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Definition of a Broadcasts and Accumulation

**Definition of Broadcast:**

Given are $G = (V, E)$ and $v \in V$.

- $v$ has information $I(v)$
- no node from $V \setminus \{v\}$ knows $I(v)$.
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Types of Communication

- **Telegraph-Mode**: Communication is directed.
  - Is also called one-way communication.
- **Telephone-Mode**: Information is exchanged.
  - Is also called two-way communication.
- Communication only between neighbours.
- Communication is done in rounds.
- In each round the active edges are a matching.
- Each round uses one time-unit.
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Types of Communication

- In the broadcast-problem the information of one node is transferred to all others.
- The accumulation-problem is a “inverse” broadcast.
- A gossip distributes the sum of all informations to all nodes.
- In each round the communication is done by a matching.
- The communication on an edge may be one-way or two-way, depending on the mode.
- The size of send date is ignored.
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Definition

- By $comm(A)$ we denote the complexity (number of rounds) of a communication-algorithm.

- $r(G) = \min \{comm(A) | A \text{ is a one-way algorithm for the gossip-problem on } G\}$

- $r_2(G) = \min \{comm(A) | A \text{ is a two-way algorithm for the gossip-problem on } G\}$

- $b(v, G) = \min \{comm(A) | A \text{ is a one-way algorithm for the broadcast-problem on } G \text{ and } v\}$

- $a(v, G) = \min \{comm(A) | A \text{ is a one-way algorithm for the accumulations-problem on } G \text{ and } v\}$
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- $b(\nu, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the broadcast-problem on } G \text{ and } \nu\}$

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First Results

- For each graph $G$ and $v \in V$ we have:
  - $a(v, G) = b(v, G)$
  - $a(G) = b(G)$
  - $\text{mina}(G) = \text{minb}(G)$

- Note: reverse broadcast is accumulation.

- There exists a graph $G$ with: $r(G) = 2 \cdot r_2(G)$.
  - Note: 2-clique or cycle of length four.

- The following holds: $\text{minb}(G) \leq b(G) \leq r_2(G) \leq r(G) \leq 2 \cdot r_2(G)$.

- The inequalities result from the definitions.

- $\text{minb}(L(n)) = \lceil n/2 \rceil$

- Optimal broadcast on a line start in the center of the line.

- $b(L(n)) = n - 1$

- A message from the left has to traverse all edges.
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- $b(L(n)) = n - 1$

- A message from the left has to traverse all edges.
First Results

- For each graph $G$ and $v \in V$ we have:
  - $a(v, G) = b(v, G)$
  - $a(G) = b(G)$
  - $\text{mina}(G) = \text{minb}(G)$

  Note: reverse broadcast is accumulation.

- There exists a graph $G$ with: $r(G) = 2 \cdot r_2(G)$.
  Note: 2-clique or cycle of length four.

- The following holds: $\text{minb}(G) \leq b(G) \leq r_2(G) \leq r(G) \leq 2 \cdot r_2(G)$.

- The inequalities result from the definitions.

- $\text{minb}(L(n)) = \lceil n/2 \rceil$

- Optimal broadcast on a line start in the center of the line.

- $b(L(n)) = n - 1$

- A message from the left has to traverse all edges.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
- Let $B = F_1, F_2, \cdots F_z$ be the corresponding one-way accumulation-algorithm.
- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:
- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
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- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

1. $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
2. $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

1. Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
2. Let $A = E_1, E_2, \ldots, E_z$ be the corresponding one-way broadcast-algorithm.
3. Let $B = F_1, F_2, \ldots, F_z$ be the corresponding one-way accumulation-algorithm.
4. Then is $F_1, F_2, \ldots, F_z, E_1, E_2, \ldots, E_z$ one-way gossip-algorithm.
5. Note: in the two-way case holds: $F_z = E_1$.
6. Note: For $L(2 \cdot n)$ we have equality.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
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- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:
For each graph $G$ with $|V| \geq 2$ we have:
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- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
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- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
First Results II

Lemma:
For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \minb(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \minb(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \minb(G) = \mina(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
- Let $B = F_1, F_2, \cdots F_z$ be the corresponding one-way accumulation-algorithm.
- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

1. $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
2. $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$. 
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm. 
- Let $B = F_1, F_2, \cdots F_z$ be the corresponding one-way accumulation-algorithm. 
- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm. 
- Note: in the two-way case holds: $F_z = E_1$. 
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

Proof: Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
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- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
**Lemma:**

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

**Proof:** Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
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- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- **Note:** For $L(2 \cdot n)$ we have equality.
Lemma:

For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and

\[ b(G) = r(G) \]

Proof (for \( n = 8 \)):
Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and $b(G) = r(G)$

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Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and $b(G) = r(G)$

Proof (for $n = 8$):

\begin{center}
\begin{tikzpicture}
\node[vertex, label=above:$a_1$] (a1) at (0, 0) {};
\node[vertex, label=above:$a_2$] (a2) at (1, 0) {};
\node[vertex, label=above:$a_3$] (a3) at (2, 0) {};
\node[vertex, label=above:$a_4$] (a4) at (2,-1) {};
\node[vertex, label=above:$a_5$] (a5) at (3, 0) {};
\node[vertex, label=above:$a_6$] (a6) at (4, 0) {};
\node[vertex, label=above:$a_7$] (a7) at (5, 0) {};
\node[vertex, label=above:$a_8$] (a8) at (6, 0) {};
\draw (a1) -- (a2);
\draw (a3) -- (a4);
\draw (a5) -- (a6);
\draw (a7) -- (a8);
\end{tikzpicture}
\end{center}
First Results III

Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

\[ b(G) = r(G) \]

Proof (for $n = 8$):
Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

$$b(G) = r(G)$$

Proof (for $n = 8$):

Both broadcasts together are a gossip-algorithm.
Lemma:

For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

\[ b(G) = r(G) \]

Proof (for $n = 8$):

Both broadcasts together are a gossip-algorithm.
Lemma:

For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

\[ b(G) = r(G) \]

Proof (for $n = 8$):

![Diagram of a graph with 8 nodes and two broadcast processes]
Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and
\[ b(G) = r(G) \]

Proof (for $n = 8$):

Both broadcasts together are a gossip-algorithm.
First Results III

**Lemma:**
For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and \( b(G) = r(G) \)

**Proof (for \( n = 8 \)):**

Both broadcasts together are a gossip-algorithm.
Lemma:

For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and

$$b(G) = r(G)$$

Proof (for $n = 8$):

Both broadcasts together are a gossip algorithm.
Lemma:
For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and
\[ b(G) = r(G) \]

Proof (for \( n = 8 \)):
Lemma:

For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and \( b(G) = r(G) \).

Proof (for \( n = 8 \)):
**Lemma:**

For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and $b(G) = r(G)$

Proof (for $n = 8$):

\[ \begin{align*} 
\text{broadcast with } a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \\
\text{gossip with } a_1 \rightarrow a_3 \rightarrow a_5 \rightarrow a_7 \\
\text{broadcast with } a_2 \rightarrow a_3 \rightarrow a_4 \\
\text{gossip with } a_6 \rightarrow a_3 \rightarrow a_5 \\
\end{align*} \]
Lemma:
For each even $n$ with $n \geq 8$ exists a Graph $G$ with $n$ nodes and
\[ b(G) = r(G) \]

Proof (for $n = 8$):

Both broadcasts together are a gossip-algorithm.
First Results IV

- \( \text{rad}(G) \leq \text{minb}(G) \).
- \( \text{rad}(G) \leq \text{diam}(G) \leq b(G) \).
- Let \( G = (V, E) \) and \( H = (V, F) \) with \( F \subset E \). Then we have:
  - \( b(G) \leq b(H) \).
  - \( \text{minb}(G) \leq \text{minb}(H) \).
  - \( r(G) \leq r(H) \).
  - \( r_2(G) \leq r_2(H) \).

- \( \text{minb}(G) \leq (\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \).
- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G) \).
- \( r(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 2 \).
- \( r_2(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \).

\[
\text{diam}(G) = \max\{\text{dist}(u, v) \mid u, v \in V\}
\]
\[
\text{rad}(v, G) = \max\{\text{dist}(v, x) \mid x \in V\}
\]
\[
\text{rad}(G) = \min\{\text{rad}(v, G) \mid v \in V\}
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First Results IV

- \( \text{rad}(G) \leq \min b(G) \).
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Let \( G = (V, E) \) and \( H = (V, F) \) with \( F \subset E \). Then we have:
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- \( \min b(G) \leq (\deg(G) - 1) \cdot \text{rad}(G) + 1 \).
- \( b(G) \leq (\deg(G) - 1) \cdot \text{diam}(G) + 1 \).
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\text{diam}(G) & = \max \{ \text{dist}(u, v) \mid u, v \in V \} \\
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First Results IV

- $\text{rad}(G) \leq \text{minb}(G)$.
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First Results IV

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\[
\begin{align*}
\text{diam}(G) &= \max \{ \text{dist}(u, v) \mid u, v \in V \} \\
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\end{align*}
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First Results IV

- \( \text{rad}(G) \leq \min b(G) \).
- \( \text{rad}(G) \leq \text{diam}(G) \leq b(G) \).
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First Results IV

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First Results IV

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- \( \text{rad}(G) \leq \text{minb}(G) \).
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- Let \( G = (V, E) \) and \( H = (V, F) \) with \( F \subseteq E \). Then we have:
  - \( b(G) \leq b(H) \).
  - \( \text{minb}(G) \leq \text{minb}(H) \).
  - \( r(G) \leq r(H) \).
  - \( r_2(G) \leq r_2(H) \).

- \( \text{minb}(G) \leq (\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \).
- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G) \).
- \( r(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 2 \)
- \( r_2(G) \leq 2(\text{deg}(G) - 1) \cdot \text{rad}(G) + 1 \)

\[
\begin{align*}
\text{diam}(G) & = \max\{\text{dist}(u, v) \mid u, v \in V\} \\
\text{rad}(v, G) & = \max\{\text{dist}(v, x) \mid x \in V\} \\
\text{rad}(G) & = \min\{\text{rad}(v, G) \mid v \in V\}
\end{align*}
\]
Lemma

Let $G = (V, E)$ be a graph with $n$ nodes. Then we have:

- $b(G) \geq \min b(G) \geq \lceil \log n \rceil$

Proof:

- Let $A(t)$ be the number of informed nodes after $t$ rounds.
- $A(0) = 1$
- $A(t+1) \leq 2 \cdot A(t)$
- $A(t) \leq 2^t$
- At the end $2^t \geq n$ must hold.
Lower Bound

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Optimal Broadcast-Tree

Each informed node has to send in each round the information to a non-informed node:

A tree $T_i$ is a broadcast-tree, iff
- the root of $T_i$ has $i$ successors $v_0, v_1, \cdots, v_{i-1}$ and
- $v_j$ is the root of a $T_j$. 
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Each informed node has to send in each round the information to a non-informed node:

![Diagram of a broadcast tree]

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First Results

Lemma

We have:

- \( \min b(K(n)) = b(K(n)) = \lceil \log n \rceil \) and
- \( \min b(HQ(m)) = b(HQ(m)) = m. \)

Proof \((K(n))\):

for \( t = 1 \) to \( \lceil \log n \rceil \) do

for all \( i \in \{0, 1, \ldots, 2^{t-1} - 1\} \) do in parallel

if \( i + 2^{t-1} \leq n \) then

\( i \) sends to \( i + 2^{t-1} \)

Proof \((HQ(m))\):

for \( i = 1 \) to \( m \) do

for all \( a_1, a_2, \ldots, a_{i-1} \in \{0, 1\} \) do in parallel

\( a_1 a_2 \cdots a_{i-1} 00 \cdots 0 \) sends to \( a_1 a_2 \cdots a_{i-1} 10 \cdots 0 \)
Lemma

We have:
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Proof \((K(n))\):

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\text{for } t = 1 \text{ to } \lceil \log n \rceil \text{ do} \\
\quad \text{for all } i \in \{0, 1, \ldots, 2^{t-1} - 1\} \text{ do in parallel} \\
\qquad \text{if } i + 2^{t-1} \leq n \text{ then} \\
\qquad \quad i \text{ sends to } i + 2^{t-1}
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Proof ($K(n)$):

for $t = 1$ to $\lceil \log n \rceil$ do
    for all $i \in \{0, 1, \ldots, 2^{t-1} - 1\}$ do in parallel
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for $i = 1$ to $m$ do
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        $a_1a_2\cdots a_{i-1}00\cdots 0$ sends to $a_1a_2\cdots a_{i-1}10\cdots 0$
Lemma

For all $k, m \geq 2$ we have: $\min b(T_k(m)) = k \cdot m$.

Idea of proof:

- $b(\varepsilon, T_k(m)) = k \cdot m$.
- $b(\varepsilon, T_k(m)) \leq b(\nu, T_k(m))$.
- Note that $\nu$ has to inform $\varepsilon$.
- and $\varepsilon$ has to inform the other successors.
First Results II

Lemma

For all $k, m \geq 2$ we have: $\min_b(T_k(m)) = k \cdot m$.

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For all $k, m \geq 2$ we have: $\min(b(T_k(m))) = k \cdot m$.

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Lemma

We have:

- \( b(\text{CCC}(k)) \leq 5k + O(1) \)
- \( b(\text{BF}(k)) \leq 4.5k + O(1) \)
- \( b(\text{SE}(k)) \leq 4k + O(1) \)
- \( b(\text{DB}(k)) \leq 3k + O(1) \)

Proof: Use the following statements:

- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1. \)
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Theorem:

We have: ⌈5k/2⌉ − 2 ≤ minb(CCC(k)) = b(CCC(k)) ≤ ⌈5k/2⌉ − 1.

The following parts are proven:

- minb(CCC(k)) ≥ ⌈5k/2⌉ − 2
- Algorithm for ⌈5k/2⌉ − 1 will be presented.
Theorem:
We have: \(\lceil \frac{5k}{2} \rceil - 2 \leq \min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil \frac{5k}{2} \rceil - 1\).

- The following parts are proven:
  - \(\min b(\text{CCC}(k)) \geq \lceil \frac{5k}{2} \rceil - 2\)
  - Algorithm for \(\lceil \frac{5k}{2} \rceil - 1\) will be presented.
Theorem:
We have: \(\lceil \frac{5k}{2} \rceil - 2 \leq \min_b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil \frac{5k}{2} \rceil - 1\).

- The following parts are proven:
  - \(\min_b(\text{CCC}(k)) \geq \lceil \frac{5k}{2} \rceil - 2\)
  - Algorithm for \(\lceil \frac{5k}{2} \rceil - 1\) will be presented.
CCC, Proof $\text{minb}(\text{CCC}(k)) \geq \left\lceil 5 \cdot k/2 \right\rceil - 2$

- $\text{diam}(\text{CCC}(k)) = \left\lceil 5/2 \cdot k \right\rceil - 2$
- The statement holds for even $k$.
- Let $k$ be odd.
- Let $(0, 00 \cdots 0)$ be the origin of the message.
- The nodes $(\lfloor k/2 \rfloor, 11 \cdots 1)$ and $(\lfloor k/2 \rfloor + 1, 11 \cdots 1)$ are both in distance $(\lfloor 5 \cdot k/2 \rfloor - 2)$.
- Thus we need one round more then the diameter.
- The statement hold, because the CCC is node-symetric.
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- Thus we need one round more than the diameter.
- The statement hold, because the CCC is node-symmetric.
Algorithm \textsc{Broadcast-CCC}_k

$(0,00\ldots0)$ sends to $(0,10\ldots0)$;

\textbf{for } $i = 0$ \textbf{to } $k - 1$ \textbf{do begin}

\textbf{for all } $a_0,\ldots,a_{i-1} \in \{0,1\}$ \textbf{do in parallel}

$(i-1,a_0\ldots a_{i-1}00\ldots0)$ sends to $(i,a_0\ldots a_{i-1}00\ldots0)$;

\textbf{for all } $a_0,\ldots,a_{i-1} \in \{0,1\}$ \textbf{do in parallel}

$(i,a_0\ldots a_{i-1}00\ldots0)$ sends to $(i,a_0\ldots a_{i-1}10\ldots0)$;

\textbf{end;}

\textbf{for all } $\alpha \in \{0,1\}^k$ \textbf{do in parallel}

Broadcast on cycle $C_\alpha(k)$ starting from $(k-1,\alpha)$;

\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Theorem:

We have: \( \min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5 \cdot k/2 \rceil - 2. \)

Idea of proof: Change the first phase and send in both directions.
Theorem:
We have: $\min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5 \cdot k/2 \rceil - 2$.

Idea of proof: Change the first phase and send in both directions.
Theorem:

We have: \(\min_b(SE(k)) = b(SE(k)) = 2 \cdot k - 1\)

Proof:

- The diameter provides the lower bound.
- Note \(SE(k)\) is not node-symmetric.
- We have to provide an algorithm for any node \(v\).
- Algorithm has to be without conflicts.
- And we do now show it here in detail.
Theorem:
We have: \( \min b(SE(k)) = b(SE(k)) = 2 \cdot k - 1 \)

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We have: \( \min b(\text{SE}(k)) = b(\text{SE}(k)) = 2 \cdot k - 1 \)

Proof:
- The diameter provides the lower bound.
- Note \( \text{SE}(k) \) is not node-symmetric.
- We have to provide an algorithm for any node \( v \).
- Algorithm has to be without conflicts.
- And we do now show it here in detail.
Theorem:
We have: \( \lceil \frac{3m}{2} \rceil \leq \min_b(\text{BF}(m)) = b(\text{BF}(m)) \leq 2 \cdot m \)

- The diameter gives the lower bound.
- Algorithm will be provided in the following.
Theorem:

We have: \( \lfloor 3m/2 \rfloor \leq \min_b(BF(m)) = b(BF(m)) \leq 2 \cdot m \)

- The diameter gives the lower bound.
- Algorithm will be provided in the following.
BF (Idea of proof)

- Distribute the information in two ways:
  - Prefer in the first strategy the cycle-edges.
  - Prefer in the second strategy the cross-edges.

- Split the butterfly into two isomorph parts.
- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

\[ \left\lfloor \frac{3m}{2} \right\rfloor \leq \min_b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]
BF (Idea of proof)

- Distribute the information in two ways:
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\[
\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
BF (Idea of proof)

- Distribute the information in two ways:
  - Prefer in the first strategy the cycle-edges.
  - Prefer in the second strategy the cross-edges.
- Split the butterfly into two isomorph parts.
- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

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\[ \lfloor \frac{3m}{2} \rfloor \leq \min\{b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]
BF (Idea of proof)

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- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

\[
\lfloor \frac{3m}{2} \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
**BF (Proof I)**

- **Splitting of BF(m) in F₀ and F₁:**
  - **F₀** has nodes: \( \{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\} \).
  - **F₁** has nodes: \( \{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\} \).
  - **F₀** and **F₁** are isomorphic.

- **♯₀(w)** denotes the number of 0’en in \( w \).
- **♯₁(w)** denotes the number of 1’en in \( w \).

\[
\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
BF (Proof I)

- Splitting of $BF(m)$ in $F_0$ and $F_1$:
  - $F_0$ has nodes: $\{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
  - $F_1$ has nodes: $\{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
  - $F_0$ and $F_1$ are isomorphic.
- $\#_0(w)$ denotes the number of 0’en in $w$.
- $\#_1(w)$ denotes the number of 1’en in $w$.

$$\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$$
BF (Proof I)

Splitting of $BF(m)$ in $F_0$ and $F_1$:
- $F_0$ has nodes: $\{(l, \alpha 0) | 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_1$ has nodes: $\{(l, \alpha 1) | 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_0$ and $F_1$ are isomorphic.
- $\#_0(w)$ denotes the number of 0’en in $w$.
- $\#_1(w)$ denotes the number of 1’en in $w$.

$\lfloor \frac{3m}{2} \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$
\[ \left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]

- Splitting of $BF(m)$ in $F_0$ and $F_1$:
  - $F_0$ has nodes: $\{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
  - $F_1$ has nodes: $\{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
  - $F_0$ and $F_1$ are isomorphic.
- $\#_0(w)$ denotes the number of 0’en in $w$.
- $\#_1(w)$ denotes the number of 1’en in $w$. 

![Diagram of BF (Proof 1)](attachment:bf_diagram.png)
Splitting of $BF(m)$ in $F_0$ and $F_1$:

- $F_0$ has nodes: $\{(l, \alpha 0) | 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_1$ has nodes: $\{(l, \alpha 1) | 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
- $F_0$ and $F_1$ are isomorph.

- $\#_0(w)$ denotes the number of 0’en in $w$.
- $\#_1(w)$ denotes the number of 1’en in $w$. 

$$\lfloor 3m/2 \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$$
BF (Proof I)

- Splitting of $BF(m)$ in $F_0$ and $F_1$:

  - $F_0$ has nodes: $\{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.
  - $F_1$ has nodes: $\{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\}$.

- $F_0$ and $F_1$ are isomorphic.

- $\#_0(w)$ denotes the number of 0’en in $w$.

- $\#_1(w)$ denotes the number of 1’en in $w$.
Consider $F_0$: from node $v_0 = (0, 00 \cdots 00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

Consider $F_1$: from node $v_1 = (m - 1, 00 \cdots 01)$ exists a unique path of length $m - 1$ to $w_1 = (0, \alpha 1)$ for $\alpha \in \{0, 1\}^{m-1}$.

First step of the algorithm $v_0$ informs $v_1$.

Then we use in $F_0$ and $F_1$ two different strategies.
Consider $F_0$: from node $v_0 = (0, 00 \cdots 00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

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Consider \( F_0 \): from node \( v_0 = (0, 00 \cdots 00) \) exists a unique path of length \( m - 1 \) to \( w_0 = (m - 1, \alpha 0) \) for \( \alpha \in \{0, 1\}^{m-1} \).

Consider \( F_1 \): from node \( v_1 = (m - 1, 00 \cdots 01) \) exists a unique path of length \( m - 1 \) to \( w_1 = (0, \alpha 1) \) for \( \alpha \in \{0, 1\}^{m-1} \).

First step of the algorithm \( v_0 \) informs \( v_1 \).

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Consider $F_0$: from node $v_0 = (0, 00 \cdots 00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

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First step of the algorithm $v_0$ informs $v_1$.

Then we use in $F_0$ and $F_1$ two different strategies.
Aim: Inform in ⌊3m/2⌋ steps the nodes \(w_0 = (m - 1, \alpha_0)\) and \(w_1 = (0, \alpha_1)\) for \(\alpha \in \{0, 1\}^{m-1}\).

If a node \(w_0 = (m - 1, \alpha_0)\) gets informed, then it informs in the next step \(w_1 = (0, \alpha_1)\) (if necessary).

If a node \(w_1 = (0, \alpha_1)\) gets informed, then it informs in the next step \(w_0 = (m - 1, \alpha_0)\) (if necessary).
Aim: Inform in $\lfloor 3m/2 \rfloor$ steps the nodes $w_0 = (m - 1, \alpha_0)$ and $w_1 = (0, \alpha_1)$ for $\alpha \in \{0, 1\}^{m-1}$.

If a node $w_0 = (m - 1, \alpha_0)$ gets informed, then it informs in the next step $w_1 = (0, \alpha_1)$ (if necessary).

If a node $w_1 = (0, \alpha_1)$ gets informed, then it informs in the next step $w_0 = (m - 1, \alpha_0)$ (if necessary).
BF (Proof III)

\[
\left\lceil \frac{3m}{2} \right\rceil \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]

- **Aim:** Inform in \(\left\lceil \frac{3m}{2} \right\rceil\) steps the nodes \(w_0 = (m - 1, \alpha 0)\) and \(w_1 = (0, \alpha 1)\) for \(\alpha \in \{0, 1\}^{m-1}\).

- If a node \(w_0 = (m - 1, \alpha 0)\) gets informed, then it informs in the next step \(w_1 = (0, \alpha 1)\) (if necessary).

- If a node \(w_1 = (0, \alpha 1)\) gets informed, then it informs in the next step \(w_0 = (m - 1, \alpha 0)\) (if necessary).
In $F_0$ a informed node $(l, \alpha 0)$ sends first to $(l + 1, \alpha 0)$ and then to $(l + 1, \alpha(l)0)$. [$\alpha(l) = \alpha_1 \ldots \bar{\alpha}_l \ldots$]

In $F_1$ a informed node $(l, \alpha 1)$ sends first to $(l + 1, \alpha(l)1)$ and then to $(l + 1, \alpha 1)$.

The time to inform from $v_0 = (0, 00 \cdots 00)$ a node $w_0 = (m - 1, \alpha 0)$ is: $1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha)$.

The time to inform from $v_1 = (m - 1, 00 \cdots 01)$ a node $w_1 = (0, \alpha 1)$ is: $1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha)$.
In $F_0$ a informed node $(l, \alpha_0)$ sends first to $(l + 1, \alpha_0)$ and then to $(l + 1, \alpha(l)0)$. $[\alpha(l) = \alpha_1 \ldots \bar{\alpha}_l \ldots]$

In $F_1$ a informed node $(l, \alpha_1)$ sends first to $(l + 1, \alpha(l)1)$ and then to $(l + 1, \alpha_1)$.

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The time to inform from $v_1 = (m - 1, 00 \cdots 01)$ a node $w_1 = (0, \alpha_1)$ is: $1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha)$.
\[ \lfloor \frac{3m}{2} \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]

- In \( F_0 \) a informed node \((l, \alpha 0)\) sends first to \((l + 1, \alpha 0)\) and then to \((l + 1, \alpha(l)0)\). \([\alpha(l) = \alpha_1 \ldots \bar{\alpha}_l \ldots]\)

- In \( F_1 \) a informed node \((l, \alpha 1)\) sends first to \((l + 1, \alpha(l)1)\) and then to \((l + 1, \alpha 1)\).

- The time to inform from \(v_0 = (0, 00 \cdots 00)\) a node \(w_0 = (m - 1, \alpha 0)\) is:
  \[ 1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha) \]

- The time to inform from \(v_1 = (m - 1, 00 \cdots 01)\) a node \(w_1 = (0, \alpha 1)\) is:
  \[ 1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha) \]
BF (Proof IV)

\[ \lfloor \frac{3m}{2} \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]

- In \( F_0 \) a informed node \((l, \alpha 0)\) sends first to \((l + 1, \alpha 0)\) and then to \((l + 1, \alpha(l)0)\). \([\alpha(l) = \alpha_1 \ldots \bar{\alpha}_l \ldots ]\)
- In \( F_1 \) a informed node \((l, \alpha 1)\) sends first to \((l + 1, \alpha(l)1)\) and then to \((l + 1, \alpha 1)\).
- The time to inform from \( v_0 = (0, 00 \cdots 00) \) a node \( w_0 = (m - 1, \alpha 0) \) is:
  \[ 1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha). \]
- The time to inform from \( v_1 = (m - 1, 00 \cdots 01) \) a node \( w_1 = (0, \alpha 1) \) is:
  \[ 1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha). \]
**Case 1: m is odd:**

- **Case 1.1: \( \#_1(\alpha) < (m - 1)/2: \)**
  
  Node \( w_0 \) will be informed from \( v_0 \) at time 
  
  \[ m + \#_1(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor. \]

  After this \( w_0 \) sends to \( w_1 \).

- **Case 1.2: \( \#_0(\alpha) < (m - 1)/2: \)**
  
  Node \( w_1 \) will be informed from \( v_0 \) at time 
  
  \[ m + \#_0(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor. \]

- **Case 1.3: \( \#_0(\alpha) = \#_1(\alpha) = (m - 1)/2: \)**
  
  \( w_0 \) is informed at time 
  
  \[ m + \#_1(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor. \]

\( w_1 \) is informed at time 

\[ m + \#_0(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor. \]
Case 1: $m$ is odd:

Case 1.1: $\#_1(\alpha) < (m - 1)/2$:
Node $w_0$ will be informed from $v_0$ at time
$m + \#_1(\alpha) < (3m - 1)/2 = [3m/2]$.
After this $w_0$ sends to $w_1$.
$w_1$ is informed at time $[3m/2]$.

Case 1.2: $\#_0(\alpha) < (m - 1)/2$:
node $w_1$ will be informed from $v_0$ at time
$m + \#_0(\alpha) < (3m - 1)/2 = [3m/2]$.
$w_0$ will be informed from $w_1$ at time $[3m/2]$.

Case 1.3: $\#_0(\alpha) = \#_1(\alpha) = (m - 1)/2$:
$w_0$ is informed at time
$m + \#_1(\alpha) = (3m - 1)/2 = [3m/2]$.
$w_1$ is informed at time $m + \#_0(\alpha) = (3m - 1)/2 = [3m/2]$. 

$[3m/2] \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$
BF (Proof V)

Case 1: \( m \) is odd:

Case 1.1: \( \#_1(\alpha) < \frac{(m - 1)}{2} \):
Node \( w_0 \) will be informed from \( v_0 \) at time
\[ m + \#_1(\alpha) < \frac{(3m - 1)}{2} = \lfloor \frac{3m}{2} \rfloor. \]
After this \( w_0 \) sends to \( w_1 \).
\( w_1 \) is informed at time \( \lfloor \frac{3m}{2} \rfloor \).

Case 1.2: \( \#_0(\alpha) < \frac{(m - 1)}{2} \):
Node \( w_1 \) will be informed from \( v_0 \) at time
\[ m + \#_0(\alpha) < \frac{(3m - 1)}{2} = \lfloor \frac{3m}{2} \rfloor. \]
\( w_0 \) will be informed from \( w_1 \) at time \( \lfloor \frac{3m}{2} \rfloor \).

Case 1.3: \( \#_0(\alpha) = \#_1(\alpha) = \frac{(m - 1)}{2} \):
\( w_0 \) is informed at time
\[ m + \#_1(\alpha) = \frac{(3m - 1)}{2} = \lfloor \frac{3m}{2} \rfloor. \]
\( w_1 \) is informed at time \( m + \#_0(\alpha) = \frac{(3m - 1)}{2} = \lfloor \frac{3m}{2} \rfloor \).

\[ \lfloor \frac{3m}{2} \rfloor \leq \min(b(BF(m))) = b(BF(m)) \leq 2 \cdot m \]
Case 1: $m$ is odd:

- **Case 1.1: $\#_1(\alpha) < (m - 1)/2$:**
  Node $w_0$ will be informed from $v_0$ at time $m + \#_1(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
  After this $w_0$ sends to $w_1$.
  $w_1$ is informed at time $\lfloor 3m/2 \rfloor$.

- **Case 1.2: $\#_0(\alpha) < (m - 1)/2$:**
  node $w_1$ will be informed from $v_0$ at time $m + \#_0(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
  $w_0$ will be informed from $w_1$ at time $\lfloor 3m/2 \rfloor$.

- **Case 1.3: $\#_0(\alpha) = \#_1(\alpha) = (m - 1)/2$:**
  $w_0$ is informed at time $m + \#_1(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
  $w_1$ is informed at time $m + \#_0(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor$. 

\[ [3m/2] \leq \min_b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]
Case 2: $m$ is even:

Case 2.1: $\#_1(\alpha) \leq (m - 2)/2$:
node $w_0$ will be informed from $v_0$ at time $m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
Thus node $w_1$ will be informed at time $\lfloor 3m/2 \rfloor$.

Case 2.2: $\#_0(\alpha) \leq (m - 2)/2$:
node $w_1$ will be informed from $v_0$ at time $m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
Thus node $w_0$ will be informed at time $\lfloor 3m/2 \rfloor$.

In the last phase we distribute the information on the cycles.

Running time is: $\lceil m/2 \rceil$ rounds.

Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
**Case 2: m is even:**

- **Case 2.1:** $\#_1(\alpha) \leq (m - 2)/2$:
  node $w_0$ will be informed from $v_0$ at time $m + \#_1(\alpha) \leq 3m/2 - 1 < [3m/2]$. Thus node $w_1$ will be informed at time $[3m/2]$.

- **Case 2.2:** $\#_0(\alpha) \leq (m - 2)/2$:
  node $w_1$ will be informed from $v_0$ at time $m + \#_0(\alpha) \leq 3m/2 - 1 < [3m/2]$. Thus node $w_0$ will be informed at time $[3m/2]$.

- In the last phase we distribute the information on the cycles.
- Running time is: $\lceil m/2 \rceil$ rounds.
- Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
Case 2: \( m \) is even:

- **Case 2.1: \( \#_1(\alpha) \leq (m - 2)/2: \)**
  node \( w_0 \) will be informed from \( v_0 \) at time
  \( m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor \).
  Thus node \( w_1 \) will be informed at time \( \lfloor 3m/2 \rfloor \).

- **Case 2.2: \( \#_0(\alpha) \leq (m - 2)/2: \)**
  node \( w_1 \) will be informed from \( v_0 \) at time
  \( m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor \).
  Thus node \( w_0 \) will be informed at time \( \lfloor 3m/2 \rfloor \).

In the last phase we distribute the information on the cycles.

Running time is: \( \lceil m/2 \rceil \) rounds.

Total running time: \( \lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m \)
Case 2: $m$ is even:

- **Case 2.1:** $\#_1(\alpha) \leq (m - 2)/2$:
  node $w_0$ will be informed from $v_0$ at time
  $m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_1$ will be informed at time $\lfloor 3m/2 \rfloor$.

- **Case 2.2:** $\#_0(\alpha) \leq (m - 2)/2$:
  node $w_1$ will be informed from $v_0$ at time
  $m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
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In the last phase we distribute the information on the cycles.

Running time is: $\lceil m/2 \rceil$ rounds.

Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
BF (Proof V)

\[ \left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]

- **Case 2: \( m \) is even:**
  - **Case 2.1:** \( \#_1(\alpha) \leq \frac{(m - 2)}{2} \):
    - node \( w_0 \) will be informed from \( v_0 \) at time \( m + \#_1(\alpha) \leq 3m/2 - 1 < \left\lfloor \frac{3m}{2} \right\rfloor \).
    - Thus node \( w_1 \) will be informed at time \( \left\lfloor \frac{3m}{2} \right\rfloor \).
  - **Case 2.2:** \( \#_0(\alpha) \leq \frac{(m - 2)}{2} \):
    - node \( w_1 \) will be informed from \( v_0 \) at time \( m + \#_0(\alpha) \leq 3m/2 - 1 < \left\lfloor \frac{3m}{2} \right\rfloor \).
    - Thus node \( w_0 \) will be informed at time \( \left\lfloor \frac{3m}{2} \right\rfloor \).

- In the last phase we distribute the information on the cycles.
- **Running time is:** \( \left\lceil \frac{m}{2} \right\rceil \) rounds.
- **Total running time:** \( \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor = 2m \)
Case 2: $m$ is even:

- **Case 2.1:** $\#_1(\alpha) \leq (m - 2)/2$:
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  $m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
  Thus node $w_0$ will be informed at time $\lfloor 3m/2 \rfloor$.

In the last phase we distribute the information on the cycles.

Running time is: $\lceil m/2 \rceil$ rounds.

**Total running time:** $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
Theorem:
We have: \( d \leq \min b(DB(d)) = b(DB(d)) \leq \left\lfloor \frac{3}{2} \cdot (d + 1) \right\rfloor. \)

Proof:
- Idea \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \overline{y_1})\).
- The order is given by the parity.
- Let \(\alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2\).
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \overline{\alpha})\).
- \((0011000)\) informs first \((0110000)\) and then \((0110001)\).
Theorem:
We have: \( d \leq \min b(\text{DB}(d)) = b(\text{DB}(d)) \leq \lfloor 3/2 \cdot (d + 1) \rfloor. \)

Proof:
- **Idea** \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \bar{y_1})\).
- The order is given by the parity.
- Let \( \alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2. \)
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \bar{\alpha})\).
- \((0011000)\) informs first \((0110000)\) and then \((0110001)\).
Theorem:

We have: \( d \leq \min b(DB(d)) = b(DB(d)) \leq \lfloor 3/2 \cdot (d + 1) \rfloor. \)

Proof:

- Idea \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \overline{y_1})\).
- The order is given by the parity.
- Let \( \alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2. \)
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \overline{\alpha})\).
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Degree of the Nodes

**Theorem:**

Let $n \geq 5$ and $G = (V, E)$ be a graph with $n$ nodes:

- If $\Delta(G) = 3$ holds, we have: $b(G) \geq \min b(G) \geq 1.4404 \log(n) - 3$.
- If $\Delta(G) = 4$ holds, we have: $b(G) \geq \min b(G) \geq 1.1374 \log(n) - 2$.

**Proof:**

- Let $A$ be a broadcast-algorithm.
- Let $\text{Broad}_i^A(v_0)$ be the set of nodes, which are informed from $v_0$ by $A$ in $i$ rounds.
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Building the Idea

We consider here only the case $\Delta(G) = 3$. The case $\Delta(G) = 4$ is similar.

- The initial node may send at most three times.
- The initial node sends only in rounds $1, 2, 3$.
- Any other nodes will be informed at time $t$ via an edge $e$.
- No further node may be informed via $e$.
- Thus any other node may send at most two times.
- If a node $v$ is informed in round $t$ by $w$, then did $w$ receive the information at round $t - 1$ or $t - 2$.
- Thus the number of newly informed nodes in round $t > 3$, is at most the number of nodes which got informed in rounds $t - 1$ and $t - 2$. 
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Building the Idea

We consider here only the case $\Delta(G) = 3$. The case $\Delta(G) = 4$ is similar.

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Proof

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- $A(3) = 4$
- $A(i) = A(i - 1) + A(i - 2)$ für $i \geq 4$.
- Show by induction: $A(i) \leq 1.61804^i$ for $i \geq 0$. 
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More Results

Consequence:
\[ b(DB_k) \geq \min b(DB_k) \geq 1.1374 \cdot k - 2 \]

Theorem:
\[ b(BF_m) = \min b(BF_m) > 1.7396m \text{ for large enough } m. \]

Idea of Proof: Check the number of nodes in distance \( k \).

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## Overview

| Graph   | $|V|$  | Diameter | Lower Bound  | Upper Bound  |
|---------|------|----------|--------------|--------------|
| $K_n$   | $n$  | 1        | $\lceil \log_2 n \rceil$ | $\lceil \log_2 n \rceil$ |
| $HQ_k$  | $2^k$ | $k$      | $k$          | $k$          |
| $CCC_k$ | $k \cdot 2^k$ | $\lfloor 5k/2 \rfloor - 2$ | $\lfloor 5k/2 \rfloor - 2$ | $\lfloor 5k/2 \rfloor - 2$ |
| $SE_k$  | $2^k$ | $2k - 1$ | $2k - 1$     | $2k - 1$     |
| $DB_k$  | $2^k$ | $k$      | $1.4404k$    | $\frac{3}{2}(k + 1)$ |
| $BF_k$  | $k \cdot 2^k$ | $\lfloor 3k/2 \rfloor$ | $1.7609k$    | $2k - \frac{1}{2} \log \log k + c$ |
Definition

In edge-disjoint-path communication the information is passed on a set of edge-disjoint paths between the endpoint of each path. A sending or receiving node may not forward any information at the same time.

A edge-disjoint communication algorithm for $G$ is a sequence of rounds $A_1, A_2, \ldots, A_k$, where each $A_i$ is a correct edge-disjoint-path communication.

- $edp-b(G) =$ maximal time to broadcast in edp-mode in $G$.
- $edp-a(G) =$ maximal time to accumulate in edp-mode in $G$.
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Idea ($edp-a(G)$)

**Definition**

A set of vertices $K \subset V$ is called **knowledge set**, if the pieces of information residing in the vertices of $K$ form the cumulative message.

**Definition**

Let $T = (V, E)$ be some tree, we will refer to any vertex of degree $> 2$ in $T$ as a critical vertex, while all other vertices are called non-critical.

- Collect within two rounds all pieces of information in a subset $K$ of non-critical vertices with $|S| \leq \lfloor n/2 \rfloor$.
- In each of $\lceil \log_2 n \rceil - 1$ communication rounds, reduce the size of a given knowledge set $K$ by a factor of two.
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Lower Bound

Lemma:

Let $G = (V, E)$ a graph with $n$ nodes. Then we have:

$$r(G) \geq r_2(G) \geq \begin{cases} \lceil \log_2 n \rceil & n \text{ even}, \\ \lceil \log_2 n \rceil + 1 & n \text{ odd}. \end{cases}$$

Proof: Only the case, where $n$ is odd, has to be proven.

- Show: $r_2(G) \geq \lceil \log_2 n \rceil + 1$.

- Let $A$ be a communication-algorithm for the gossip-problem. $A$ has communication rounds (matchings) $E_1, E_2, \ldots, E_k$.

- Show by induction: After $i$ rounds has each node at most $2^i$ pieces of information.
  
  - $i = 0$: Each node has $2^0 = 1$ pieces of information.
  
  - $i - 1 \rightarrow i$: at most $2^{i-1} + 2^{i-1} = 2^i$ pieces of information may be collected by any node.

- In round $k$ is at least one node $v$ inactive.

- $v$ has after $k$ rounds at most $2^{k-1}$ pieces of information.
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- **Show:** $r_2(G) \geq \lceil \log_2 n \rceil + 1$.

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- **Show by induction:** After $i$ rounds has each node at most $2^i$ pieces of information.
  - $i = 0$: Each node has $2^0 = 1$ pieces of information.
  - $i - 1 \rightarrow i$: at most $2^{i-1} + 2^{i-1} = 2^i$ pieces of information may be collected by any node.

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For any graph $G = (V, E)$ with $|V| = n$ we have:

- $r(G) \leq 2n - 2$, and
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Proof: Follows from the following known statements:

- $\minb(G) \leq n - 1$ for any graph $G = (V, E)$ with $|V| = n$.
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Lemma:

We have:

- \( r(T_k(1)) = 2k \)
- \( r_2(T_k(1)) = 2k - 1 \)

Proof:

- Show: \( r(T_k(1)) \geq 2k \).
- \( r(T_k(1)) \) has one root and \( k \) leaves.
- The maximal matching is 1.
- In each round is only one leaf active.
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Gossip on Lines

**Theorem:**

We have:

- $r_2(L(n)) = n - 1$ for any even number $n \geq 2$,
- $r_2(L(n)) = n$ for any odd number $n \geq 3$,
- $r(L(n)) = n$ for any even number $n \geq 2$ and
- $r(L(n)) = n + 1$ for any odd number $n \geq 3$.

**Proof:**

- All are more or less easy.
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Gossip on arbitrary Trees

Lemma:

For any tree $T$ we have:

- $r(T) = 2 \cdot \text{minb}(T)$
- $r_2(T) = 2 \cdot \text{minb}(T) - 1$

Idea of the proof:

- We have already for any graph $G$: $r(G) \leq 2 \cdot \text{minb}(G)$.
- We have to show: $r(G) \geq 2 \cdot \text{minb}(G)$.
- Let $W = \bigcup_{w \in V} I(v)$ be the total information.
- Let $A$ be any communication algorithm on $T$.
- Let $t$ be the point in time, when some node knows $W$.
- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
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- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
Consider any tree $T$. We have:

- $r(T) = 2 \cdot \text{minb}(T)$
- $r_2(T) = 2 \cdot \text{minb}(T) - 1$

**Lemma:**

For any tree $T$, we have:

- $r(T) = 2 \cdot \text{minb}(T)$
- $r_2(T) = 2 \cdot \text{minb}(T) - 1$

**Idea of the proof:**

- We have already for any graph $G$: $r(G) \leq 2 \cdot \text{minb}(G)$.
- We have to show: $r(G) \geq 2 \cdot \text{minb}(G)$.
- Let $W = \cup_{v \in V} I(v)$ be the total information.
- Let $A$ be any communication algorithm on $T$.
- Let $t$ be the point in time, when some node knows $W$.
- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
Gossip on arbitrary Trees (Proof I)

- Let $u \neq v$ be an other node which knows $W$ after $t$ steps.
- Let $(u, y_1, y_2, \ldots, y_k, v)$ be the unique path connecting $u$ and $v$.
- If $v$ sends to $y_k$ at time $t$, then $v$ did know $W$ at time $t - 1$.
- So we have to consider the case: $y_k$ sends to $v$ at time $t$:
  - In this case $y_k$ sends $v$ some missing information.
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- Contradiction, the node $u$ does not exist.
- Thus we have: $t \geq \min b(T) = b(v, T)$.
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![Diagram](image-url)
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![Diagram of a tree with nodes $u$, $y_1$, $y_2$, $y_3$, $y_k$, $v$.]
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![Diagram of a tree showing nodes and edges connecting them.](attachment://tree_diagram.png)
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- Consider the situation at node $v$ after round $t$.
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- Let $T_1, T_2, \ldots, T_k$ be the subtrees with roots $v_1, v_2, \ldots, v_k$.
- In each subtree $T_i$ is some information $w_i$ missing.
- Only the node $v$ knows $\bigcup_{j=1}^{k} w_j$.
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- Consider the two-way mode: by a similar way we may prove:
- At time $t$ only two neighbors nodes $u$ and $v$ know the total information. We get in the similar way the second statement.
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Lemma:

For all $m \geq 1$ and $k \geq 2$ we have:

- $r(T_k(m)) = 2 \minb(T_k(m)) = 2 \cdot k \cdot m$.
- $r_2(T_k(m)) = 2 \minb(T_k(m)) - 1 = 2 \cdot k \cdot m - 1$. 
Gossip on Cycles

Theorem:

We have:

- \( r_2(C(k)) = \frac{k}{2} \) for even \( k \).
- \( r_2(C(k)) = \lceil \frac{k}{2} \rceil + 1 \) for odd \( k \).

Idea of the proof (\( k \) even): [\( k \) odd: an easy exercise]

- Let \( k \) be even.
- \( r_2(C(k)) \geq k/2 \) results by the diameter.
- \( r_2(C(k)) \leq k/2 \) is true by the following algorithm:
  1. \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \cdots, \{2i, 2i + 1\}, \cdots, \{n - 2, n - 1\}\}
  2. \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \cdots, \{2i - 1, 2i\}, \cdots, \{n - 1, 0\}\}
  3. \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \cdots, \{2i, 2i + 1\}, \cdots, \{n - 2, n - 1\}\}
  4. \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \cdots, \{2i - 1, 2i\}, \cdots, \{n - 1, 0\}\}
  5. \cdots

- Note: After \( i \) rounds knows each node 2 \( \cdot \) \( i \) Informationen.
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**Idea of the proof (\( k \) even):** [\( k \) odd: an easy exercise]

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  - \( 2 \): \( \{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2i-1,2i\}, \ldots, \{n-1,0\} \)
  - \( 3 \): \( \{0,1\}, \{2,3\}, \{4,5\}, \ldots, \{2i,2i+1\}, \ldots, \{n-2,n-1\} \)
  - \( 4 \): \( \{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2i-1,2i\}, \ldots, \{n-1,0\} \)
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- Note: After $i$ rounds knows each node $2 \cdot i$ Informationen.
Gossip on Cycles

Theorem:

We have:

- $r_2(C(k)) = k/2$ for even $k$.
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Idea of the proof ($k$ even): [k odd: an easy exercise]

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\[
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1-Way Gossip on Cycles (Idea)

- **Messages should traverse in both directions.**
- Activate each $f(n)$-th node on the cycle.
- This will result in an additional $\Theta(f(n))$ steps.
- During the distribution we get $\Theta\left(\frac{n}{2f(n)}\right)$ delays.
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- Split the cycle in $\Theta(\sqrt{n})$ blocks $B_i$.
- Within block $B_i$ ($i \in \{1, 2, 3, \ldots, k\}$ with $k \in \Theta(\sqrt{n})$) do the following:
  - Phase 1:
    - The nodes $v_i$ [$u_i$] start a “wave” to the left [right].
    - The messages of $v_i$ and $u_i$ are delayed $\Theta(\sqrt{n})$ times by the other messages.
    - After $n/2 + \Theta(\sqrt{n})$ round know nodes $z_i$ the total information.
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- Note: If $n$ is even, we have always a delay of one and the synchronization is easy.
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Gossip on Cycles (Idea)

Theorem:

We have:

- \( r(C(n)) \leq n/2 + \sqrt{2n} - 1 \) for even \( n \).
- \( r(C(n)) \leq \left\lceil n/2 \right\rceil + \left\lceil 2 \cdot \sqrt{\left\lceil n/2 \right\rceil} \right\rceil - 1 \) for odd \( n \).
- \( r(C(n)) \geq n/2 + \sqrt{2n} - 1 \) for even \( n \).
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Proof: See literature.
Gossip on the Hypercube

**Theorem:**
For all \( m \in \mathbb{N} \) we have: \( r_2(HQ(m)) = m \)

**Proof:**
- The lower bound is the diameter.
- Upper bound by the following algorithm:
  
  ```
  for i = 1 to m do
    for all \( a_1, a_2, \ldots, a_{m-1} \in \{0, 1\} \) do in parallel
      \( a_1a_2\cdots a_{i-1}0a_ia_{i+1}\cdots a_{m-1} \) sends to
      \( a_1a_2\cdots a_{i-1}1a_ia_{i+1}\cdots a_{m-1} \)
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**Corollary:**
For all \( m \in \mathbb{N} \) we have: \( r_2(K(2^m)) = m \)
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**Corollary:**
For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
CCC and BF (Idea)

- Consider one-way mode:
  - Start with the first phase of the gossip-algorithm for cycles on all cycles.
  - Then each $\Theta(\sqrt{n})$-th node on each cycle knows the total information of its cycles.
  - In $\Theta(\sqrt{n})$ waves distribute this information down and between the cycles.
  - After $\Theta(n)$ steps knows each $\Theta(\sqrt{n})$-th node of each cycle the total information.
  - The final part is the second phase of the gossip-algorithm of cycles on all cycles.
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Consider two-way mode:

- Start with the gossip algorithm for cycles on all cycles.
- Each node of the cycle knows now the total information of its cycle.
- In $\Theta(n/2)$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each node the total information.
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Theorem:

Let $k \geq 3$, then we have:

1. $r(\text{CCC}(k)) \leq r(C(k)) + 3k - 1 \leq \left\lceil \frac{7k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 2$.

2. $r(\text{BF}(k)) \leq r(C(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 1$.

3. $r_2(\text{CCC}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.

4. $r_2(\text{CCC}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$.

5. $r_2(\text{BF}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.

6. $r_2(\text{BF}(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$. 
Theorem:

Let $k \geq 3$, then we have:

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- $r_2(\text{CCC}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.

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CCC and BF

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- $r(\text{BF}(k)) \leq r(\text{C}(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 1$.
- $r_2(\text{CCC}(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.
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CC and BF

Theorem:

Let $k \geq 3$, then we have:

- $r(\text{CCC}(k)) \leq r(\text{C}(k)) + 3k - 1 \leq \left\lceil \frac{7k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 2.$

- $r(\text{BF}(k)) \leq r(\text{C}(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 1.$

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Theorem:

Let $k \geq 3$, then we have:

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Gossip on Graphs with $2 \cdot m$ Nodes (0. Idea)
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Implication:
For all $m \in \mathbb{N}$ we have:
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Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)

- Too many nodes where inactive for too long time.
- These nodes could not double their information.
- Idea: Try to double the information of any node.
- Detailed idea: In each step each node has an "interval" of information.
- To make the doubling easy split the nodes into two groups.
- Both groups should be the same size.
- In the first step pairs of node from each group share their information.
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Gossip on Graphs with $2 \cdot m$ Nodes

**Theorem:**

For all $m \in \mathbb{N}$ we have: $r_2(K(2m)) = \lceil \log 2m \rceil$

**Proof:** Split the nodes in groups $Q[i]$ and $R[i]$ ($0 \leq i \leq m - 1$).

- **algorithm:**
  
  ```text
  for all $i \in \{0, \cdots, m - 1\}$ do in parallel
    Exchange the information between $Q[i]$ and $R[i]$
  for $t = 1$ to $\lceil \log_2 m \rceil$ do
    for all $i \in \{0, \cdots, m - 1\}$ do in parallel
      Exchange the information between $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$
  ```

- **Invariant:**
  
  - Let $\alpha[i]$ be the information of $Q[i]$ and $R[i]$ after their initial exchange.
  - After round $t$ know nodes $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$: 
    $\cup_{0 \leq j \leq 2^{t-1}} \alpha[(i + j) \mod m]$
  - The invariant is easy to be shown.
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We need an extra round.

A nice proof with this idea will become complicated.

We will try to put some structure into the proof.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (a try)

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Gossip on Graphs with \(2 \cdot m + 1\) Nodes (Idea)

- **How could this be an idea?**
- We only have the edges of the first step.
- Idea: We could now choose a small even number of Nodes, which together have the total information.
- These nodes may perform the above gossip algorithm.
- In the last step we repeat the first round.
**Gossip on Graphs with** \(2 \cdot m + 1\) **Nodes (Idea)**

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Let $n = 2 \cdot m + 1$.

Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be all nodes.

For all $i \in \{0, 1, \cdots, m - 1\}$ the node $v_{m+2+i}$ sends to $v_i$.

The node $\{v_0, v_1, v_2, \cdots, v_m\}$ have now the total information.

If $m + 1$ is even, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_m\}$.

If $m + 1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_{m+1}\}$.

For all $i \in \{0, 1, \cdots, m - 1\}$ the nodes $v_i$ send to $v_{m+2+i}$.

Correctness follows direct by the construction.

Running time for $m + 1$ even:
\[
    r_2(K(m+1)) + 2 = \lceil \log_2(m+1) \rceil + 2 = \lceil \log_2 \left( \frac{n+1}{2} \right) \rceil + 2
\]
\[
    = \lceil \log_2(n+1) \rceil + 1 = \lceil \log_2 n \rceil + 1
\]

Running time for $m + 1$ odd:
\[
    r_2(K(m+2)) + 2 = \lceil \log_2(m+2) \rceil + 2 = \lceil \log_2 \left( \frac{n+3}{2} \right) \rceil + 2
\]
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Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be all nodes.
- For all $i \in \{0, 1, \cdots, m-1\}$ the node $v_{m+2+i}$ sends to $v_i$.
- The node $\{v_0, v_1, v_2, \cdots, v_m\}$ have now the total information.
- If $m + 1$ is even, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_m\}$.
- If $m + 1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_{m+1}\}$.
- For all $i \in \{0, 1, \cdots, m-1\}$ the nodes $v_i$ send to $v_{m+2+i}$.
- Correctness follows direct by the construction.

Running time for $m + 1$ even:
$$r_2(K(m+1)) + 2 = \lfloor \log_2(m + 1) \rfloor + 2 = \lfloor \log_2 (\frac{n+1}{2}) \rfloor + 2$$
$$= \lfloor \log_2(n + 1) \rfloor + 1 = \lfloor \log_2 n \rfloor + 1$$

Running time for $m + 1$ odd:
$$r_2(K(m+2)) + 2 = \lfloor \log_2(m + 2) \rfloor + 2 = \lfloor \log_2 (\frac{n+3}{2}) \rfloor + 2$$
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Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be all nodes.
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Running time for $m + 1$ even:

$$r_2(K(m + 1)) + 2 = \lceil \log_2(m + 1) \rceil + 2 = \lceil \log_2 \left(\frac{n+1}{2}\right) \rceil + 2$$

$$= \lceil \log_2(n + 1) \rceil + 1 = \lceil \log_2 n \rceil + 1$$

Running time for $m + 1$ odd:

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Gossip on Graphs with $2 \cdot m + 1$ Nodes

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- If $m + 1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_{m+1}\}$.
- For all $i \in \{0, 1, \cdots, m - 1\}$ the nodes $v_i$ send to $v_{m+2+i}$.
- Correctness follows direct by the construction.

Running time for $m + 1$ even:
$$r_2(K(m + 1)) + 2 = \lceil \log_2(m + 1) \rceil + 2 = \lceil \log_2 \left(\frac{n+1}{2}\right) \rceil + 2$$

Running time for $m + 1$ odd:
$$r_2(K(m + 2)) + 2 = \lceil \log_2(m + 2) \rceil + 2 = \lceil \log_2 \left(\frac{n+3}{2}\right) \rceil + 2$$
Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be all nodes.
- For all $i \in \{0, 1, \cdots, m - 1\}$ the node $v_{m+2+i}$ sends to $v_i$.
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$$= \lceil \log_2(n + 1) \rceil + 1 = \lceil \log_2 n \rceil + 1$$

Running time for $m + 1$ odd:
$$r_2(K(m + 2)) + 2 = \lceil \log_2(m + 2) \rceil + 2 = \lceil \log_2 \left(\frac{n+3}{2}\right) \rceil + 2$$
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Running time for $m + 1$ odd:
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**Running time for $m + 1$ even:**
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\begin{align*}
    r_2(K(m+1)) + 2 &= \lceil \log_2(m + 1) \rceil + 2 \\
                   &= \lceil \log_2(n + 1) \rceil + 1 \\
                   &= \lceil \log_2 n \rceil + 1
\end{align*}
\]

**Running time for $m + 1$ odd:**
\[
\begin{align*}
    r_2(K(m+2)) + 2 &= \lceil \log_2(m + 2) \rceil + 2 \\
                   &= \lceil \log_2(n + 3) \rceil + 1 \\
                   &= \lceil \log_2 n \rceil + 1
\end{align*}
\]
$1^{st}$ Idea (Let the Knowledge grow)

- We need more rounds.
- A nice proof with this idea will become complicated.
- We will try to put some structure into the proof.
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We need an additional two rounds.

\( \nu_x \) and \( w_y \) alternate as sender and receiver.

The information grows in blocks (intervals) in the nodes.

With this idea we may do the proof.

Only the first two rounds are special.
2\textsuperscript{nd} Idea (Let the Knowledge grow in a structured way)

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2\textsuperscript{nd} Idea (Let the Knowledge grow in a structured way)

- After the first two rounds some node-pairs share their information.
- Consider this situation as the start:
  - All $v_x$ and $w_x$ have one information pair.
  - $v_i$ sends to $w_j$ and the $w_x$ have 2 information pairs.
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  - $v_i$ sends to $w_j$ and the $w_x$ have 5 information pairs.
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  - Thus the grow-rate and the algorithm is clearly visible.
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2nd Idea (Let the Knowledge grow in a structured way)

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Let $n = 2m$.

Gossip-Algorithm:

1. $t := 0$;
2. for all $i \in \{0, \ldots, m-1\}$ do in parallel $R[i]$ sends to $Q[i]$;
3. for all $i \in \{0, \ldots, m-1\}$ do in parallel $Q[i]$ sends to $R[i]$;
4. while $fib(2t+1) < m$ do begin
   1. $t := t + 1$;
   2. for all $i \in \{0, \ldots, m-1\}$ do in parallel
      1. $R[(i + fib(2t-1)) \mod m]$ sends to $Q[i]$;
      2. if $fib(2t) < m$ then
         1. for all $i \in \{0, \ldots, m-1\}$ do in parallel
            1. $Q[(i + fib(2t)) \mod m]$ sends to $R[i]$
   end;
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- for all $i \in \{0, \ldots, m-1\}$ do in parallel $Q[i]$ sends to $R[i]$;
- while $fib(2t + 1) < m$ do begin
  - $t := t + 1$;
  - for all $i \in \{0, \ldots, m-1\}$ do in parallel
    - $R[(i + fib(2t - 1)) \mod m]$ sends to $Q[i]$;
  - if $fib(2t) < m$ then
    - for all $i \in \{0, \ldots, m-1\}$ do in parallel
      - $Q[(i + fib(2t)) \mod m]$ sends to $R[i]$
  end;
algorithm

- Let \( n = 2m \).
- Gossip-Algorithm:
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  \( \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel } Q[i] \text{ sends to } R[i]; \)
  \( \text{while } fib(2t+1) < m \text{ do begin} \)
  \( t := t + 1; \)
  \( \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel} \)
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  \( \text{if } fib(2t) < m \text{ then} \)
  \( \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel} \)
  \( Q[(i + fib(2t)) \mod m] \text{ sends to } R[i] \)
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\[ fib(0) = fib(1) = 1 \]
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Gossip-Algorithm:

$\begin{align*}
& t := 0; \\
& \text{for all } i \in \{0, \ldots, m - 1\} \text{ do in parallel } R[i] \text{ sends to } Q[i]; \\
& \text{for all } i \in \{0, \ldots, m - 1\} \text{ do in parallel } Q[i] \text{ sends to } R[i]; \\
& \text{while } \text{fib}(2t + 1) < m \text{ do begin} \\
& \quad t := t + 1; \\
& \quad \text{for all } i \in \{0, \ldots, m - 1\} \text{ do in parallel} \\
& \quad \quad R[(i + \text{fib}(2t - 1)) \mod m] \text{ sends to } Q[i]; \\
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1. $t := 0$;
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One-Way-Gossip

Theorem:

Let $n = 2m$ and $k = \min\{x \mid \text{fib}(x) \geq m\}$. Then we have $r(K(n)) \leq k + 1$.

Proof:

- The algorithm stops, if $\text{fib}(2t + 1) \geq m$ or $\text{fib}(2t) \geq m$ holds.
- The number of rounds within the loop is $2t$ or $2(t - 1) + 1$.
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- Correctness may be proven by the following invariant:
- Let $a[i]$ be the information, which share $R[i]$ and $Q[i]$ after two rounds.
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Proof: Using the same idea as for the two-way mode.

Theorem:
Let $n$ even. Then we have: $r(K(n)) \geq 2 + \lceil \log_{\frac{1}{2}}(1 + \sqrt{5}) \frac{n}{2} \rceil$.

Proof: See literature (Idea is given the following).

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  - Algorithm with “fibonacci growth”.
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- **Construction of a lower bound:**
  - Start with an arbitrary algorithm.
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  - Abstract.

- We will now try to do the abstraction.

- Try the get the core-problem.

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**Definition:**

The **Network Counting Problem**:

- **Given a directed graph** \( G = (V, E) \).
- Each node stores a number.
- Initial just the number 1 is stored.
- The receiver add the number from the sender to his number after one communication.
- The objective is: all nodes should store a number larger then \( |V| \).
- With \( nc(G) \) we denote the minimal rounds to achieve this objective.

**Lemma:**

For any graph \( G \) we have: \( r(G) \geq nc(G) \).
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2. Abstraction

- Let $G = (\{v_1, v_2, v_3, \ldots , v_n\}, E)$ be a directed Graph.
- Each node $v_i$ stores after $t$ rounds the number $z_i^t$.
- One situation of the network counting problem could be described by a vector:
  - Initial: $(1, 1, 1, \ldots , 1)^T$.
  - After $t$ rounds: $(z_1^t, z_2^t, z_3^t, z_n^t)^T$.
- One round of an algorithm for the network counting problem is given by a matrix $B$:
  - $A$ is a $n \times n$ matrix.
  - $a_{ij} = 1$ node $j$ sends to node $i$.
  - $A$ contains on the diagonal only ones.
  - $A$ has in each row at most two ones.
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We consider now matrices of the above form.

These are matrices $A$, for which there is a transformation $T$ with:

\[
TAT^{-1} = \begin{pmatrix}
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& B & & \\
& & \ddots & \\
0 & & & 1
\end{pmatrix}.
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and $B = \begin{pmatrix} 11 \\ 01 \end{pmatrix}$.

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Recollection (Norm, 3. Abstraction)

- Let \(\|\cdot\|\) be the vector norm over \(\mathbb{R}^n\). Then we have:
  - \(\|x\| = 0 \iff x = 0^n\),
  - \(\|\alpha \cdot x\| = |\alpha| \cdot \|x\|\),
  - \(\|x + y\| \leq \|x\| + \|y\|\)
  - this holds for all \(\alpha \in \mathbb{R}, x, y \in \mathbb{R}^n\)

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Recollection (Norm, 3. Abstraction)

- Let $||\cdot||$ be the vector norm over $\mathbb{R}^n$. Then we have:
  - $||x|| = 0 \iff x = 0^n$,
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2. Abstraction (Continuation)

- We compute the spectral norm:
  
  \[ \|A\| = \|TAT^{-1}\| = \|B\| \]

  \[ B^T \cdot B = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \begin{pmatrix} 11 \\ 01 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix} \]

  \[ \Rightarrow (2 - \lambda)(1 - \lambda) - 1 = 0 \]

  \[ \Rightarrow \lambda^2 - 3\lambda + 1 = 0 \]

  \[ \Rightarrow \lambda_{\text{max}}(B^T B) = \frac{3}{2} + \sqrt{\frac{5}{4}} \]

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Theorem:

A algorithm, solving the network counting problem needs \(2 + \lceil \log_{\frac{1}{2}}(1+\sqrt{5}) \frac{n}{2} \rceil\) rounds.

Proof:

- Let \(A_j, 1 \leq j \leq r\) be matrices, which solve the problem in \(r\) rounds.
- \(\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdots A_2 A_1 (1, 1, \cdots, 1)\).
- \(\|\alpha\| \leq (\prod_{i=1}^{r-2} \|A_i\|) \cdot \|(1, \ldots, 1)\| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}\)
- Let \(\inf(i, t)\) be the number, which have the nodes \(v_i\) after \(t\) rounds.
- After round \(t\) we have: \(\inf(i, t) \geq n\) for all \(i \in \{1, 2, \cdots, n\}\).
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Continuation

\[ \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)^T = A_{r-2} \cdots A_2 \cdot A_1 \cdot (1, 1, \ldots, 1) \]

- Let
  - \(c_1\) be the number of cases with: \(\alpha_i \geq n\),
  - \(c_2\) be the number of cases with: \(\alpha_i < n\) and \(\alpha_j \geq n\),
  - \(c_3\) be the number of cases with: \(\alpha_i < n\), \(\alpha_j < n\) and \(\alpha_i + \alpha_j \geq n\).

- Then we have: \(c_1 \geq c_2\) and \(c_1 + c_2 + c_3 \geq n/2\).

- Thus we also get: \(2c_1 + c_3 \geq \frac{n}{2}\)

- \[\|\alpha\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \sqrt{c_1 n^2 + c_3 \cdot 2 \cdot \frac{n^2}{4}} \geq n \cdot \sqrt{\frac{1}{2} (2c_1 + c_3)} \geq \frac{n}{2} \sqrt{n} .\]

- We already have:
  \[\|\alpha\| \leq (\prod_{i=1}^{r-2} \|A_i\|) \cdot \|(1, \ldots, 1)\| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}.\]

- And we get:
  \[\frac{n}{2} \cdot \sqrt{n} \leq \|\alpha\| \leq \Phi^{r-2} \cdot \sqrt{n},\]

- From which we conclude:
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- $c_1$ be the number of cases with: $\alpha_i \geq n$,
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$$||\alpha|| = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \sqrt{c_1 n^2 + c_3 \cdot 2 \cdot \frac{n^2}{4}} \geq n \cdot \sqrt{\frac{1}{2} (2c_1 + c_3)} \geq \frac{n}{2} \sqrt{n}.$$  

We already have:

$$||\alpha|| \leq (\prod_{i=1}^{r-2} ||A_i||) \cdot ||(1, ..., 1)|| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}.$$  

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Continuation

\( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)^T = A_{r-2} \cdot \ldots \cdot A_2 \cdot A_1 \cdot (1, 1, \ldots, 1) \)

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  \[ r \geq 2 + \left\lceil \log_2 \frac{1}{2} (1 + \sqrt{5}) \frac{n}{2} \right\rceil \]
Continuation

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Continuation

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Quality of these Bounds

**Lemma:**

Let \( n = 2m \) and let:

- \( t_1 := 1 + k \), with \( k \) is the smallest number with \( m \leq F(k) \) and
- \( t_2 := 2 + \lceil \log_{\frac{1}{2}} (1 + \sqrt{5}) m \rceil \).

Then we have \( t_1 = t_2 \) for infinite many \( m \) and \( t_1 \leq t_2 + 1 \) for all \( m \).

**Proof:**

- Let \( \Phi = \frac{1}{2} (1 + \sqrt{5}) \).
- Then we have: \( \Phi^2 = \Phi + 1 \).
- Furthermore we have \( \Phi^{i-2} \leq F(i) \leq \Phi^{i-1} \) for all \( i \geq 2 \).
- Consider \( n \in \mathbb{N} \) with: \( n = 2 \cdot F(k) \) for some \( k \).
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- Let $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
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Proof:

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Lemma:

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- Let $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
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Proof:

- Let \( \Phi = \frac{1}{2} \left( 1 + \sqrt{5} \right) \).
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- Furthermore we have \( \Phi^{i-2} \leq F(i) \leq \Phi^{i-1} \) for all \( i \geq 2 \).
- Consider \( n \in \mathbb{N} \) with: \( n = 2 \cdot F(k) \) for some \( k \).
  - Then we have: \( t_1 = k + 1 \) and
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Let $n = 2m$ and let:

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Proof:

- Let $\Phi = \frac{1}{2} (1 + \sqrt{5})$.
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- Consider $n \in \mathbb{N}$ with: $n = 2 \cdot F(k)$ for some $k$.
  - Then we have: $t_1 = k + 1$ and $t_2 = 2 + \lceil \log_{\Phi} F(k) \rceil = 2 + k - 1 = k + 1$.
  - From which we get: $t_1 = t_2$ for these $n$. 
Quality of these Bounds

Lemma:

Let $n = 2m$ and let:
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Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Let $\Phi = \frac{1}{2} (1 + \sqrt{5})$.
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Quality of these Bounds

Lemma:

Let $n = 2m$ and let:

- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
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Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Let $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
- Then we have: $\Phi^2 = \Phi + 1$.
- Furthermore we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Consider $n \in \mathbb{N}$ with: $n = 2 \cdot F(k)$ for some $k$.
  - Then we have: $t_1 = k + 1$ and
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  - From which we get: $t_1 = t_2$ for these $n$. 
Quality of these Bounds (Part 2)

Lemma:

Let $n = 2m$ and let:

- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{\frac{1}{2}}(1 + \sqrt{5}) \cdot m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Setze $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
- Then we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Let $n = 2 \cdot m$ arbitrary.
  - Let $i$ be defined by: $\Phi^{i-1} < m \leq \Phi^i$, then we have: $t_2 = 2 + i$.
  - Let $k$ be the smallest number with $F(k) \geq m$.
  - Note: $\Phi^{k-2} \leq F(k) \leq \Phi^{k-1}$.
  - Then we have: $i = k - 1$ oder $i = k - 2$.
  - From which we conclude: $t_1 = k + 1 \leq i + 3$. 
Lemma:

Let $n = 2m$ and let:
- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{1/2} \left(1 + \sqrt{5}\right) m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Setze $\Phi = \frac{1}{2} (1 + \sqrt{5})$.
- Then we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Let $n = 2 \cdot m$ arbitrary.
  - Let $i$ be defined by: $\Phi^{i-1} < m \leq \Phi^i$, then we have: $t_2 = 2 + i$.
  - Let $k$ be the smallest number with $F(k) \geq m$.
  - Note: $\Phi^{k-2} \leq F(k) \leq \Phi^{k-1}$.
  - Then we have: $i = k - 1$ oder $i = k - 2$.
  - From which we conclude: $t_1 = k + 1 \leq i + 3$. 
Lemma:

Let \( n = 2m \) and let:

- \( t_1 := 1 + k \), with \( k \) is the smallest number with \( m \leq F(k) \) and
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Quality of these Bounds (Part 2)

Lemma:

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Then we have \( t_1 = t_2 \) for infinite many \( m \) and \( t_1 \leq t_2 + 1 \) for all \( m \).

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### Summary (Telefon-Mode)

| Graph   | $|V|$       | diam | Lower Bound                          | Upper Bound                        |
|---------|------------|------|--------------------------------------|------------------------------------|
| $K_n$   | $n$        | 1    | $\lceil \log_2 n \rceil + \text{odd}(n)$ | $\lceil \log_2 n \rceil + \text{odd}(n)$ |
| $H_k$   | $2^k$      | $k$  | $n - \text{even}(n)$                | $n - \text{even}(n)$              |
| $P_n$   | $n$        | $n - 1$ | $\lceil \frac{n}{2} \rceil + \text{odd}(n)$ | $\lceil \frac{n}{2} \rceil + \text{odd}(n)$ |
| $C_n$   | $n$        | $\lfloor \frac{5k}{2} \rfloor - 2$ | $2k - 1$                            | $2k - 1$                          |
| $CCC_k$ | $k \cdot 2^k$ | $\lfloor \frac{3k}{2} \rfloor$ | $1.9770k$                           | $2.25 \cdot k + o(k)$             |
| $SE_k$  | $2^k$      | $k$  | $1.5965k$                           | $2k + 5$                          |
| $BF_k$  | $k \cdot 2^k$ | $k$  | $1.5965k$                           | $2k + 5$                          |
| $DB_k$  | $2^k$      |      | $1.5965k$                           | $2k + 5$                          |
# Summary (Telegraph-Mode)

| Graph  | $|V|$   | diam | Lower Bound            | Upper Bound          |
|--------|--------|------|------------------------|----------------------|
| $K_n$  | $n$    | 1    | $1.44 \log_2 n$        | $1.44 \log_2 n$      |
| $H_k$  | $2^k$  | $k$  | $1.44 k$               | $1.88 k$             |
| $P_n$  | $n$    | $n-1$| $n + \text{odd}(n)$    | $n + \text{odd}(n)$  |
| $C_n$  | $n$ even | $\lceil \frac{n}{2} \rceil$ | $\frac{n}{2} + \lceil \sqrt{2n} \rceil - 1$ | $\frac{n}{2} + \lceil \sqrt{2n} \rceil - 1$ |
|        | $n$ odd | $\lceil \frac{n}{2} \rceil$ | $\lceil \frac{n}{2} \rceil + \lceil \sqrt{2n - \frac{1}{2}} \rceil - 1$ | $\lceil \frac{n}{2} \rceil + 2 \sqrt{\lceil \frac{n}{2} \rceil} - 1$ |
| $CCC_k$| $k \cdot 2^k$ | $\lceil \frac{5k}{2} \rceil - 2$ | $\lceil \frac{5k}{2} \rceil - 2$ | $\lceil \frac{7k}{2} \rceil + 2 \sqrt{\lceil \frac{k}{2} \rceil} - 2$ |
| $SE_k$ | $2^k$  | $2k-1$ | $2k-1$                | $3k+3$              |
| $BF_k$ | $k \cdot 2^k$ | $\lceil \frac{3k}{2} \rceil$ | $1.9770k$             | $\lceil \frac{5k}{2} \rceil + 2 \sqrt{\lceil \frac{k}{2} \rceil} - 1$ |
| $DB_k$ | $2^k$  | $k$  | $1.5965k$             | $3k+3$              |
Results

Lemma

\[ \text{edp}-r_1(G) \leq \min_{u \in V(G)} \{ \text{edp}-a_u(G) + \text{edp}-b_u(G) \} = 2 \cdot \text{edp}-b_{\min}(G) \]
\[ \text{edp}-r_2(G) \leq 2 \cdot \text{edp}-b_{\min}(G) - 1 \]

Lemma

For any graph \( G_n \) of \( n \) nodes, \( n \geq 2 \),

- \( \lceil \log_2 n \rceil \leq \text{edp}-r_2(G_n) \leq 2 \cdot \lceil \log_2 n \rceil + 1 \),
- \( \log_b(\lfloor n/2 \rfloor) + 2 \leq \text{edp}-r_1(G_n) \leq 2 \cdot \lceil \log_2 n \rceil + 2 \).
Results

**Lemma**

For each complete binary tree $T_2^h$ of depth $h \geq 3$ (and $n = 2^{h+1} - 1$ nodes),

- $2h + 3 = 2 \cdot \lceil \log_2 n \rceil + 1 \leq edp-r_1(C2T_h) \leq 2h + 4$,
- $2h + 2 = 2 \cdot \lceil \log_2 n \rceil \leq edp-r_2(C2T_h) \leq 2h + 3$.

**Lemma**

\[
edp-r_2(Gr_n^2) = 1.5 \cdot \log_2 n - \log_2 \log_2 n \pm O(1) \\
edp-r_2(Pl(n, h)) \geq 1.5 \log_2 n - \log_2 \log_2 n - 0.5 \log_2 h - 2
\]
Lemma

For $d \geq 3$

(i) $edp-r_2(Gr_n^d) = (1 + 1/d) \cdot \log_2 n - \log_2 n \log_2 n \pm O(d)$,

(ii) $edp-r_1(Gr_n^d) \leq (\log_2 b + (2 - \log_2 b)/d) \cdot \log_2 n + O(d)$

$= (1.44\ldots + 0.56\ldots/d) \cdot \log_2 n + O(d)$.

Lemma

For every $X_k \in \{BF_k, CCC_k, Q_k\}$ of $n$ nodes and dimension $k$, $edp-r_1(X_k) \leq r_1(K_n) + O(\log_2 \log_2 n)$.

Lemma

For every $Y_k \in \{BF_k, CCC_k\}$ of $n$ nodes and dimension $k$, $edp-r_2(Y_k) \leq r_2(K_n) + O(\log_2 \log_2 n)$. 
Literature

Legende

■ : Nicht relevant
■ : Grundlagen, die implizit genutzt werden
■ : Idee des Beweises oder des Vorgehens
■ : Struktur des Beweises oder des Vorgehens
■ : Vollständiges Wissen