There exists planar graph that is not 4-choosable

Proof: $G(\alpha, \beta)$ with $\alpha = \{5, 6, 7, 8\}$

$\beta = \{9, 10, 11, 12\}$

Claim: $G(\alpha, \beta)$ has no compatible list-coloring

Take 16 graphs $G(\alpha, \beta)$

identify $\alpha$-vertices $L = \{5, 6, 7, 8\}$

identify $\beta$-vertices $L = \{9, 10, 11, 12\}$

**Theorem**

Every outer-planar graph is 3-choosable

Proof by induction on $|V|$

- Every outer-planar graph has vertex of degree $\leq 2$
- Color $G - v$ inductively, put $v$ back & pick color

Ramsey theory

Message = "total chaos is impossible"

="every very large, very irregular structure contains a large regular substructure"

Warm-up #1: Among any six persons, there are three pairwise friends or three pairwise enemies.

Warm-up #2: Every sequence of $n^2 + 1$ reals contains monotone subsequence of length $n + 1$

Proof: Consider sequence $a_{1}, \ldots, a_{n^2}$
• Let $x_i(y_j)$ denote the length of largest increasing (decreasing) sub-sequence that ends with $x_i$.
• $(x_i, y_j) \neq (x_j, y_j)$ for $i \neq j$.
• There are $n^2$ pairs over $1, \ldots, n$.

**Def** For integers $a, b \geq 1$, $R(a, b)$ is the smallest integer $N$ so that every red-blue coloring of the edges of $K_N$ contains red $K_a$ or blue $K_b$.

**Ex**  $R(a, b) = R(b, a)$
• $R(a, 1) = R(1, a) = 1$
• $R(a, 2) = R(2, a) = a$
• $R(3, 3) = 6$

**Thm** (Ramsey, 1930)
$R(a, b)$ exists for all $a, b \geq 1$.

**Proof** by induction on $a+b$.
• Consider $a, b \geq 3$ and let $N' = R(a, b-1) + R(a-1, b)$.
• Consider arbitrary red-blue coloring of $K_N$.
• Pick vertex $v$ of $N-1$ incident edges $R(a, b-1)$ have color blue or $R(a-1, b)$ have color red.
• Wlog $R(a, b-1)$ edges of color blue.
• Consider the subgraph induced by "blue neighbours" of $v$.
• By inductive assumption, this subgraph contains red $K_a$ or blue $K_{b-1}$.

**Lemma** $R(a, b) \leq \left( \frac{a+b-2}{a-1} \right) 2^{a+b-2}$.

**Thm** (Erdős, 1947)
If $\binom{n}{a} < 2^{\frac{a-1}{2}}$, then $R(a, a) > n$. 
Proof: Color edges of $K_n$ randomly and independently with $\Pr(e=\text{blue}) = \frac{1}{2}$

- For $T \in V$ with $|T|=a$, define event $A_T$'s clique over $T$ is monochromatic.
- $\Pr[A_T] = \frac{2 \cdot 3^{a-1}}{2(3)^a} = \frac{1}{2} \cdot \frac{3^a}{3^a}$
- $\Pr[\text{at least one } A_T] = \sum_T \Pr[A_T] = 2^{a-1} \cdot \frac{3^a}{3^a} = \frac{3^a}{2^a} < 1$ \quad \Box

Lemma: $R(a,a) > 2^{a/2}$ for $a \geq 3$

Proof: For $n = 2^{a/2}$ we have $\binom{n}{a} \cdot 2^{1-\frac{a}{2}} \leq \frac{n^a}{a!} \cdot 2^{1-\frac{a}{2}} = \frac{1}{a!} 2^{\frac{a}{2}} 2^{1-\frac{a}{2}} < 1$ \quad \Box
\[ R(a, b) \leq R(a, b-1) + R(a-1, b) \]

**Ramsey bounds**

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+ Lower + Upper Bounds

**Theorem (Kim, 1995)**

\[ R(a, 3) = \Theta \left( \frac{a^2}{\log a} \right) \]

**Proof (Lower Bound)**

Red edges in \( K_8 \)

(i) No red triangle \( K_3 \)

(ii) No \( K_4 \)

\[ \Rightarrow R(3, 4) \leq 9 \]

(Upper Bound)

\[ R(a, 5) \neq R(3, 4) \leq R(3, 3) + R(2, 4) = 10 \]

Consider vertex \( v \) in \( K_3 \)

- \( R(2, 4) = 4 \Rightarrow v \) has \( < 4 \) red edges
- \( R(3, 3) = 6 \Rightarrow v \) has \( < 6 \) blue edges

Remaining case: Every \( v \) is incident to 3 red edges

**Variant**

\[ R(a, b, c) \]

Why do these values exist?

\[ R(a, b, c) \leq R(a, R(b, c)) \]
Def: For integer $r \geq 1$ and finite set $S$, let $(S)_r$ denote the system of $r$-element subsets of $S$.

- Let $f : (S)_r \to \{1, \ldots, k\}$ be a $k$-coloring. A subset $T \subseteq S$ is called **mono-chromatic** under $f$ if all $X, Y \in (S)_r$ have $f(X) = f(Y)$.

Def: For integers $r$ and $p_1, p_2, \ldots, p_r$, the **Ramsey number** $R_r(p_1, p_2, \ldots, p_r)$ is the smallest integer $N$ s.t. for every $k$-coloring of $(S)_r$ with $|S| = N$, there exists an $i$ and $T \subseteq S$ s.t. $|T| = p_i$ and $T$ is mono-chromatic under $f$.

**Example:**

- $r = 2$
- $r = 1$

\[ R_1(p_1, \ldots, p_r) = \sum p_i - (k - 1) \]

\[ R_1(a, b) = a + b - 1 \]

**Theorem:** $R_r(a, b)$ exists for $r \geq 2$

**Proof:** Outer induction on $r$ and inner induction on $ab$

- $r = 2$ done
- Cases with $(a < r)$ or $(b < r)$ trivial

- $R_r(r, x) = R_r(x, r) = x$

**Inductive step:**

- Let $x = R_r(a - 1, b)$
- $y = R_r(a, b - 1)$
- Let $N = 2^{R-1}(x, y) + 1$
- Consider set $S$ with $|S| = N$
- Let $f : (S') \rightarrow \{\text{red, blue}\}$
- Let $v \in S$
- Define $S' = S - v$
- Define $f' : (S') \rightarrow \{\text{red, blue}\}$ via $f'(T) = f(T \cup v)$
- Inductive hypothesis yields
  red, $x$-set on
  blue, $y$-set of $S'$ under $f'$
- W.l.o.g. $T' \subset S'$ with $|T'| = x$ and $T'$ is red under $f'$.
- Consider $(T') \subset S$ under $f$
  By inductive hypothesis there is
  (i) red subset of size $a - 1$ under $f$ or
  (ii) blue subset of size $b$ under $f$.
- Case (ii) settles the theorem.
- Case (i) yields a red set $T'' \subset T'$ of size $a - 1$ under $f$.
  All $(R - 1)$-subsets of $T''$ are red under $f'$
  $\Rightarrow$ all $R$-subsets of $T'' \cup v$ that contain $v$ are red under $f$.
  $\Rightarrow$ all $R$-subsets of $T'' \cup v$ are red under $f$. 

\[\square\]
Def: $R_k(3) = \frac{R_k(3)}{k \text{ times}}$

Ex: $R_2(3) = 6$

Thm: $R_k(3) \leq k \cdot (R_{k-1}(3) - 1) + 2$ for $k \geq 3$

Proof: Consider some fixed vertex $v$ in $k$-coloring of edges of complete graph on $N = k \cdot (R_{k-1}(3) - 1) + 2$
- There are $N - 1$ incident edges
- One color occurs on at least $\frac{N-1}{k} = R_{k-1}(3)$ edges

Let $S$ be set of (other) end vertices of these edges.
(Case 1) $S$ spans a red edge $\rightarrow$ edge $tv$ forms red triangle
(Case 2) $S$ only spans non-red edges then $R_{k-1}(3)$ vertices, edges $(k-1)$-colored $\rightarrow$ monochromatic triangle in $S$

Lem: $R_k(3) \leq L \cdot k! \cdot \left\lfloor \frac{k}{2} \right\rfloor$ for $k \geq 3$

Proof: $L \cdot k! \cdot \left\lfloor \frac{k}{2} \right\rfloor = L \cdot k! \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{1}{j!} = L \cdot \frac{k!}{0!} + \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} \frac{1}{j!} = 1 + k \cdot \frac{1}{0!} \cdot \left\lfloor \frac{k}{2} \right\rfloor$

Ex: $R_3(3) \leq 3 \cdot (6 - 1) + 2 = 17$

Known: $R_3(3) = 17$
- The edges of $K_{16}$ can be partitioned into three copies of Paley graph

(Ex 4)

Def: For $k \geq 2$, let $S(k)$ denote smallest integer so that:
- in every $k$-coloring of the integers $\{1, 2, ..., S(k)\}$, there exist $x, y, z$ with same color and $x + y = z$ "Schur Number"

Ex: $S(2) = 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

\begin{align*}
1 + 2 &= 3 \\
2 + 2 &= 4 \\
1 + 3 &= 4 \\
1 + 4 &= 5 = 5 \\
\end{align*}

Thm (Schur 1946)
- $S(k)$ exists for $k \geq 1$
Proof Let \( n = \bar{R}_4(3) \)

- Consider \( k \)-coloring \( c \) of \( S(1, 2, \ldots, n) \)
- Consider \( K_n \) on \( S(1, 2, \ldots, n) \)
  - Color edge \([i, j] \) by color \( c([i, j]) \)
- Ramsey yields existence of monochromatic triangle \( i,j,k \)
  - Edges \([i, j], [i, k], [j, k] \) have same color
  - Integers \( i-j, i-k, j-k \) have same color

WLOG \( i > j \geq k \) Then \( i-k = [i-j-j] + [j-k-k] \)

**Theorem** \( S(k) \geq 3S(k-1) - 1 \) for \( k \geq 2 \)

**Proof** by induction on \( k \)

- \( S(2) = 5 \) and \( S(4) = 2 \)
- Inductive step \( t = S(k-1) \)

1. **2** \( \ldots \), \( t-1 \)

- \( k-1 \) colors no such triple

- \( 1, 2 \ldots, t-1 \), \( 2t+1 \), \( 2t-1 \), \( 3 \ldots, 3+2 \)

- \( k-1 \) colors new color \( (k-1) \)-coloring shifted by \( 2 \times 2 \)

**Claim** New \( k \)-coloring has no monochromatic solution of \( x + y = 2 \)

- Color \( k \): \( x, y \geq t \) if \( x \leq 2t-1 \)
- Color \( c \leq k-1 \)
  
  \[ L + L = L + L \leq 2t+2 \]
  
  \[ R + R = R + R \geq 4 + 4 \]
  
  \[ L + R = R \rightarrow \text{triple in } L \]

**Example**

- \( S(3) = 14 \)
- \( S(4) = 45 \)
- \( 160 \times S(5) = 315 \)

**Definition** Point set \( P \subseteq \mathbb{R}^2 \) is in general position if no three points on common line.

**Observation** (Esther Klein, 1935)

Every \( 5 \)-element point set contains convex quadrangle

**Proof**
Theorem (Erdős, Szekeres, 1935)
For every \( k \geq 4 \), there exists \( N(k) \) so that every \( N(k) \)-element point set contains a convex \( k \)-gon.

Proof. Let \( N = R_4(k,5) \)
Consider point set \( P \) with \( |P| = N \) color every 4-element subset \( S \subseteq P \) blue if \( S \) convex, red if \( S \) not convex.
- Ramsey \( \Rightarrow \) red subset of size 5 or blue subset of size \( k \)
  (Case 1) Red subset of size 5 \( \frac{5}{2} \) (Klein)
  (Case 2) Blue subset of size \( k \) \( \Rightarrow \) \( k \)-gon convex

Conjectured \( N(k) = 2^{k+2} + 1 \)
Known \( \geq \) and \( k = 4, k = 5 \)
Thm \[ N(5) = 9 \]

Proof \( (L o w e r \ B o u n d) \)

\[ N(5) > 8 \]

(Upper Bound)

Consider \( P \subseteq \mathbb{R}^2 \) with \( |P| = 3 \). Wlog \( \text{conv} \{P\} \leq 4 \).

(Case 1) Convex hull \( H \) is 4-gon. Consider \( P-H \). Wlog \( P-H \) not convex.

- three regions
- one Region has \( \geq 2 \) points of \( H \)
  
  Together with \( \triangle \), done.

(Case 2) Convex hull \( H \) is 3-gon. Consider convex hull \( H' \) of 6 interior pts. Wlog \( |H'| \leq 4 \).

(Case 2a) \( H' \) has 4 points

- trivial

(Case 2b) \( |H'| = 3 \)

- \( \Delta PQG \) no point of \( H \) (only 3 left)
- \( \Delta PQF \) no point of \( H \) otherwise 5-gon
- \( \Delta FQG \) \( \leq 1 \) pt of \( H \)
- \( \Delta EPI \) \( \leq 2 \) pts of \( H \)

Conj \[ N(6) \geq 14 \]

\( \geq 14 \) Known
Def: For graphs $G_1, \ldots, G_k$, the graph-Ramsey-number $R(G_1, \ldots, G_k)$ is the smallest integer $N$, so that 
\every \ k-coloring of \ edges \ of \ $K_N$ \ contains \ copy \ of \ $G_i$ \ in \ color \ $i$ \ for \ some \ $i$.

Existence clear.

Ex: $R(K_3, K_3) = 6$
\[ R(G_4, G_4) = 6 \] $\Rightarrow$

Tool: Let $T$ be a tree with $k$ edges.
\[ \text{Every graph} \ G \ \text{with} \ \delta(G) \geq k \ \text{contains copy of} \ T. \]

Proof (by induction on $k$ ($k=1$ trivial))

- Inductive step: Consider tree $T$ on $k+1$ edges. Let $l$ be leaf of $T$.
  Remove $l$ and incident edge $[l,v]$ from $T$. Apply inductive statement to rest-tree $T'$ and $G$. Put back $l$ and $[l,v]$. \[ \blacksquare \]

Thm: Let $T$ be a tree on $b$ vertices.
\[ \text{Then} \ R(K_{a,1}) = (a-1)(b-1)+1 \]

Proof (Lower bound)
Take $a-1$ copies of $K_{b-1}$. All remaining edges are red.

(Upper bound) by induction on $a \geq 1$
Consider red-blue coloring of $K_{a-1}(b-1)+1$
- Consider vertex $v$.
(Case 2) $v$ has $\geq (a-2)(b-1)+1$ incident red edges.
By inductive statement red neighborhood of $v$ contains a blue $T$ or red $K_{a-1}$ ($K_{a-1}uv$ is red).

In remaining cases, every $v \in V$ has $\geq (a-1)(b-1)-(a-2)(b-1) = b-1$ blue neighbours. Done since $T$ has $b-1$ edges, use tool \[ \blacksquare \]

H2 Random graphs
Def (Erdős, Rényi 1959; Gilbert 1959)
The Erdős-Rényi random graph $G(n,p)$ has $n$ vertices. Every edge occurs (indep.)
with probability $p$. 

**Random graphs**
- $G(n, p)$
- Markov inequality

**Tool 1** Let $X \geq 0$ be integer, then \[ \Pr[X = 0] \geq 1 - \mathbb{E}[X] \]

**Theorem** For every $\varepsilon > 0$, \[ \lim_{n \to \infty} \Pr[X(G(n, \frac{1}{2}) \leq (1 + \varepsilon) \log n)] = 1 \]

**Proof** Let $k = (2 + \varepsilon) \log n$.
- Let $X_k$ count the number of $k$-cliques in $G(n, \frac{1}{2})$.
- $\mathbb{E}[X_k] \leq \binom{n}{k} 2^{\binom{k}{2}} \leq n^k 2^{k^2 - \binom{k}{2}} = 2^\frac{k}{2} (2 \log n - k + 1) = 2^\frac{k}{2} (-\varepsilon \log n + 1) \to -\infty$

Hence $\mathbb{E}[X_k] \to 0$ and $\Pr[X_k = 0] \to 1$. \qed

**Lemma** Chebyshev inequality

Let $X$ be RV with $\mathbb{E}[X] = \mu$ and $\text{Var}[X]$.

Then $\Pr[|X - \mu| \geq b] \leq \frac{\text{Var}[X]}{b^2}$

**Proof** Markov for RV $(X - \mu)^2$ ? \[ \Pr[(X - \mu)^2 \geq b^2] \leq \mathbb{E}[(X - \mu)^2] / b^2 \]

**Tool 2** Let $X \geq 0$ be integer-valued RV, then $\Pr[X = 0] \leq \frac{\text{Var}[X]}{\mathbb{E}[X]}$ (set $b = \mu$ then $X = 0$ is subcase)

"Second moment method"

**Lemma** (Technical) Let $n \geq 1$ and $k = (2 + \varepsilon) \log n$. As $n \to \infty$, sum \[ \frac{k}{\varepsilon^2} \left( \frac{n}{e^2} \right)^{n-k} \to 0 \]

**Proof** We use the bounds \[ \binom{n}{k} \leq k^k \text{ and } \binom{n-k}{n-k} \leq \frac{(n-k)^n}{n!} \]

Then the sum $S$ can be bounded \[ S \leq \sum_{k=2}^{n-1} \frac{k^k}{\varepsilon^2} \left( \frac{n}{e^2} \right)^{n-k} \leq \frac{k^k}{\varepsilon^2} \left( \frac{n}{e^2} \right)^{n-k} 2^k \leq \frac{k^k}{\varepsilon^2} \left( \frac{k^2}{2} \left( \frac{n}{e^2} \right)^{n-k} \right) \leq \frac{k^k}{\varepsilon^2} \left( \frac{k^2}{2} \left( \frac{n}{e^2} \right)^{n-k} \right) \]

Then \[ \frac{2k^k}{n} \frac{2k^2}{n} \leq \frac{2k^k}{n} \frac{2k^2}{n} \left( \frac{n}{e^2} \right)^{n-k} \leq \frac{2k^k}{n} \frac{2k^2}{n} \left( \frac{n}{e^2} \right)^{n-k} \leq \frac{2k^k}{n} \frac{2k^2}{n} \left( \frac{n}{e^2} \right)^{n-k} \]

Then \[ S \leq \frac{k^k}{\varepsilon^2} \left( \frac{n}{e^2} \right)^{n-k} \leq k \cdot n^{-\varepsilon/4} = (2 - \varepsilon) \log n \cdot n^{-\varepsilon/4} \to 0 \]
Thm. For every $\varepsilon > 0$,

$$
\lim_{n \to \infty} \text{Prob}\left[ \omega(G(n, \frac{\varepsilon}{2})) \geq (2 - \varepsilon) \log n \right] \to 1
$$

Proof: Let $h = (2 - \varepsilon) \log n$

- For $h$-element $S \subseteq V$, let $X$ be indicator RV of event "$S$ is clique".
- RV $X_h = \sum \sum X_{ij}$ counts $h$-cliques

(1) $E[X_h] = \binom{\frac{n}{2}}{h} 2^{-h}$

We bound the variance of $X_h$ as follows.

$$
\text{Var}[X_h] = E[X_h^2] - [E[X_h]]^2
$$

$$
= \sum \sum \sum \sum E[X_{ij}X_{kl}] - [E[X_{ij}]E[X_{kl}]]^2
$$

$$
\leq \sum \sum \sum E[X_{ij}X_{kl}]
$$

For fixed $l$, there are

- $\binom{n}{h}$ choices for $S$

- $\binom{l}{h}$ choices for $T_n S$

- $\binom{n - l}{h - l}$ choices for $T - S$

Hence:

$$
\text{Var}[X_h] \leq \sum \sum \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l} \cdot 2^{-h} \cdot \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l}
$$

$$
= \sum \sum \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l} \cdot 2^{-h} \cdot \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l}
$$

(2)

We use (1) and (2) in Chebyshev tool.

$$
\text{Prob}[X_h = 0] \leq \frac{\text{Var}[X_h]}{E^2[X_h]} \leq \sum \sum \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l} \cdot 2^{-h} \cdot \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l}
$$

$$
\leq \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l} \cdot 2^{-h} \cdot \binom{n}{h} \binom{l}{h} \binom{n - l}{h - l}
$$

$$
\rightarrow 0
$$
• Clique number of $G(n,\frac{\alpha}{e})$ is $2\log n$
• Chromatic number of $G(n,\frac{\alpha}{e})$ is $\frac{n}{2\log n}$ (since $\alpha=\omega=2\log n$)
• For $G(n,p)$ with $p=1-\frac{\omega}{n}$ with $\omega\geq 1$
  $\omega=2\cdot\log_\omega(n)$
  $\omega=\frac{n}{2\log n}$

**Thm.** As $n$ goes to $\infty$, $\Pr[\text{has diam } 2T]$ goes to 1.

**Proof.** For $u,v \in V$, let $X_{uv}$ be indicator RV that $u$ and $v$ have common neighbor
• Let $X=\sum X_{uv}$ count number of vertex pairs without common neighbor
• $\Pr[X_{uv}=1]=(1-\frac{1}{4})^{n-2}$
• $E(X)=\binom{n}{2}(\frac{3}{4})^{n-2} \rightarrow 0$

• $G(n,\frac{\alpha}{e})$ is connected

**Thm.** As $n$ goes to $\infty$, probability that bandwidth of $G(n,\frac{\alpha}{e})$ is at least $n-6\log n$ tends to 1.

**Proof.** We show that almost every graph has the following property:
For any choice of $S$ and $T$, $S, T \subseteq V$, $S \cap T = \emptyset$, $|S|=|T|=3\log n = k$
there is an edge between $S$ and $T$
• $\Pr[\text{no edge between } S \text{ and } T] = \left(\frac{1}{2}\right)^{n-k}$
• $E[\# \text{ edges between } S \text{ and } T] = \binom{n}{k} (\frac{1}{2})^{n-k} = n^{n-k} (\frac{1}{2})^{k} = 2^{2k} \log n - k^2 \rightarrow 0$
  since $6\log^2 n - 3\log^2 n \rightarrow -\infty$
• Markov $\Rightarrow \Pr[\text{all } S, T \text{ are good }] \rightarrow 1$
• For bandwidth, consider embedding and let $S$ contain first $3\log n$ vertices and $T$
  contain last $3\log n$ vertices. The bandwidth is $n-6\log n$.
  $n-6\log n + \epsilon$ possible
**Thm.** (Caro 1979)

A graph $G$ with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ satisfies

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}$$

**Proof:** Order vertices uniformly at random. Construct independent set $S$ by sweeping over the ordering. Vertex $v$ is included in ind. set $S$ if $v$ occurs before all its neighbors.

- Hence $S$ is independent
- $\mathbb{E}[\mid S \mid] = \sum_{v_i \in S} \frac{1}{d_i + 1}$
- $\mathbb{E}[\mid S \mid] \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}$

**Thm.** (Alon 1990)

Let $G$ be a graph, $\delta(G) = k$. Then $G$ has dominating set of size

$$n \cdot \frac{\ln (k+1)}{k+1}$$

**Proof:** Let $p = \frac{\ln (k+1)}{k+1}$. Pick every vertex $v \in V$ with probability $p$ to get random subset $S \subseteq V$.

- Let $T = V - (S \cup N(S))$
- $\mathbb{E}[\mid S \mid] = np$
- $\mathbb{E}[\mid T \mid] = \mathbb{E}[\text{no neighbor of } S \text{ in } S] \leq (1-p)^k$
- $\mathbb{E}[\mid T \mid] \leq n \cdot (1-p)^k$
- $\mathbb{E}[\mid S \cup T \mid] \leq n \cdot (p \cdot (1-p)^k) + \frac{\ln \frac{\epsilon}{n \cdot (1-p)^k}}{\ln (k+1)}$
  $$\leq np + ne^{-p(k+1)} = n \cdot \frac{\ln (k+1)}{k+1} + ne^{-\ln (k+1)}$$
A tournament is a digraph \((V, A)\) with exactly one arc between any two vertices.

• A \(S\) is a winning set if there is no \(v \in V \setminus S\) with \(v \notin A\) for all \(s \in S\).

**Theorem** For any \(k \geq 1\), there exists a winning set of size \(k\).

**Proof** In (undirected) \(K_n\), every edge is oriented in either direction with probability \(\frac{1}{2}\).

• For \(k\)-element \(S \subseteq V\), \(\text{Prob}\left[S \text{ winning}\right] = \left(1 - \frac{1}{2^k}\right)^{n-k} \leq e^{-2^k(n-k)}

• The probability that a tournament has a winning set is \(\leq \binom{n}{k} e^{-2^k(n-k)} = e^{O(n^2) - 2^k(n-k)}\) \(\xrightarrow{n \to \infty} 0\).

**First order theory of graphs**

• Boolean operators \(\land, \lor, \neg, \Rightarrow\)

• Existential, universal quantifiers

• Variables represent vertices

• Equality \((u = v, u \neq v)\), adjacency \(u \sim v\) (symmetric, anti-refl)

**Example** \(G\) contains a path of length 3

\[\exists a, b, c, d : (a \rightarrow b) \land (b \rightarrow c) \land (c \rightarrow d)\]

• Every edge is contained in a triangle

\[\forall x, y : (x \sim y) \Rightarrow \exists e : (2 \sim x) \land (2 \sim y)\]

• \(G\) is connected

• \(G\) is planar

• \(G\) is Hamiltonian

**Theorem (Fagin, 1976)** "zero-one-law"

For every first order statement \(S\)

\[\lim_{n \to \infty} \text{Prob}\left[G(n, \frac{1}{2}) \text{ satisfies } S\right] = 0 \text{ or } 1\]

**Proof** For \(k = 1\), statement \(\text{Ans}\) says "For any pairwise distinct vertices \(x_1, \ldots, x_k\) and \(y_1, \ldots, y_k\), there exists another vertex \(z\) that is adjacent to all \(x_i\) and \(y_i\) for \(i = 1, \ldots, k\)."
non-adjacent to all $y_i$.

(2) For all $r, s \geq 1$, \[ \lim_{n \to \infty} \Pr [ \text{G}(n, \frac{1}{2}) \text{ has } A_{r,s} ] = 1 \]

(3) There exists a countable graph $G$ on vertex set $\mathbb{N}$ that satisfies $A_{r,s}$ for all $r, s \geq 1$.

Proof by inductive construction

- Start with a single vertex $A$.
- Let $V_i$ be the set of vertices constructed in the first $i$ phases.
- For every $X \subseteq V_i$ for every $Y \subseteq V_i$, so that $X \neq \emptyset$, $Y \neq \emptyset$, $X \cap Y = \emptyset$, create a new vertex $v_i$ with $v_i \in X \forall x \in X$ and $v_i \in Y \forall y \in Y$.

(4) If two countable graphs $G'$ and $G''$ satisfy $A_{r,s}$ for all $r, s \geq 1$, then $G'$ and $G''$ are isomorphic.

Proof by construction of bijection

- Define $f(i), f^*(i), f(i), f^{-1}(i)$ for all $i$.
- $f(i) = A_i$
- When $f(i)$ has to be determined, $f$ has already been specified on $V_i$.
- Set $f(i) = g \in G''$, so that $\forall v_i, v'_i \in G'' \Leftrightarrow f(v_i), f(v'_i) \in G''$ for all $v_i \in V_i$.
- Existence due to $A_{r,s}$ for all $r, s \geq 1$.
- $f^{-1}(i)$ equivalent.

(5) The system consisting of all $A_{r,s}$ with $r, s \geq 1$ is complete (for every FO statement $B$, either $B$ or $\neg B$ is provable).

Proof: Suppose adding $B$ gives theory $T'$, suppose adding $\neg B$ gives theory $T''$.

- Gödel's Completeness Theorem yields countable model for consistent $T'$ and consistent $T''$.
- Model $G'$ for $T'$ and model $G''$ for $T''$ are isomorphic $G$.

(6) Let $B$ be any FO statement, provable from $A_{r,s}$ with $r, s \geq 1$.

Proofs are finite $\Rightarrow$ $B$ is provable from a finite subset of $A_{r,s}$.

\[ \lim_{n \to \infty} \Pr [ \text{G}(n, \frac{1}{2}) \text{ does not satisfy } B ] \leq \sum_{A_{r,s}} \Pr [ \text{G}(n, \frac{1}{2}) \text{ does not satisfy } A_{r,s} ] \to 0 \Rightarrow \text{G}(n, \frac{1}{2}) \text{ satisfies } B \text{ with probability } 1 \text{ as } n \to \infty. \]