 1$ (with $B = \lceil \lfloor \text{db}(C) \rfloor \rceil$).

Such an embedding exists by the previous lemma (cf. Lemma 97) and the dilation of the numbering is at most $B$. \hfill \Box
9 Colorings

We start by recalling some basic notions:

- The chromatic number $\chi(G)$ is the smallest integer such that $G$ is $\chi(G)$-colorable, i.e. there exists a partition of the vertices of $G$ into $\chi(G)$ independent sets.

- The girth $g(G)$ is the length of the shortest cycle in $G$.

Note that with increasing girth, the neighborhoods of single vertices become more tree-like, i.e. they are locally 2-colorable. This naturally leads to the conjecture that graphs with high girth have a low chromatic number. However, this conjecture is very wrong:

Definition 99 (Mycielskian, Mycielski, 1955). For a graph $G$ we define the Mycielskian $M(G)$ of $G$ as the graph on the vertex set $V \cup \{v' \mid v \in V\} \cup \{z\}$ and edge set $E \cup \{[v', z] \mid v \in V\} \cup \{[v, w'] \mid [v, w] \in E\}$.

Example 100 (Mycielskian of $C_5$). The Mycielskian of $C_5$ is given by

![Mycielskian of C5](image)

Lemma 101. For every $G$ we have $\chi(M(G)) = \chi(G) + 1$.

Proof. It is easy to give a $(\chi(G) + 1)$-coloring of $M(G)$ by taking any $\chi(G)$-coloring and coloring $v'$ like $v$ and coloring $z$ with a new coloring.

For the other direction, suppose $M(G)$ has a $\chi(G)$-coloring. Then in the induced coloring of the subgraph $G$ of $M(G)$, there exists a vertex $v$ of color $c$ adjacent to all other colors. Now since $v'$ is also adjacent to every neighbor of $v$, it follows that $v'$ must have color $c$ as well and hence, all $\chi(G)$ colors are present in the neighborhood of $z$ and thus, we cannot color $z$ without a new color.

Lemma 102. If $g(G) \geq 4$, then $g(M(G)) \geq 4$.

Proof. Any cycle including $z$ must also include two added vertices $v'$ and $w'$ and thus, such a cycle must have length at least 4. Further, a cycle including some added vertex $v'$ but not $z$ must use two distinct vertices $u, w$ from the neighborhood of $v$ which are not connected since $G$ has girth 4 (i.e. no triangle $uvw$ is allowed) and thus such a cycle too must have length 4. In fact, if there exists a vertex with 2 neighbors in $G$, then there exists a 4-cycle (namely $vuv'w$) in $M(G)$.

These results give rise to an immediate corollary: There exist triangle-free graphs (i.e. girth at least 4) and chromatic number at least $k$ for all $k$, for example $M^k(C_5)$. But indeed, a more general result for arbitrary girth exists.
Theorem 103 (Erdős, 1959). For all $k \geq 2$ there exists a graph $G$ with $\chi(G) \geq k$ and $g(G) \geq k$.

The proof of this theorem is beautiful and the first ever use of a technique pioneered by Erdős, the so-called probabilisitic method where the existence of a combinatorial object is established by showing that there exists a probability distribution on the space of all such objects and then proving that the probability of a randomly chosen object exhibiting the desirable properties is positive.

For this, we need a tool from probability theory:

Lemma 104 (Markov inequality). Let $X$ be a non-negative random variable and $a \geq 0$. Then

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$

where $E[X]$ is the expected value of $X$.

Proof. Let $\lambda$ be the probability density of $X$. By the definition of the expected value we get

$$E[X] = \int_{\mathbb{R}^+} x \, d\lambda(x) \geq \int_{\mathbb{R}^+} a \, d\lambda(x) = a \Pr[X \geq a]$$

Proof (of Theorem 103). Let $V = \{v_1, \ldots, v_n\}$ and $p = n^{-k/(k+1)}$ ($k$ from the formulation of the theorem).

We consider the probability space $\mathcal{G}(n, p)$ which contains every graph over $V$. Every edge in a graph occurs independently with probability $p$.

We begin by considering independent sets in $\mathcal{G}(n, p)$. Consider an $r$-element subset $R$ of the vertices. Then the probability that $R$ is an independent set is

$$\Pr[R \text{ is independent set}] = (1 - p)^{{\binom{r}{2}}}$$

and further we have

$$\Pr[\alpha(G) \geq r] \leq \sum_{R \in \binom{V}{r}} \Pr[R \text{ is independent set}] = \binom{n}{r} (1 - p)^{{\binom{r}{2}}} \leq n'^{(r)(1-p)^{\frac{r}{2}}} = (n(1-p)^{\frac{r}{2}})^r \leq \left(ne^{-\frac{r}{2}}\right)^r$$

\[ \square \]
where the last inequality follows from the fact that \( 1 + x \leq e^x \) holds for all \( x \).

Now, for sufficiently large \( n \), the inequality \( n^{1/(k+1)} > 6k \ln(n) \) holds and thus,

\[
p = n^{-k/(k+1)} = n^{-1/(k+1)}n^{-1} > 6k \frac{\ln(n)}{n}
\]

and by choosing \( r = \lceil n/2k \rceil \) we get \( pr > 3\ln(n) \) and plugging this into the base of our previous inequality for the probability that \( \alpha(G) \geq r \) gives

\[
ne^{-p\frac{r-1}{2}} < ne^{-3/2\ln(n)}e^{p/2} \leq nn^{-3/2}e^{1/2} = \sqrt{\frac{e}{n}}
\]

Hence, we have

\[
\Pr[\alpha(G) \geq \lceil n/2k \rceil] \leq \left( \sqrt{\frac{e}{n}} \right)^n \rightarrow 0
\]

and thus it is reasonable to say that large random graphs do not contain (very) large independent sets.

Next, we take a look of the number of cycles of length at most \( k \) in \( G \). So let \( 3 \leq i \leq k \) and \( A \subseteq V \) with \( |A| = i \). The number of possible cycles of length \( i \) through \( A \) is \( 1/2(i-1)! \) (as we have \( i! \) possible paths of length \( i \), but we do not care for reversal of order or the starting vertex). If we factor in the possible choices of \( A \), we get that there are

\[
\binom{n}{i} \frac{1}{2(i-1)!}
\]

possible cycles of length \( i \) over \( V \), and every such cycle occurs with probability \( p^i \).

Let \( X \) be a random variable counting the number of cycles of length at most \( k \) in \( G(n, p) \) and let \( X_C \) be an indicator variable for a cycle \( C \) of length at most \( k \). Thus, we have

\[
X = \sum_{C \in \binom{V}{\leq k}} X_C
\]

and thereby (note that \( \binom{n}{i} \leq n^i/i! \))

\[
\mathbb{E}[X] = \sum_{C \in \binom{V}{\leq k}} \mathbb{E}[X_C] = \sum_{i \in [3,k]} \binom{n}{i} \frac{1}{2(i-1)!}p^i \leq \frac{1}{2} \sum_{i \in [3,k]} n^i p^i = \frac{1}{2} \sum_{i \in [3,k]} (np)^i \leq \frac{1}{2} k(np)^k
\]
Colorings

where the last inequality holds since we have \( p = n^{-k/(k+1)} \), i.e. the summand of the sum is maximal for \( i = k \). Using the Markov inequality with \( a = n/2 \) yields

\[
\Pr[X \geq n/2] \leq \frac{2\mathbb{E}[X]}{n} \leq \frac{1}{n} k(np)^k = \frac{k}{n} \left( n^{1/(k+1)} \right)^k = kn^{-1/(k+1)} \rightarrow 0
\]

and thus, we get that large random graphs contain relatively few short cycles.

Wrapping up, we get that for sufficiently large \( n \) there exists a \( n \)-vertex graph \( G \) with \( \alpha(G) \leq \lceil n/2k \rceil \) and with \( \leq n/2 \) cycles of length at most \( k \). Deleting one vertex from each of those short cycles gives us a new graph \( H \) on \( t \geq n/2 \) vertices and girth at least \( k + 1 \). Since we had \( \alpha(G) \leq n/2k \), we get that \( \alpha(H) \leq t/k \) and as every color class in a proper coloring is an independent set, we get \( \chi(H) \leq t/k \), which concludes the proof.

**Definition 105.** Let \( \vec{G} \) be an orientation of an undirected graph \( G \). Then \( \ell(\vec{G}) \) is the number of vertices in the longest simple directed path in \( \vec{G} \).

**Theorem 106** (Gallai, 1968). For every \( G \),

\[
\chi(G) = \min \ell(\vec{G})
\]

**Proof.** To see that every graph possesses an orientation such that every directed path has vertex-length at most \( \chi(G) \), we start by considering an proper coloring of \( G \) with \( \chi(G) \) colors and orient every edge from the smaller to the larger color to obtain \( \vec{G} \). Then every step in a directed path in \( \vec{G} \) corresponds to an increase in the color of the vertex we stand on and there are only \( \chi(G) \) colors and hence, the path cannot consist of more than \( \chi(G) \) vertices.

For the other direction, consider some orientation \( \vec{G} \) of \( G \) and remove the minimal set of edges so that the resulting graph \( \vec{H} \) has no directed cycles. Now, let \( \ell(v) \) denote the vertex-length of the longest directed path starting in \( V \) and we get that

\[
\ell(v) \leq \ell(\vec{H}) \leq \ell(\vec{G})
\]

as removing edges can only shorten paths. We now color \( v \) by \( \ell(v) \). To see that this gives us a correct coloring of \( G \), we look at two cases:

1. The arc (i.e. directed edge) \([x, y]\) is in \( \vec{H} \). Then \( \ell(x) \geq \ell(y) + 1 \) (as \( \vec{H} \) is acyclic) and thus, \( x \) and \( y \) are colored differently.

2. The arc \([x, y]\) was removed during the construction of \( \vec{H} \). Then there exists a directed cycle using the arc \([x, y]\) in \( \vec{G} \) and \( \vec{H} \) contains a directed path from \( y \) to \( x \) and thus, \( \ell(y) > \ell(x) \).

**Lemma 107.** Let \( E_i \subseteq E \) for \( i \in [1, r] \) such that \( E = \bigcup_{i \in [1, r]} E_i \), we get

\[
\chi(G) \leq \prod_{i \in [1, r]} \chi(V, E_i)
\]
Proof. Let $c_i$ be a proper coloring of $(V, E_i)$ and color the vertex $v \in V$ as $(c_1(v), \ldots, c_r(v))$. \hfill \Box

**Theorem 108.** Let $P$ be a set of $n = k^r + 1$ points in the euclidean plane $\mathbb{R}^2$. Then there exists an almost straight line of $k + 1$ points, i.e. a sequence $p_0, p_1, \ldots, p_k \in P$ with $\angle p_{i-1}p_ip_{i+1} \geq (1 - 1/r)\pi$ for $1 \leq i \leq k - 1$.

Proof. Consider the complete graph on the vertex set $P$. We partition the edge set of said graph into $E_1, \ldots, E_r$ and orient every edge $[p_i, p_j]$ upwards (towards the point with larger $y$-coordinate in $\mathbb{R}^2$). Now, $[p_i, p_j]$ is in $E_\ell$ if the angle between the directed edge $[p_i, p_j]$ and the (positive) $x$-axis lies between $((\ell - 1)/r)\pi$ and $(\ell/r)\pi$ (for $1 \leq \ell \leq r$).

Suppose the desired point sequence does not exist. Then the subgraph $(P, E_\ell)$ has no directed path on $k + 1$ vertices and thus, by Gallai’s theorem (cf. Theorem 106), the chromatic number of the subgraph satisfies $\chi(P, E_\ell) \leq k$ and by the product bound (cf. Lemma 107) we get $\chi(P, E) \leq k^r$, but $(P, E)$ is a complete graph on $k^r + 1$ vertices and thus, we know that $\chi(P, E) = k^r + 1$ holds. Hence, the desired point sequence must exist. \hfill \Box

Our last theorem in this chapter gives a characterization of the chromatic number via subgraphs:

**Definition 109** ($k$-Hajósness). For $k \geq 2$, we inductively define the class of $k$-Hajós graphs as follows:

1. The complete graph $K_k$ is $k$-Hajós.
2. If $G$ is $k$-Hajós and $[x, y] \notin E$, then the contraction $(G + [x, y])/[x, y]$ is $k$-Hajós.
3. If $G$ and $H$ are $k$-Hajós and $G \cap H = \{x\}$ and there exist edges $[x, y] \in E_G$ and $[x, y'] \in E_H$, then

   $$(G \cup H) - [x, y] - [x, y'] + [y, y']$$

   is $k$-Hajós.

**Example 110** ($2$-Hajós graphs). By definition, the $K_2$ is $2$-Hajós. Applying the second construction with two $K_2$-copies gives another $K_2$ with an isolated vertex (and the repeated with a fresh $K_2$ application gives us one more isolated vertex every time). The first construction can only be used to delete an isolated vertex.

Thus, the class of $2$-Hajós graphs (up to isomorphism) is given by $K_2 \cup \overline{K_\ell}$ for some $\ell \geq 0$.

**Theorem 111** (Hajós, 1961). A graph $G$ has chromatic number $\chi(G) \geq k$ if and only if it has a $k$-Hajós subgraph.

Proof. We start by proving that every $k$-Hajós graph has chromatic number at least $k$. It is immediately clear that the $K_k$ has chromatic number $k$, which forms the base case for an induction on the construction of $H$.

So let $H$ be a $k$-Hajós graph with chromatic number at least $k$. Then a $(k-1)$-coloring of $(H + [x, y])/[x, y]$ gives us a $(k-1)$-coloring of $H$ by just coloring $x$ and $y$ as the contracted vertex in the constructed graph and thus, the constructed graph must also have chromatic number at least $k$. 

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Further, let $H'$ be a second $k$-Hajós graph (with $\chi(H') \geq k$) satisfying the prerequisites for the second construction (w.r.t. $H$), and assume that $(H \cup H') - [x, y] - [x, y'] + [y, y']$ is $(k - 1)$-colorable. We can now obtain a proper coloring for one of $H$ or $H'$ (induced from the constructed graph) as either $y$ or $y'$ must have a different color than $x$ and thus, the induced coloring for one of the graphs is proper (as we only removed the edges from $y$ and $y'$ to $x$) and uses at most $(k - 1)$ colors, which is a contradiction.

For the other direction, suppose $G$ is an edge-maximal counterexample and consider two cases:

1. Non-adjacency (denoted $\sim$) is an equivalence relation (i.e. reflexive, symmetric and transitive) on $V$. It is easy to see that the equivalence classes of $V$ under non-adjacency are independent sets and thus, $G$ is $\ell$-partite where $\ell$ is the index of the non-adjacency relation (i.e. $\ell$ is the number of elements of the quotient set $V/\sim$).

   Now, this means that $\chi(G) \geq k$ implies $\ell \geq k$, and thus $G$ contains $K_k$.

2. Non-adjacency is not an equivalence relation. As it must still be reflexive and symmetric, there must exist vertices $x, y, z \in V$ such that $[x, y] \notin E$ and $[y, z] \notin E$, but $[x, z] \in E$. Since $G$ is edge-maximal, the graph $G + [x, y]$ contains a $k$-Hajós subgraph $H_1$ and $G + [y, z]$ contains a $k$-Hajós subgraph $H_2$.

   Now consider the graph $H_2'$ which is obtained from $H_2$ by duplicating all vertices but $y$ in $H_1 \cap H_2$ (rename the copy of $v$ to $v'$). Apply the second construction to $H_1$ and $H_2'$ using the vertices $x, y, z$ and then use the first construction to merge the vertices $a \in H_1 \cap H_2$ and $a' \in H_2'$.

   The resulting graph $H$ has an edge $[x, z]$ but lacks $[x, y]$ and $[y, z]$ and after the mergers, the vertices shared by $H_1$ and $H_2$ have the correct connections to both subgraphs of $G$ and thus, $H$ is a subgraph of $G$ which is $k$-Hajós.

   It follows that no counterexample exists and thus, the proof is concluded. \(\square\)
10 Choosability

Our new scenario is the following: Consider some graph $G$ which we want to color properly, but for every vertex $v$ there exists a list $L(v)$ of colors we may chose from to color this vertex. The classical coloring problem is thus a special case of this choosing problem where all the lists are the same, namely the list of colors available.

**Definition 112** ($k$-Choosability). A graph $G$ is $k$-choosable if we are able to color the graph properly for any assignment of $k$-element lists to the vertices.

**Example 113.**

1. The $C_5$ is not 2-choosable as it is not even 2-colorable.
2. The $K_{4,2}$ is not 2-choosable:

   ![Graph Illustration](image)

3. The $K_\ell$ is $\ell$-choosable as every vertex has only $\ell - 1$ neighbors and thus only $\ell - 1$ entries of its list can be blocked by its neighbors.

It is a famous result that every planar graph can be colored using four colors (the *four color theorem*) and thus also using 5 colors. We will see now that planar graphs are also 5-choosable:

**Theorem 114** (Thomassen, 1994). Every planar graph is 5-choosable.

*Proof.* Without loss of generality, $G$ is connected and triangulated on the inside (i.e. not on the infinite face). We prove a stronger statement: Let $R$ be the cycle bounding the infinite face and let $x$ and $y$ be neighbors on $R$ which are properly already colored using the colors $\alpha$ (for $x$) and $\beta$ (for $y$) and let $|L(v)| \geq 3$ for $v \in R \setminus \{x, y\}$ and $|L(w)| \geq 5$. Then $G$ has a compatible proper coloring.

We use induction on $n$ (the cases for $n \leq 3$ are clear). So let $n \geq 3$ and consider two cases:

1. $R$ has a chord, i.e. two vertices in $R$ are connected via an edge not in $R$. Then we can split $G$ at the chord (put a copy of both endpoints in both resulting graphs) and color the subgraphs using the inductive hypothesis. By coloring the subgraph which contains the chord first, we fix the color of both endpoints of the chord and we can color the second one appropriately so that we can build a coloring of $G$ from the colorings for both subgraphs.

2. $R$ has no chord. Let $z \neq y$ be a neighbor of $x$ on $R$. As we have $|L(z)| \geq 3$ there are two colors $\gamma, \delta \in L(z)$ different from $\alpha$ (and thus are not blocked by $x$). Let $v_0, \ldots, v_k, x$ be the neighbors of $z$ (where $v_0$ lies on $R$) and remove $z$ from $G$ and
set \( L(v_i) = L(v_i) \setminus \{\gamma, \delta\} \) for \( i \geq 1 \).

Applying the induction hypothesis gives us a coloring for the graph with \( z \). We can now read \( z \) and since \( v_1 \) is the only neighbor which can be colored with \( \gamma \) or \( \delta \), there must be at least one color of \( L(z) \) which is still available. Thus, we get a proper coloring for \( G \).

However, this is the best we can do, as planar graphs are in general not 4-choosable:

**Theorem 115** (Voigt, 1993). There exists a planar graph which is not 4-choosable.

**Proof.** For \( \alpha \in [5, 8] \) and \( \beta \in [9, 12] \) consider the graph which consists of 16 copies (one for every pair \((\alpha, \beta)\), identified at \( \alpha \)-vertices and \( \beta \)-vertices where the values coincide) of the following graph \( G(\alpha, \beta) \):

Now, \( G(\alpha, \beta) \) has no compatible list coloring.

But if we restrict ourselves to outer-planar graphs (planar graphs which has a drawing with all vertices on the boundary of the infinite face), we can do even better:

**Theorem 116.** Every outer-planar graph is 3-choosable.

**Proof.** We use induction on \( n \) (again, the cases for \( n \leq 3 \) are easy). As every outer-planar graph possesses an vertex of degree at most 2 (else, we can find a \( K_4 \) subgraph in \( G \) which is not outer-planar). Thus we can simply color \( G - v \) by the induction hypothesis and put \( v \) back in and chose the list entry not blocked by its two neighbors. \( \square \)
11 Ramsey theory

The field of Ramsey theory formalizes the statement “Even very irregular structures (if they are very large) contain large regular substructures” to some extend. Let us start with some examples:

Example 117 (Warm-up 1). Among any 6 persons (any pair of them are either friends or enemies), there are 3 pairwise friends or 3 pairwise enemies.

Fix some person $p$ (thus there are 5 other persons). This person now has relations to 5 other people and thus, $p$ has either at least 3 friends or at least 3 enemies. Without loss of generality, we assume that $p$ has 3 friends $a, b$ and $c$.

Now either (at least) one of the two friends of $p$ have a friendly relation ship or none do. In the first case there are 3 pairwise friends and in the second one, the friends of $p$ are three pairwise enemies.

This example can also be stated in graph-theoretic terms: Any 2-coloring of the edges of the $K_6$ produces a triangle in one of the colors.

Consider a second example:

Example 118 (Warm-up 2). Every sequence of $n^2 + 1$ reals contains a monotone subsequence of length $n + 1$.

Consider some sequence $a = (a_i)_{i \in [1,n^2+1]}$. Let $x_i$ denote the length of the longest increasing subsequence (of $a$) ending in $a_i$ and $y_i$ the length of the longest decreasing subsequence ending in $a_i$.

As we either have $a_i \leq a_j$ or $a_i \geq a_j$ (for say $i < j$), we also must have $(x_i, y_i) \neq (x_j, y_j)$ as we can append $a_j$ to at least one of the subsequences ending in $a_i$. But there are only $n^2$ pairs in $[1,n]^2$, and thus at least one pair $(x_i, y_i)$ must contain a number larger than $n$, giving us a monotone subsequence of length at least $n + 1$.

Definition 119 (Ramsey numbers). For integers $a, b \geq 1$ define $R(a, b)$ as the smallest integer $n$ so that every 2-coloring of the edges of $K_n$ contains a red $K_a$ or a blue $K_b$.

Example 120.

1. $R(a, b) = R(b, a)$ by symmetry.

2. $R(1, a) = 1$ as any graph contains a red $K_1$ since it has no edges.

3. $R(2, a) = a$, because if we cannot have a red edge, we must color every edge blue. This works for $K_{a-1}$, but not $K_a$, as this would produce a blue $K_a$.

4. $R(3, 3) = 6$ by Example 117 (Warm-up 1).

Now, the first question with regards to the definition of the Ramsey numbers has to be about their existence. Do they even exist for all values of $a$ and $b$?

Theorem 121 (Ramsey, 1930). The Ramsey number $R(a, b)$ exists for all $a, b \geq 1$.  

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Proof. We use induction on \(a + b\). The base of our induction is given by the previous example (cf. Example \[120\]), so we only consider \(a, b \geq 3\) and let \(n = R(a, b-1) + R(a-1, b)\).

Now, consider an arbitrary red-blue-coloring of the edges of \(K_n\) and pick some vertex \(v\). The vertex \(v\) has \(n - 1\) incident edges and due to our choice of \(n\), this means that either (at least) \(R(a, b-1)\) edges must be colored blue or (at least) \(R(a-1, b)\) edges colored red.

We only consider the first case (as the second one is completely analogous), i.e. at least \(R(a, b-1)\) edges incident to \(v\) are colored blue. Consider the subgraph induced the “blue” neighborhood of \(v\). By the inductive hypothesis, this subgraph contains a red \(K_a\) or a blue \(K_{b-1}\) and thus, there either exists a red \(K_a\) or a blue \(K_{b-1}\) (as every vertex in the \(K_{b-1}\) has a blue edge to \(v\)) in \(K_n\).

From this proof, we get an immediate corollary:

**Lemma 122.** For all \(a, b \geq 2\), we have \(R(a, b) \leq R(a - 1, b) + R(a, b - 1)\).

**Lemma 123.** For all \(a, b \geq 1\), we have

\[
R(a, b) \leq \binom{a + b - 1}{a - 1} \leq 2^{a+b-2}
\]

**Theorem 124** (Erdős, 1947). If

\[
\binom{n}{2} < 2^{\binom{a}{2} - 1}
\]

holds, then \(R(a, a) > n\) holds.

Proof. Color the edges randomly (and independently) with \(\Pr[\text{e is blue}] = 1/2\). For \(T \subseteq V\) of size \(a\), consider the event \(A_T\) representing the fact that the clique on \(T\) is colored monochromatically. The event \(A_T\) occurs with probability

\[
\Pr[A_T] = 2^{-\binom{a}{2} + 1}
\]

and the probability that \(A_T\) occurs for any choice of \(T\) is

\[
\Pr[A_T \text{ occurs for at least one choice of } T] \leq \sum_{|T| = a} \Pr[A_T] = \binom{n}{a} 2^{-\binom{a}{2} + 1} < 1
\]

where the last inequality holds by the prerequisite. This means that that the probability that there exists a coloring of the edges of \(K_n\) which produces no monochromatic \(K_a\) is nonzero and since we are studying a finite probability space, such a coloring must exist. This gives us the desired inequality.

**Lemma 125.** \(R(a, a) > 2^{a/2}\) holds for all \(a \geq 4\).

Proof. Let \(n = 2^{a/2}\). By a completely analogous procedure as in the previous proof (cf. proof of Theorem \[124\]), we obtain

\[
\binom{n}{a} 2^{-\binom{a}{2} + 1} \leq \frac{n^a}{a!} 2^{-\binom{a}{2} + 1}
\]

and

\[
= \frac{1}{a!} 2^{2a^2/2 - 2 + a^2/2 + a/2}
\]

\[
= \frac{1}{a!} 2^{a + a/2} < 1
\]
as we have \( \frac{2^3}{4!} = \frac{1}{3} \) and increasing \( a \) corresponds to multiplying this value by \( \sqrt{2}/a < 1 \) (as \( a \geq 4 \)).

**Observation 126** (Values and bounds on \( R(a, b) \)). Some known values and bounds for the Ramsey numbers:

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For some values, there exists an asymptotic result:

**Theorem 127** (Kim 1995). For all \( a \), we have \( R(3, a) \in \Theta(a^2/\ln(a)) \).

We now prove \( R(3, 4) = 9 \).

**Theorem 128.** \( R(3, 4) = 9 \)

**Proof.** To see that 9 is a lower bound, consider the following coloring of \( K_8 \) (red edges displayed, blue edges are implicitly given by non-edges):

![Graph with red edges forming a triangle and no blue K_4](image)

which has no red triangle and no blue \( K_4 \) (for any four vertices at least one edge between them is red).

For the upper bound, note that our recursive bound (cf. Lemma 122) is not strong enough as it yields 10. So consider some vertex \( v \) in an edge-colored \( K_9 \). If \( v \) has at least 4 red edges, then its neighborhood either forms a blue \( K_4 \) or we have a red triangle (by \( R(2, 4) = 4 \)) and similarly, we get that \( v \) must have less than 6 blue edges (by \( R(3, 3) = 6 \)).

Thus, every vertex has to have exactly 3 red and 5 blue incident edges, which is a contradiction to the handshake lemma and thus, no contradicting coloring can exist. □

The concept of the classical Ramsey numbers can be generalized by introducing more colors, i.e. considering \( R(a, b, c) \) (which clearly satisfies \( R(a, b, c) \leq R(a, R(b, c)) \) and thus exists).

**Definition 129.** For some integer \( r \geq 1 \) and a finite set \( S \), let \( \binom{S}{r} \) denote the set of \( r \)-element subsets of \( S \).

Let \( f \) be a \( k \)-coloring of \( \binom{S}{r} \) and \( T \subseteq S \). We call \( T \) monochromatic under \( f \) if for all \( X, Y \in \binom{T}{r} \) we have \( f(X) = f(Y) \).
Definition 130. For integers \( r \) and \( p_1, \ldots, p_k \), the Ramsey number \( R_r(p_1, \ldots, p_k) \) is the smallest integer \( n \) such that for every \( k \)-coloring of \( \binom{S}{r} \) (with \( |S| = n \)) there exists an \( i \) and a subset \( T \subseteq S \) of size \( p_i \) such that every \( r \)-element subset of \( T \) is colored with \( i \) (i.e. \( T \) is monochromatic with color \( i \) under the coloring).

Example 131. We have

\[
R_1(p_1, \ldots, p_k) = \sum_{i \in [1, k]} p_i - k + 1
\]

by the pigeonhole principle.

Theorem 132. For all \( r \geq 2 \), \( R_r(a, b) \) exists.

Proof. We use nested induction. The outer induction is performed over \( r \), the inner one over \( a + b \) (notice that the case for \( r = 2 \) is given by the usual Ramsey numbers).

For the inductive step, let \( x = R_r(a - 1, b) \), \( y = R_r(a, b - 1) \) and \( n = R_{r-1}(x, y) + 1 \).

We consider a set \( S \) of size \( n \) and let \( f \) be a red-blue coloring of \( \binom{S}{r} \). Fix some \( v \in S \) and let \( S' = S \setminus \{v\} \) and likewise, let \( f' \) be a red-blue coloring of \( \binom{S'}{r-1} \) satisfying \( f'(T) = f(T \cup \{v\}) \).

Now, the (outer) inductive hypothesis yields a red set of size \( x \) or a blue set of size \( y \) under \( f' \). We consider a red set \( T' \subseteq S' \) of size \( x = R_r(a - 1, b) \) (the other case follows suit) under \( f \). Thus, by the (inner) inductive hypothesis we get two cases:

1. There exists a blue subset of size \( b \) under \( f \) and we are done.
2. There exists a red subset \( T'' \) of size \( a - 1 \) under \( f \). Then all \( (r - 1) \)-element subsets of \( T'' \) are red under \( f' \) and all \( r \)-element subsets of \( T'' \cup \{v\} \) which contain \( v \) are red under \( f \). Thus (by the definition of \( f' \)) all \( r \)-element subsets of \( T'' \cup \{v\} \) are red under \( f \) and we are done.

We continue our studies of the generalized Ramsey numbers with the following class:

Definition 133. \( \overline{R}_k(3) = R_2(3, \ldots, 3) \) \( k \) times

Example 134. \( \overline{R}_2(3) = R(3, 3) = 6 \)

Theorem 135. For all \( k \geq 3 \), \( \overline{R}_k(3) \leq k(\overline{R}_{k-1}(3) - 1) + 2 \) holds.

Proof. Consider some fixed vertex \( v \) in a \( k \)-coloring of the edges of the complete graph on \( n = k(\overline{R}_{k-1}(3) - 1) + 2 \) vertices. There are \( n - 1 \) edges incident to \( v \) and one color (say red) occurs on at least \( \overline{R}_{k-1}(3) \) edges. Denote the red neighborhood of \( v \) by \( S \).

We distinguish two cases:

1. \( S \) has an red edge. Then we have a red triangle and are done.
2. \( S \) has no red edge. Then \( S \) is colored using \( k - 1 \) colors and since it consists of \( \overline{R}_{k-1}(3) \) vertices, there must exist a monochromatic triangle in \( S \).
Solving this recurrence equation gives us the following result:

**Lemma 136.** For $k \geq 3$, we have $R_k(3) \leq \lceil ek! \rceil + 1 \in 2^{\Theta(k \ln(k))}$.

**Proof.**

\[
\lceil ek! \rceil = \left\lceil \sum_{j=0}^{\infty} \frac{k!}{j!} \right\rceil = \left\lceil \sum_{j=0}^{k} \frac{k!}{j!} + \sum_{j=k+1}^{\infty} \frac{k!}{j!} \right\rceil = \sum_{j=0}^{k} \frac{k!}{j!} = 1 + k \sum_{j=0}^{k-1} \frac{(k-1)!}{j!} = 1 + k \lfloor e(k-1)! \rfloor
\]

where we used $\sum_{j=k+1}^{\infty} k!/j! < 1$.

**Example 137.** Using this bound gives us $R_3(3) \leq 17$ and it is known that $R_3(3) = 17$ (edges of $K_{16}$ can be partitioned into three copies of the triangle-free Chlebsch graph).

**Definition 138** (Schur numbers). For every $k \geq 1$, let $s(k)$ denote the smallest integer so that in every $k$-coloring of the integers in $[1, s(k)]$ there exist three integers $x, y, z$ with the same color satisfying $x + y = z$.

**Example 139.** We determine $s(2)$.

Without loss of generality, color 1 red. As we have $1 + 1 = 2$, we know that 2 must be colored blue, but 3 could be colored either way. However, by $2 + 2 = 4 = 1 + 3$, it is clear that 4 must be colored red and 3 must be colored blue.

Now we have $1 + 4 = 5 = 2 + 3$, so any coloring produced a conflict and thus, we get $s(2) = 5$.

**Theorem 140** (Schur 1916). For all $k \geq 1$, the Schur number $s(k)$ exists.

**Proof.** Let $n = R_k(3)$ and consider a $k$-coloring of $[1, n]$. Further, introduce a $K_n$ on $[1, n]$ where the edge $[i, j]$ is colored $|i - j|$.

Now, by the definition of $R_k(3)$ we get the existence of a monochromatic triangle $i, j, k$ in said $K_n$, i.e. $[i, j], [j, k]$ and $[i, k]$ have the same color. This gives us $|i - j| = |j - k| = |i - k|$ and w.l.o.g. $i > j > k$ holds and thereby

\[(i - k) = (i - j) + (j - k)\]

\[\square\]

**Theorem 141.** For all $k$, we have $s(k) \geq 3s(k - 1) - 1$. 

Proof. We use induction on \( k \) (base case is \( s(1) = 2, s(2) = 5 \)).

So let \( k \geq 3 \) and define \( t = s(k - 1) \). Thus, there exists a \((k - 1)\)-coloring with no monochromatic solution to \( x + y = z \) for the numbers \([1, t - 1] \). Now, color the numbers \([1, t - 1] \) using the inductive hypothesis using \( k - 1 \) colors and reuse this coloring (just shifted by \( 2t - 1 \)) for \([2t, 3t - 2] \). Further, color \([t, 2t - 1] \) using a new color.

This yields a \( k \)-coloring of \([1, 3t - 1] \) and it remains to show that the coloring is valid.

As only the numbers \([t, 2t - 1] \) are colored \( k \), they cannot produce a monochromatic solution as we have \( t + t > 2t - 1 \). Further, adding any two numbers from \([2t, 3t - 2] \) produces a number greater than \( 3t - 1 \) and hence, they cannot produce a monochromatic solution either. And since we colored \([1, t - 1] \) so that it has no monochromatic solution, the only way one can produce one is by using some numbers from \([2t, 3t - 2] \). But if this gives us a monochromatic solution, then we can shift the values taken from \([2t, 3t - 2] \) by \(-2t + 1\) and obtain a monochromatic solution using only values in \([1, t - 1] \), which is a contradiction.

As \([1, 3t - 2] \) has \( 3t - 1 \) elements, the desired inequality follows. \( \square \)

**Example 142** (Bound vs. actual Schur numbers). It is known that \( s(3) = 14 \) (bound gives \( \geq 14 \)), \( s(4) = 45 \) (bound gives \( \geq 41 \)) and \( 160 \leq s(5) \leq 315 \) (bound gives \( \geq 134 \)).

**Definition 143** (General position). A point set \( P \subseteq \mathbb{R}^2 \) is in general position if no three points of \( P \) lie on a common line.

From now on, we assume every point set to be in general position.

**Observation 144** (Klein 1935). Every 5-element point set contains a convex quadrangle.

**Theorem 145** (Erdős, Szekeres 1935). For every \( k \geq 4 \) there exists an integer \( n(k) \) so that every \( n(k) \)-element point set contains a convex \( k \)-gon.

**Proof.** Let \( k \geq 4 \) and \( n(k) = R_4(k, 5) \) and consider a point set of size \( n(k) \). Color every 4-element subset \( S \subseteq P \) blue if \( S \) is convex and red if it is not.

Ramsey implies a red subset of size 5 or a blue subset of size \( k \) and we distinguish the cases:

1. We have a red subset of size 5. Then there exists a set of 5 points which do not contain a convex quadrangle which is a contradiction to Klein’s observation (cf. Observation 144).

2. There exists a blue subset of size \( k \). Then there exists a convex \( k \)-gon.

\( \square \)

It is conjectured that \( n(k) = 2^{k-2} + 1 \) holds and it is known to be a lower bound (and exact for \( k = 4 \) and \( k = 5 \)).

**Proof for** \( k = 5 \). We want to prove that \( n(5) = 2^3 + 1 = 9 \).

The lower bound is given by the following point set, which does not contain a convex 5-gon:
For the upper bound, consider some point set $P \subseteq \mathbb{R}^2$ of size 9 and assume that $P$ has no convex 5-gon. Thus, the convex hull of $P$ also contains at most four points.

We distinguish cases:

1. The convex hull $H$ contains exactly four points. As $P \setminus H$ contains 5 points, it is not convex and there exists a non-convex set of four points, giving us the following situation:

Now, the innermost point forms a convex set with any two of the three points it forms a non-convex set with and the points on the convex hull of $P$ which are in the corresponding sector. As there are three sectors and four points on the convex hull, at least one sector contains two such points and thus $a$ forms a convex 5-gon.

2. The convex hull $H$ is a 3-gon. Consider the convex hull $H'$ of the 6 interior points and distinguish two subcases:

   (a) $H'$ contains four elements. Then there exist two points in the interior of $H'$ which can form one of the two following structures:
In the first structure (depicted on the left), we have a convex 5-gon formed by the interior points and three points from $H'$.

In the second structure, there either is a point forming a convex 5-gon with $IPQG$ or with $EPQF$ or else all three points of the outer convex hull are distributed onto $IPE$ and $GQF$, which means that at least one of these regions holds two points of $H$ and thus forms a convex 5-gon with them.

(b) $H'$ contains three points. The argument is essentially the same as for the second structure in the previous subcase:

![Diagram](image)

To not get a convex 5-gon, $GPE$ and $GQF$ cannot contain more than one point of the outer convex hull $H$ each, but $EPQF$ cannot contain any point. This gives a contradiction and a convex 5-gon must exist.

Thus we find a convex 5-gon in and point set of size at least 9.

**Definition 146** (Graph-Ramsey numbers). For graphs $G_1, \ldots, G_k$, the graph-Ramsey number $R(G_1, \ldots, G_k)$ is the smallest integer $n$ so that every $k$-coloring of the edges of $K_n$ contains a copy of $G_i$ in color $i$.

Notice that the existence of the graph-Ramsey numbers is easily reduced to the existence of the Ramsey numbers.

**Example 147.**

1. We have $R(K_3, K_3) = 6$ (cf. Example 117).

2. We have $R(C_4, C_4) = 6$ (coloring the outer border red of $K_5$ and the inner parts blue gives the lower bound 6).

**Lemma 148** (Auxilliary Tool). Let $T$ be a tree with $k$ edges. Then every graph $G$ with $\delta \geq k$ contains a copy of $T$.

**Proof.** We use induction on $k$ (note that the case $k = 1$ is easy).

So let $k \geq 1$ and $T$ be a tree on $k + 1$ edges. Further, let $G$ be a graph with $\delta \geq k + 1$ and let $\ell$ be a leaf of $T$. Remove $\ell$ and its incident edge $[\ell, v]$ from $T$ and apply the inductive statement to the remaining tree. Putting $\ell$ and its edge back (possible as we have $\delta \geq k + 1$) gives the desired statement.

**Theorem 149.** Let $T$ be a tree on $b$ vertices. Then $R(K_a, T) = (a - 1)(b - 1) + 1$.

**Proof.** To see that $R(K_a, T) \geq (a - 1)(b - 1) + 1$ holds, take $a - 1$ copies of the $K_{b-1}$ and color them blue and the remaining edges red.
For the converse statement $R(K_a, T) \leq (a - 1)(b - 1) + 1$, we use induction. So consider some coloring of $K_{(a-1)(b-1)+1}$ and fix some vertex $v$. Suppose $v$ has at least $(a - 2)(b - 1) + 1$ incident red edges. Then by the induction hypothesis, the red neighborhood of $v$ contains a blue $T$ or a red $K_{a-1}$ and using $v$ we find a red $K_a$.

In the remaining cases, every $v \in V$ has at least $b - 1$ blue neighbors and using the auxiliary tool (cf. Lemma 148) we get a blue $T$. 

\[\square\]
12 Random graphs

Finally, we look at properties of random graphs. In the following, we will denote the base-2-logarithm as $\log(n)$.

**Definition 150** (Erdős, Rényi 1959, Gilbert 1959). The Erdős–Rényi random graph $G(n, p)$ has $n$ vertices and every possible edge occurs (independently) with probability $p$.

Based on the Markov inequality (cf. Lemma 104), we derive the following tool:

**Lemma 151.** Let $X \geq 0$ be an integer-valued random variable. Then $\Pr[X = 0] \geq 1 - \mathbb{E}[X]$  

**Proof.** We have $\Pr[X = 0] = 1 - \Pr[X \geq 1] \geq 1 - \mathbb{E}[X]$. \hfill $\square$

We now derive a strong statement on the clique number of $G(n, 1/2)$ in two theorems:

**Theorem 152.** For every $\varepsilon \geq 0$,

$$\lim_{n \to \infty} \Pr[\omega(G(n, 1/2)) \leq (2 + \varepsilon) \log(n)] = 1$$

**Proof.** Let $k = (2 + \varepsilon) \log(n)$ and let $X_k$ be a random variable counting the number of $k$-cliques in $G(n, 1/2)$.

Then we have

$$\mathbb{E}[X_k] = \binom{n}{k} 2^{-\binom{k}{2}} \leq n^k 2^{-\binom{k}{2}} = 2^{k \log(n) - \binom{k}{2}}$$

and the exponent satisfies

$$k \log(n) - \binom{k}{2} = \frac{k}{2} (2 \log(n) - k + 1) = \frac{k}{2} (-\varepsilon \log(n) + 1)$$

which diverges to $-\infty$ as $n$ goes to $\infty$ (as $k \geq 0$). Thus, the expected value $\mathbb{E}[X_k]$ converges to 0 as $n$ approaches $\infty$ and our tool (cf. Lemma 151) gives the desired statement. \hfill $\square$

We introduce another bound derived from the Markov inequality for our next proof. Recall that the variance of a random variable $X$ is given by $\mathbb{E}[(X - \mathbb{E}(X))^2]$, the expected value of the squared distance to the expected value of $X$.

**Lemma 153** (Chebyshev inequality). Let $X$ be a random variable with expected value $\mu = \mathbb{E}[X]$ and variance $\text{Var}[X]$. Then for non-negative $b$, we get

$$\Pr[|X - \mu| \geq b] \leq \frac{\text{Var}[X]}{b^2}$$
Proof. Use the Markov inequality for the random variable \((X - \mu)^2\). □

We are going to need another tool:

**Lemma 154.** Let \(X \geq 0\) be an integer-valued random variable. Then \(\Pr[X = 0] \leq \text{Var}[X]/\mathbb{E}^2[X]\).

Now we can prove a technical lemma that will aid us.

**Lemma 155 (Technical lemma).** Let \(n \geq 1\) and \(k = (2 - \varepsilon) \log(n)\). As \(n\) goes to infinity,

\[
S = \binom{n}{k}^{-1} \sum_{\ell=2}^{k} \binom{k}{\ell} \binom{n-k}{k-\ell} 2^{\binom{\ell}{2}}
\]

approaches 0.

**Proof.** We use

\[
\binom{k}{\ell} \leq k^\ell
\]

and

\[
\binom{n-k}{k-\ell} \leq \frac{(n-k)^{k-\ell}}{(k-\ell)!}
\]

and

\[
\binom{\ell}{2} \leq \frac{\ell^2}{2}
\]

and

\[
\binom{n}{k} \geq \frac{(n-k)^k}{k!}
\]

Plugging these values into the given expression gives

\[
S \leq \sum_{\ell=2}^{k} k^\ell \frac{(n-k)^{k-\ell}}{(k-\ell)!} 2^{\binom{\ell}{2}} \frac{k!}{(n-k)^k}
\]

\[
\leq \sum_{\ell=2}^{k} \frac{k^{2\ell}}{(n-k)^\ell} 2^{\binom{\ell}{2}}
\]

where we used \(k!/(k-\ell)! \leq k^\ell\). Using \(\ell \leq k\) we further get

\[
\leq \sum_{\ell=2}^{k} \frac{k^{2\ell}}{(n-k)^\ell} 2^{\binom{\ell}{2}}
\]

\[
= \sum_{\ell=2}^{k} \left(\frac{k^{2\ell/2}}{n-k}\right)^\ell
\]

\[
\leq \sum_{\ell=2}^{k} \left(\frac{2k^{2\ell/2}}{n}\right)^\ell
\]

as we have \(n-k \geq n/2\) for large \(n\). Then

\[
\frac{2}{n} \frac{k^{2\ell/2}}{n} = \frac{2k^{2\ell/2}}{n} 2^{(1-\varepsilon/2)\log(n)}
\]

\[
= 2k^{2} n^{-\varepsilon/2}
\]

\[
\leq n^{-\varepsilon/4}
\]
and thus, 
\[ S \leq \sum_{\ell=2}^{k} n^{-\ell/4} \leq kn^{-\ell/4} \]
which approaches 0 as \( n \) grows towards infinity.

And finally, we are able to complete our considerations of the clique number started in Theorem \[152\].

**Theorem 156.** For every \( \varepsilon > 0 \),
\[ \lim_{n \to \infty} \Pr[\omega(G(n, 1/2)) \leq (2 - \varepsilon) \log(n)] = 1 \]

**Proof.** Let \( k = (2 - \varepsilon) \log(n) \) and for a \( k \)-element subset \( S \) of \( V \), let \( X_S \) be the indicator random variable of the statement “\( S \) is a clique in \( G \)”. Thus, the random variable \( X_k = \sum_{S \subseteq (V)_k} X_S \) counts the number of \( k \)-cliques and we have
\[ \mathbb{E}[X_k] = \binom{n}{k} 2^{-\binom{k}{2}} \]
and we can bound the variance of \( X_k \) as follows:
\[ \text{Var}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}^2[X_k] \]
\[ = \sum_{S \subseteq (V)_k} \sum_{T \subseteq (V)_k} \mathbb{E}[X_S X_T] - \mathbb{E}[X_S] \mathbb{E}[X_T] \]
We introduce \( \ell = |S \cap T| \) and get
\[ \text{Var}[X_k] \leq \sum_{\ell=2}^{k} \sum_{S \subseteq (V)_k} \sum_{T \subseteq (V)_k, |T \cap S| = \ell} \mathbb{E}[X_S X_T] - \mathbb{E}[X_S] \mathbb{E}[X_T] \]
For the case \( \ell = 0 \) we have independent cliques which satisfy \( \mathbb{E}[X_S X_T] - \mathbb{E}[X_S] \mathbb{E}[X_T] = 0 \).

Now, if we fix \( \ell \), there are \( \binom{n}{k} \) choices for \( S \), \( \binom{k}{\ell} \) choices for \( S \cap T \) and \( \binom{n-k}{k-\ell} \) choices for \( T \setminus S \) and we can bound the variance further as
\[ \text{Var}[X_k] \leq \sum_{\ell=2}^{k} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} \mathbb{E}[X_S X_T] \]
As \( X_S X_T \) only takes the values 0 and 1 (if both \( S \) and \( T \) are \( k \)-cliques), we get
\[ \mathbb{E}[X_S X_T] = \Pr[S \text{ a clique}, T \text{ a clique}] = 2^{-2\binom{k}{2} + \binom{\ell}{2}} \]
and plugging it into our previous bound on the variance gives
\[ \text{Var}[X_k] \leq \sum_{\ell=2}^{k} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} 2^{-2\binom{k}{2} + \binom{\ell}{2}} \]

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Using the tool we got from the Chebyshev inequality (cf. Lemma 154), \( E[X_k] \) and our bound on the variance we get
\[
\Pr[X_k = 0] \leq \frac{\text{Var}[X_k]}{E[X_k]^2}
\leq \sum_{\ell=2}^{k} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} 2^{-2\binom{\ell}{2}} \frac{2^2(\ell)}{\binom{n}{k}}
= \binom{n}{k}^{-1} \sum_{\ell=2}^{k} \binom{k}{\ell} \binom{n-k}{k-\ell} 2^{\binom{\ell}{2}}
\]
which we know to converge to 0 by our technical lemma (cf. Lemma 155). Thus, we find a clique of the desired size with very high probability.

Hence, we know that \( G(n, 1/2) \) has a clique number of \( 2 \log(n) \) with high probability (as \( n \) gets large). The proof method we used is called second moment method (i.e. we used the second moment to prove that some probability is positive).

It is also known that \( G(n, 1/2) \) has a chromatic number of \( n/(2 \log(n)) \) (for large \( n \) with high probability) and for \( p = 1 - 1/c \) it is known that \( G(n, p) \) has a clique number of \( 2 \log \frac{1}{c-1}(n) \) and a chromatic number of \( n/(2 \log_e(n)) \) (for large \( n \) with high probability).

Next up is a result about the diameter of \( G(n, 1/2) \):

**Theorem 157.** As \( n \) goes to infinity, \( \Pr[G(n, 1/2) \text{ has diameter } 2] \) converges to 1.

**Proof.** For \( u, v \in V \) let \( X_{uv} \) be the indicator variable that \( u \) and \( v \) have no common neighbor and let \( X = \sum X_{uv} \) count the number of vertex pairs without common neighbors. Then we have \( \Pr[X_{uv} = 1] = (1 - 1/4)^{n^2} \) and thus \( \mathbb{E}[X] = \frac{n}{2}(3/4)^{n^2} \) which converges to 0 as \( n \) grows towards infinity, giving us the desired statement.

**Theorem 158.** As \( n \) goes to infinity, \( \Pr[\text{bw}(G(n, 1/2)) \geq n - 6 \log(n)] \) converges to 1 (where \( \text{bw}(G) \) denotes the bandwidth of \( G \)).

**Proof.** We show that almost all random graphs have the following property: “For any choice of disjoint \( S, T \subseteq V \) of size \( k = 3 \log(n) \), there is an edge between \( S \) and \( T \) with high probability”

The probability that there is no edge between \( S \) and \( T \) is \( (1/2)^{k^2} \) and thus, we get
\[
\mathbb{E}[\text{number of such } S, T \text{ without connecting edge}] = \binom{n}{k} \binom{n-k}{k} \frac{1}{2}^{k^2}
\leq n^{2k} \left( \frac{1}{2} \right)^{k^2}
= 2^{k \log(n) - k^2}
\]
which converges to 0 as we have \( 2k \log(n) - k^2 = -3 \log^2(n) \). By the Markov inequality (cf. Lemma 151), the probability that all such \( S, T \) have a connection goes to 1 for large \( n \).

For the bandwidth, consider some embedding and let \( S \) contain the first \( 3 \log(n) \) vertices and \( T \) the last \( 3 \log(n) \) vertices. Thus almost all graphs have an edge between all possible choices for \( S \) and \( T \) (for large \( n \)) and thus, every embedding has a dilation of at least \( n - 6 \log(n) \).
Moving on from the Erdős-Rényi random graphs, there are other interesting probabilistic statements or theorems with a probabilistic proof:

**Theorem 159** (Caro 1979). A graph $G$ satisfies

$$
\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{\deg(v_i) + 1}
$$

with high probability (where $\alpha(G)$ is the size of the largest independent set of $G$).

*Proof.* Order the vertices uniformly at random and construct an independent set by sweeping over the ordering and including $v_i$ if and only if it occurs before all its neighbors in the ordering (which obviously gives an independent set).

Now, the probability of $v_i$ being included is $\frac{1}{\deg(v_i) + 1}$ as we basically randomly permute the set $\{v_i\} \cup \Gamma(v_i)$ (which has $\deg(v_i) + 1$ elements) and include $v_i$ if it comes first in this permutation.

Thus, we have

$$
\mathbb{E}[|S|] = \sum_{i=1}^{n} \frac{1}{\deg(v_i) + 1}
$$

\[\square\]

**Theorem 160** (Alon, 1990). Let $G$ be a graph and $\delta(G) = k$. Then $G$ has a dominating set of size

$$
n \frac{1 + \ln(k + 1)}{k + 1}
$$

*Proof.* Let $p = \ln(k + 1)/(k + 1)$ and pick every vertex $v \in V$ with probability $p$ to get a random subset $S$ of $V$ (thus, we have $\mathbb{E}[|S|] = np$).

Set $T = V \setminus (S \cup \Gamma(S))$ gives (recall $\delta(G) = k$)

$$
\Pr[v \in T] = \Pr[v \notin S, \Gamma(v) \cap S = \emptyset] \leq (1 - p)^{k+1}
$$

and for the expected value of the size of $T$, we thus get $\mathbb{E}[|T|] = n(1 - p)^{k+1}$.

Using the linearity of expectation (and the fact $1 + x \leq e^x$ we learned previously), we get

$$
\mathbb{E}[|S \cup T|] \leq np + n(1 - p)^{k+1}
\leq np + ne^{-p(k+1)}
= n(p + e^{-p(k+1)})
= n \left( \frac{\ln(k + 1)}{k + 1} + e^{-\ln(k+1)} \right)
= n \left( \frac{\ln(k + 1)}{k + 1} + \frac{1}{k + 1} \right)
= n \frac{1 + \ln(k + 1)}{k + 1}
$$

and as $S \cup T$ is a dominating set of $G$, we get the desired statement. \[\square\]
Picking a random set which may not have the desired properties (but is close enough) and fixing it up adequately (by adding/deleting elements) is called the deletion method.

**Definition 161** (Tournaments, Winning sets). A tournament \((V, A)\) is an orientation of a complete graph.

A set \(S \subseteq V\) is a winning set if there is no \(v \in V \setminus S\) with \([v, s] \in A\) for all \(s \in S\).

**Example 162.**

In the left tournament, the set \(\{b, c\}\) is a winning set whereas in the right tournament it is not.

**Theorem 163.** For any \(k \geq 1\), there exists a tournament without a winning set of size \(k\).

*Proof.* We start with an undirected \(K_n\) and orientate each edge (both directions are equiprobable at \(p = 1/2\)) to obtain a random tournament on \(n\) vertices. For a \(k\)-element subset \(S\) of \(V\), we have

\[
\Pr[S \text{ a winning set}] = \left(1 - \frac{1}{2^k}\right)^{n-k} \leq e^{2^{-k}(n-k)}
\]

by \(1 + x \leq e^x\) and thus, the probability that our tournament has a winning set of size \(k\) is

\[
\Pr[\text{Tournament has winning set of size } k] \leq \binom{n}{k} e^{-2^{-k}(n-k)} \leq n^k e^{-2^{-k}(n-k)} = e^{k \ln(n) - 2^{-k}(n-k)}
\]

which converges to 0 as \(n\) grows large. \(\Box\)

At this point, our journey ends at a remarkable theorem about the first-order theory of graphs

**Theorem 164** (Zero-one law). For every first-order formula \(\varphi\),

\[
\lim_{n \to \infty} \Pr[G(n, 1/2) \models \varphi]
\]

is either 0 or 1.

*Proof.* We prove the theorem in 6 steps.
1. For \( r, s \geq 1 \) define the first-order formula \( \varphi_{r,s} \) stating that for pairwise distinct vertices \( x_1, \ldots, x_r \) and \( y_1, \ldots, y_s \), there exists another vertex \( z \) such that \( z \) is adjacent to all \( x_i \) but to no \( y_i \).

2. For all \( r, s \geq 1 \), we have
\[
\lim_{n \to \infty} \Pr[G(n, 1/2) \models \varphi_{r,s}] = 1
\]
which can be proven just like the previous theorem (cf. Theorem 163).

3. There exists a countable graph \( G \) on the vertex set \( \mathbb{N} \) which models \( \varphi_{r,s} \) for all \( r, s \geq 1 \).

To see this, we construct \( G \) inductively. Start with the single vertex 1 and let \( V_i \) be the set of vertices constructed in the first \( i \) phases. Now, for every \( X \subseteq V_i \) and \( Y \subseteq V_i \) so that \( X \cap Y \neq \emptyset \), we create a new vertex \( v \) connected to all of \( X \) but to no vertex of \( Y \).

4. If two countable graphs \( G, H \) satisfy \( \varphi_{r,s} \) for all \( r, s \geq 1 \), they are isomorphic.

We can (again inductively) construct an isomorphism by defining \( f(1), f^{-1}(1), f(2) \) and so on:

For this, we set \( f(1) = 1 \) and when \( f(i) \) has to be determined, then \( f \) has already been specified on all of \( V_i \) (from the previous step) and we set \( f(i) = y \in V_H \) so that \( [v, i] \in E_G \) if and only if \( [f(v), y] \in E_H \) for all \( v \in V_i \). The existence of \( y \) is guaranteed due to the fact that both \( G \) and \( H \) satisfy \( \varphi_{r,s} \) for all \( r, s \geq 1 \).

5. The theory of the sentence set \( \Phi \) consisting of all sentences \( \varphi_{r,s} \) (for \( r, s \geq 1 \)) is complete, i.e. for any first-order sentence \( \psi \) we either have \( \Phi \models \psi \) or \( \Phi \models \neg \psi \).

For this, suppose adding \( \psi \) to \( \Phi \) gives the consistent theory \( T' \) and adding \( \neg \psi \) gives another consistent theory \( T'' \) (i.e. we suppose \( \psi \) is independent from \( \Phi \)). Then by Gödel’s completeness theorem (and the Löwenheim-Skolem theorem), we get the existence of countable models for the theories \( T' \) and \( T'' \).

These models are \( G \) and \( H \) from the previous step, but they are isomorphic. Since isomorphic first-order structures satisfy the same sentences, we get a contradiction (as we would have to satisfy \( \psi \) and \( \neg \psi \) at the same time).

6. Let \( \psi \) be any first-order sentence provable from the axiom system \( \Phi \) consisting of all \( \varphi_{r,s} \) for \( r, s \geq 1 \).

By the compactness theorem, there exists a finite subset \( S^* \) of \( \Phi \) implying \( \psi \) and we have
\[
\Pr[G(n, 1/2) \not\models \psi] \leq \sum_{\varphi_{r,s} \in S^*} \Pr[G(n, 1/2) \not\models \varphi_{r,s}]
\]
which converges to 0 (step 2). Hence \( G(n, 1/2) \) satisfies \( \psi \) with probability 1 as \( n \) grows large (and analogous for \( \Phi \not\models \psi \)).