Planar Graphs

Walter Unger

Lehrstuhl für Informatik 1

13:00 , April 19, 2016
<table>
<thead>
<tr>
<th>Section</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic Definitions</td>
<td>1-2</td>
</tr>
<tr>
<td>- Graphs</td>
<td></td>
</tr>
<tr>
<td>- Special Graphs</td>
<td></td>
</tr>
<tr>
<td>- Connectivity of Graphs</td>
<td></td>
</tr>
<tr>
<td>- Statements</td>
<td></td>
</tr>
<tr>
<td>Introduction to planar Graphs</td>
<td>3-5</td>
</tr>
<tr>
<td>- Definitions</td>
<td></td>
</tr>
<tr>
<td>- Theorems on planar Graphs</td>
<td></td>
</tr>
<tr>
<td>- Definitions on outer-planar graphs</td>
<td></td>
</tr>
<tr>
<td>- Theorems on outer-planar Graphs</td>
<td></td>
</tr>
<tr>
<td>- Theorems on SP-Graphs</td>
<td></td>
</tr>
<tr>
<td>- Homeomorph Graphs</td>
<td></td>
</tr>
<tr>
<td>Separators</td>
<td>6</td>
</tr>
<tr>
<td>- Motivation</td>
<td></td>
</tr>
<tr>
<td>- Definition</td>
<td></td>
</tr>
<tr>
<td>- Examples</td>
<td></td>
</tr>
<tr>
<td>- Alternative Definition</td>
<td></td>
</tr>
<tr>
<td>- Introduction to planar Separators</td>
<td></td>
</tr>
<tr>
<td>- Overview</td>
<td></td>
</tr>
<tr>
<td>- Preparation</td>
<td></td>
</tr>
<tr>
<td>- Planare-Graph-Separator Theorem</td>
<td></td>
</tr>
<tr>
<td>Applications</td>
<td>7</td>
</tr>
<tr>
<td>- Independent Set on planar Graphs</td>
<td></td>
</tr>
</tbody>
</table>
Definition: Graph

Let $V(G) = \{v_1, ..., v_n\}$ be a non-empty set of nodes and $E(G)$ be a set or multiset of pairs from $V(G)$ (set of edges).

The sets $V(G)$ and $E(G)$ define the graph $G = (V(G), E(G))$.

If $G$ is uniquely determined, then we just write: $V$ and $E$.

Or in other words $G = (V, E)$.

We always use as default writing: $n = |V|$ and $m = |E|$.
**Way of Speaking for Graphs**

- Let $G = (V(G), E(G))$ and $e = (v, w) \in E(G)$.
- The nodes $v, w$ are called **connected** (adjacent) by an edge $e$.
- An edge $e$ is called **loop**, if $v = w$ holds.
- Two edges are called **parallel**, if they are the same.
- A graph without parallel edges is called **simple**.

As long as we do not state differently we will use in the following simple graph without loops.
Degree of a Node

**Definition (Degree of a Node)**

Let $v \in V(G)$.

With

$$\deg(v) = |\{e \in E(G) \mid e = (v, v'), v' \in V(G) \setminus \{v\}\}|$$

we denote the degree of a Node (degree) of $v$.

- $\deg(v_0) = 4$.
- $\deg(v_1) = 3$.
- $\deg(v_4) = 6$.
- $\deg(v_5) = 6$. 
### Regular and Complete

**Definition (Regular)**

A graph $G$ is called $k$-regular, iff for all $v \in V(G)$ we have: $d(v) = k$.

**Definition (Complete)**

A graph $G$ is called complete, iff all pairs of nodes $a, b$ from $V$ holds: $(a, b) \in E$.

- Notation: $K_n$. 
Definition (Bipartite)
A Graph $G$ is called **bipartite**, iff $V$ may be split into disjoint sets $V'$, $V''$, such that each edge connects only nodes from both partitions.

- Notation: $G = (V', V'', E)$

Definition (Complete bipartite)
A Graph $G$ is called **complete bipartite**, iff $V$ may be split into disjoint sets $V'$, $V''$, and $E = \{ (a, b) \mid a \in V', b \in V'' \}$.

- Notation: $K_{p,q}$ with $p = |V'|$ and $q = |V''|$.
- Star, iff $S_n = K_{1,n-1}$. 
Examples

1. Graph 1:
- Nodes: a0, a1, a2, a3, a4, b0, b1, b2, b3, b4
- Edges: a0-b0, a0-a1, a0-a2, a0-a3, a0-a4, a1-b1, a1-a2, a1-a3, a1-a4, a2-b2, a2-a3, a2-a4, a3-b3, a3-a4, a4-b4

2. Graph 2:
- Nodes: a0, a1, a2, a3, a4, b0, b1, b2, b3, b4
- Edges: a0-b0, a0-a1, a0-a2, a0-a3, a0-a4, a1-b1, a1-a2, a1-a3, a1-a4, a2-b2, a2-a3, a2-a4, a3-b3, a3-a4, a4-b4
Definition (Subgraph)

A Graph $H = (V(H), E(H))$ is called a subgraph of $G = (V(G), E(G))$, iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
**Subgraphs**

**Definition (node-induced subgraph)**

A graph \( H = (V(H), E(H)) \) is a node-induced subgraph of \( G = (V(G), E(G)) \), iff \( V(H) \subseteq V(G) \) and \( E(H) = \{ (a, b) \in E(G) \mid a, b \in V(H) \} \).
A graph \( G = (V, E) \) is called connected, iff between any two different nodes \( a, b \) exists a path from \( a \) to \( b \).
Definition

Let $G = (V, E)$, $V' \subset V$ is called a node-separator (vertex cut), iff $G - V'$ is not connected.

Notation: $G - V' := (V \setminus V', \{(a, b) \in E \mid a, b \in V \setminus V'\})$

Definition

If $\{v\}$ is a node-separator, then $v$ is called articulation point.

Theorem

Only cliques $K_n$ do not have any node-separator.
Example
Edge-Separator

**Definition**

Let $G = (V, E)$. $E' \subset E$ is called edge-separator (edge cut), iff $G - E'$ is not connected.

Notation: $G - E' := (V, E \setminus E')$

**Definition**

If $\{v, w\}$ is an edge-separator, then $\{v, w\}$ is called a bridge.

**Theorem**

An minimal edge-separator $E'$ of $G = (V, E)$ induces a 2-partite graph. Or in other words: $G = (V, E')$ is a 2-partite graph.
Example
Connectivity

**Definition**

A Graph $G = (V, E)$ is called $k$-connected, iff $\forall V' \subset V : |V'| = k - 1$ we have $G - V'$ is connected.

A $k$-connected Graph is also $k - 1$-connected.

**Notation:** $\kappa(G) = k$

**Definition**

Let $G = (V, E)$ and $k$ minimal with: $\exists E' \subset E : |E'| = k$ and $G - E'$ is not connected or trivial. Then we call $G$ $k$-edge-connected.

A $k$-edge-connected Graph is also $k - 1$-edge-connected.

**Notation:** $\lambda(G) = k$
Statements on Connectivity

Theorem

For any graph $G = (V, E)$ we have:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Notation: $\delta(G) := \min\{\deg(v) \mid v \in V\}$

Theorem

For all integer numbers $0 < a \leq b \leq c$ there are graphs $G$ with:

$$\kappa(G) = a, \ \lambda(G) = b, \ \delta(G) = c$$

Theorem

Let $G = (V, E)$ be a graph with: $|V| = n$ and $\delta(G) \geq n/2$. Then we have:

$$\lambda(G) = \delta(G)$$
Statements on Node-Connectivity

**Theorem**

Let $G = (V, E)$ with: $|V| = n$ and $|E| = m$. Then is the maximal connectivity (maximal $k$ with $G$ is $k$-connected) of $G$:

- $0$ falls if $m < n - 1$
- $2 \cdot \frac{m}{n}$ if $m \geq n - 1$

**Theorem**

Let $G = (V, E)$ connected. The following statements are equivalent:

1. $v \in V$ is a node-separator.
2. $\exists a, b \in V: a, b \neq v$: each path from $a$ to $b$ traverses via $v$.
3. $\exists A, B: A \cup B = V \setminus \{v\}$ and each path from $a \in A$ to $b \in B$ traverses via $v$. 
Statements on Edge-Connectivity

Theorem

Let $G = (V, E)$ be connected. The following statements are equivalent:

1. $e \in E$ is a edge-separator.
2. $e$ is not in any simple cycle of $G$.
3. $\exists a, b \in E$: each path from $a$ to $b$ traverses via $e$.
4. $\exists A, B$: $A \cup B = V$ and each path from $a \in A$ to $b \in B$ traverses via $e$. 
Definition

Let $G = (V, E)$ and $(a, b) = e \in E$. The subdivision of an edge $e$ results in graph $G = (V \cup \{v\}, E \cup \{(a, v), (v, b)\} \setminus \{e\})$.

Definition

A set of paths of $G = (V, E)$ is called intern-node-disjoint, iff no two paths share an internal-node. The internal nodes are all except the start and the end node.
Let $G = (V, E)$ with $|V| \geq 3$. The following statements are equivalent:

1. $G$ is 2-connected.
2. Each node pair is connected by two intern-node-disjoint paths.
3. Each node pair is on a common simple cycle.
4. There exits an edge and each node together with this edge is on a common simple cycle.
5. There exit two edges and each pair of edges is on a common simple cycle.
6. For each pair of nodes $a, b$ and an edge $e$ exists a simple path from $a$ to $b$ traversing $e$.
7. For three nodes $a, b, c$ exists a path from $a$ to $b$ traversing $c$.
8. For three nodes $a, b, c$ exists a path from $a$ to $b$ avoiding $c$. 
**Theorem**

Let $G = (V, E)$ $k$-connected. Then any $k$ nodes are on a common simple cycle.

**Notation:** Let $(G = V, E)$ and $(H = W, F)$ graphs

$G + W = (V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\})$

**Theorem**

A graph $G$ is 3-connected, iff $G$ may be constructed from the wheel $W_i = K_1 + C_i$ $(i \geq 4)$ by the following operations:

1. **Adding a new edge.**
2. **Splitting a node of degree $\geq 4$ into two connected nodes of degree $\geq 3$.**
Theorem (Menger’s Theorem)

\( G \) is \( k \)-connected, iff any two nodes are connected by \( k \) intern-node-disjoint paths.

Theorem (Menger’s Theorem)

\( G \) is \( k \)-edge-connected, iff any two nodes are connected by \( k \) edge-disjoint paths.
Theorem

The 1-connectivity of a graph may be computed by DFS/BFS.

Theorem

The 1-edge-connectivity of a graph may be computed by DFS/BFS.

Theorem

The 2-connectivity of a graph may be computed by flow algorithms.

Theorem

The k-edge-connectivity of a graph may be computed by flow algorithms.
Definitions

Definition

A graph $G = (V, E)$ is called planar, iff it could be drawn in the plane without crossing edges. A connected area of such an embedding is called window. The unlimited window is called outer window.

Definition

A graph $G = (V, E)$ is called maximal planar, iff the adding of an edge makes $G$ non-planar.
Example: planar Graph
Result I

**Theorem**

If \( G = (V, E) \) is planar and 2-connected, then each window is a simple cycle and each edge separates two different windows.

**Theorem (Euler)**

Let \( G = (V, E) \) be a planar graph with \(|V| = n\), \(|E| = m\). Let \( f \) be the number of windows and \( k \) be the number of connected components. Then the following holds:

\[
 n - m + f = 1 + k.
\]

Proof by simple induction.
Proof

- $n - m + f = 1 + k$ holds for a single node.
- new node: $(n + 1) - m + f = 1 + (k + 1)$
- new edge connects components: $n - (m + 1) + f = 1 + (k - 1)$ or
- new edge separates window: $n - (m + 1) + (f + 1) = 1 + k$. 
Let $G = (V, E)$ be a planar graph with $|V| = n$, $|E| = m$ and each window is a simple cycle of length $k$. Then the following holds:

$$m = k \cdot \frac{n - 2}{k - 2}$$

Note: $k \cdot f = 2 \cdot m$ and $n - m + f = 2$

Let $G = (V, E)$ be a planar graph with $|V| = n$, $|E| = m$ and each window is a $3$-clique. Then the following holds: $m = 3 \cdot n - 6$.

If each window is a simple cycle of length $4$, then we get: $m = 2 \cdot n - 4$.

Let $G = (V, E)$ be a planar graph with $|V| = n \geq 3$, $|E| = m$. Then we get: $m \leq 3 \cdot n - 6$. If $G$ contains no triangles, then we have: $m \leq 2 \cdot n - 4$. 
Theorem

\( K_5 \) and \( K_{3,3} \) are non-planar graphs.

Theorem

Let \( G = (V, E) \) be a planar graph with \( |V| \geq 4 \). Then \( G \) contains at least four nodes with degree \( \leq 5 \).

Theorem

Let \( G = (V, E) \) be a planar graph. Then each window could become the outer window.
Let $G = (V, E)$ be a maximal planar graph with $|V| \geq 4$. Then $G$ is 3-connected.

Each 3-connected planar graph is embeddable in a unique way on the sphere.

Any planar graph could be drawn with straight lines on the plane.
Theorem

The recognition-problem for planar graphs is solvable in linear time.
A planar graph $G$ is called outer-planar, iff it could be drawn without crossing in the plane, such that all nodes are on one (the outer) window.

A graph $G = (V, E)$ is called maximal outer-planar, iff the addition of any edge makes $G$ non-outer-planar.
Example: outer-planar Graph
A planar graph $G$ is called outer-planar, iff it could be drawn without crossing in the plane, such that all nodes are on one (the outer) window.

A graph $G = (V, E)$ is called maximal outer-planar, iff the addition of any edge makes $G$ non-outer-planar.

A planar graph $G = (V, E)$ is called $k$-outer-planar, iff it could be drawn in the plane, such that

- no two edges cross and
- after deletion $k - 1$ times the nodes of the outer window,
- the remaining is a embedded outer-planar graph.
Results I

Theorem

Let $G = (V, E)$ be a maximal outer-planar graph with $|V| = n \geq 3$. Then $G$ will have $n - 2$ inner windows.

Theorem

Let $G = (V, E)$ be a maximal outer-planar Graph with $|V| = n$ and $|E| = m$. Then the following holds:

1. $2 \cdot n - 3 = m$
2. At least three nodes have a degree of $\leq 3$.
3. At least two nodes have a degree of two.
4. $G$ is exactly two-connected.

Theorem

$K_4$ and $K_{2,3}$ are not outer-planar graphs.
SP-Graphs

Definition

A SP-graph is constructed by a sequence of series and parallel operations from the graphs \((\{a, b\}, \{(a, b)\})\) and \((\{a, b\}, \emptyset)\).

The parallel operation merges the corresponding connector nodes.
The series operation merges two connector nodes. This new may not be used as a connector node in any future operation.

Theorem

\(K_4\) is not a SP-graph, but the \(K_{2,3}\) is a SP-graph.
Definition

Two graphs $G$ and $H$ are called homeomorph, iff they could be constructed from the same graph by a sequence of subdivisions.
Results I

Theorem

\( G = (V, E) \) is outer-planar, iff no subgraph is homeomorph to the \( K_4 \) or the \( K_{2,3} \) with the exception of the \( K_4 - e \).

Theorem

\( G = (V, E) \) is a SP-graph, iff no subgraph is homeomorph to the \( K_4 \) with the exception of the \( K_4 - e \).

Theorem (Kuratowski)

\( G = (V, E) \) is planar, iff no subgraph is homeomorph to the \( K_5 \) or \( K_{3,3} \).

Theorem

A outer-planar graph is a SP-graph.
A SP-Graph is an outer-planar graph.
### Results I

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any planar graph is 5-colourable.</td>
<td></td>
</tr>
<tr>
<td>Any planar graph is 4-colourable.</td>
<td></td>
</tr>
<tr>
<td>Any planar graph with at most two triangles is 3-colourable.</td>
<td></td>
</tr>
</tbody>
</table>
A Proof

Theorem

*Any planar graph is 5-colourable.*

Idea of Proof:

- Choose a node $v$ of degree less than 6.
- Colour recursively $G - \{v\}$.
- If $\deg(v) < 5$ holds, $v$ can be coloured.
- If all neighbours of $v$ use just four colours, $v$ can be coloured.
- If $\deg(v) = 5$ holds and all neighbours of $v$ are coloured with different colours, note:
  - Within $G - \{v\}$ there is a component, which uses just two colours and can be recoloured.
  - A short case discussion shows:
    - There exists two colours and a component using these colours, such that just one neighbour of $v$ receives a new colour.
Recolouring one Component
Theorem

A planar graph is 4-colourable, iff each hamilton planar graph is 4-colourable.

Theorem

A planar graph is 4-colourable, iff each cubic planar graph without bridges is 3-colourable.

Theorem

The 3-colouring-problem on planar graphs if degree \( \leq 4 \) is NP-complete.
Idea and Structure of Proof

**Theorem**

The 3-colouring-problem on planar graphs if degree $\leq 4$ is NP-complete.

- Problem $L_1$ is easier than $L_2$: $L_1 \leq_P L_2$.
- If $L_2$ is in $\mathcal{P}$, then is also $L_1$ in $\mathcal{P}$.
- If $L_1$ is hard, i.e. $L_1 \in \mathcal{NPC}$, then is also $L_2 \in \mathcal{NP}$.

**Structure of proof:**

- Let $L_1 \in \mathcal{NPC}$ and we assume $L_2 \in \mathcal{P}$.
- We transform input of $L_1$ with function $f$ into input for $L_2$ such that:
  - $x \in L_1 \iff f(x) \in L_2$.
  - If $f \in \mathcal{P}$ holds, then we get $L_1 \in \mathcal{P}$, which is a contradiction.

Here we have: $L_1$ is the 3-colouring-problem and $L_2$ 3-colouring-problem on planar graphs of degree $\leq 4$. 
Idea and Structure of Proof

Theorem

The 3-colouring-problem on planar graphs if degree $\leq 4$ is NP-complete.

- Let $G = (V, E)$ be the input of the 3-colouring-problem
- Construct planar $f(G)$ as input of the 3-colouring-problem
  - Draw $G$ in the plane. We get some crossings.
  - Replace each crossing with a 3-colorable planar graph, such that $G$ is 3-colorable, $f(G)$ is 3-colorable.

Lemma

There exists a planar graph $H$ with nodes $a, c, b, d$:

- The nodes $a, c, b, d$ are on the outer face in that order.
- The nodes $a, b$ take in any 3-coloring of $H$ the same color.
- The nodes $c, d$ take in any 3-coloring of $H$ the same color.
Proof (planar)

- The central nodes are coloured w.l.o.g. as follows.
- Case 1: Colour $a$ blue.
Proof (planar 2.case)

- The central nodes are coloured w.l.o.g. as follows.
- Case 2: Colour a red.
Proof (planar)

Each crossing is replaced by such a component.
Proof (planar, degree 4)

There exists a component $H$ with three nodes $a, h, d$ of degree 2 which are coloured the same in each 3-colouring of $H$. 

\[ \Sigma = 0 \]
Proof (planar, degree 4)

There exists a component $H_x$ with $x$ nodes of degree 2 which are coloured the same in each 3-colouring of $H_x$. 
Summary (Proof)

- Replace edge-crossings by the above construction, such that
  - each crossing is replaced by one component.
  - I.e. an edge with $x$ crossings will be replaced by $x$ components
  - and one edge.

- Replace a node of degree $g > 4$ by $\lceil (g - 6)/2 \rceil + 1$ components of the second construction.
  - Note: $x$ tree-wise connected components have $x + 2$ nodes of degree 2 coloured by the same colour.
  - $2 \cdot (\lceil (g - 6)/2 \rceil + 1 + 2) \geq 2 \cdot ((g - 6)/2 + 3) = g$
Introduction

- Basis for all divide and conquer algorithms.
- We would like to have small separators.
- Split the graph at the separator.
- Solve the problem recursively on the disconnected components.
- Construct the solution by using the sub-solutions.
- Here: separators for planar graphs.
Definition

Let $G = (V, E)$ be a graph and $n = |V|$.

Let $0 \leq \alpha \leq 1$ be a constant.

Let $f(n)$ be a function.

We call $C \subset V$ a $(f(n), \alpha)$-separator, iff
- $|C| \leq f(n)$ and
- each component of $G[V \setminus C]$ contains at most $\alpha \cdot n$ nodes.
Example 1

Lemma

A tree \( T \) has a \((1, 1/2)\)-separator.

Proof:

- Choose a arbitrary node \( c \) as a candidate.
- Let \( T_i \) be the trees in \( T - c \).
- If one component \( T_i \) contains more than \( n/2 \) nodes,
- then choose a new candidate \( c := \Gamma(c) \cap V(T_i) \).
- After such a step the size of the largest component decreases by at least one.
- Repeat till a separator is found.
Example Outer-planar Graph
Example 2

**Lemma**

A outer-planar graph $G$ has a $(3, 1/2)$-separator.

**Proof:**

- Maximise the outer-planar graph $G$.
- Use the above technique.
- Thus use the tree of inner windows.
- Choose as separator the node of the selected window.
Example Tree

Graph representation with nodes labeled from 00 to 63 and edges connecting them. Nodes are marked with various letters (a, c, e, r, s, u) and numbers. The graph is divided into two main sections, with nodes 31 and 41 highlighted in red and green, respectively. The graph is a planar graph illustrating the concept of separators in such graphs.
**Definition**

Let $G = (V, E)$ be a graph and $n = |V|$.

Let $f(n)$ be a function.

Then $C \subset V$ is called a $f(n)$-separator, iff

- $V$ may be split in $C$, $T_1$, $T_2$.
- $|C| \leq f(n)$.
- $T_1$, $T_2$ are not connected.
- $T_i$ has at most $2/3 \cdot n$ nodes.
Comparing above Definitions

**Lemma**

\[ G \text{ has a } (f(n), \frac{2}{3}) \text{-separator, iff } G \text{ has a } f(n) \text{-separator.} \]

**Show \( \Leftarrow \)**

- Each component \( K \) contains at most in one \( T_i \).
- Thus \( |V(T_i)| \leq \frac{2}{3} \cdot n \) holds.

**Show \( \Rightarrow \)**

- If a component \( K \) contains at least \( \frac{1}{3} \cdot n \) nodes, then choose \( T_1 = K \).
- If all components contain less then \( \frac{1}{3} \cdot n \) nodes, then enlarge \( T_1 \) step by step till \( T_1 \) contains more than \( \frac{1}{3} \cdot n \) nodes.
- Then \( T_2 \) contains at most \( \frac{2}{3} \cdot n \) nodes.
Planar graphs are important with many applications.

How large could be a minimal separator in a planar Graph?

First example:
Planar graphs are important with many applications.

There is no separator of constant size.

Aim: $O(\sqrt{n})$-separator.

Consider maximal planar graphs.

Consider cycles as separators.
### Overview

#### Theorem (Lipton, Tarjan 1979)

*Each planar graph with $n$ nodes has a $(2 \cdot \sqrt{2n}, 2/3)$-separator.*

#### Theorem (Lipton, Tarjan 1979)

*A $(2 \cdot \sqrt{2n}, 2/3)$-separator can be constructed on planar graphs in time $O(n)$.**

#### Theorem (Lipton, Tarjan 1979)

*Let $G = (V, E)$ be a planar graph and $\varepsilon \leq 1$ with $\varepsilon \cdot n \geq 1$. Then contains $G$ a $(2 + \sqrt{2/(\varepsilon \cdot n)}) \cdot \sqrt{6} \sqrt{n/\varepsilon}, \varepsilon)$-separator, which could be constructed in time $O(n \log 1/\varepsilon)$.**
Basic Idea

- We could hope for a good separator.
- But in general we may need $O(\sqrt{n})$.
- In the worse case the planar graph is maximal.
Definition (Diameter and Radius)

- The diameter of $G = (V, E)$ is:
  \[
  \text{diam}(G) = \max\{\text{dist}(v, w) \mid v, w \in V\}.
  \]

- The radius of a node $v \in V$ is:
  \[
  \text{rad}(v, G) = \max\{\text{dist}(v, x) \mid x \in V\}
  \]

- The radius of $G$ is:
  \[
  \text{rad}(G) = \min\{\text{rad}(v, G) \mid v \in V\}.
  \]
Lemma

Let $G = (V, E)$ be a planar graph and $B = (V, T)$ be a spanning-tree of $G$ with radius $s$. Then $G$ contains a $(2 \cdot s + 1, 2/3)$-separator.

Proof:

- Let $G$ be triangulated and embedded in the plane as a planar Graph.
- Let $e \in E \setminus T$.
- $e$ assembles with some edges from $T$ a unique cycle $C_e$.
- By $int(C_e)$ we denote the number of nodes which are inside $C_e$.
- $ext(C_e)$ we denote the number of nodes which are outside $C_e$.
- Aim: Search $e$ with $int(C_e) \leq 2/3 \cdot n$ and $ext(C_e) \leq 2/3 \cdot n$.
- Then is $C_e$ a $(2 \cdot s + 1, 2/3)$-separator.
Example
Proof (continued)

- Search step by step for an edge $e$ with $\text{int}(C_e) \leq 2/3 \cdot n$ and $\text{ext}(C_e) \leq 2/3 \cdot n$.
- Choose any $e$.
- If $\text{int}(C_e) \leq 2/3 \cdot n$ and $\text{ext}(C_e) \leq 2/3 \cdot n$ holds, terminate.
- Let w.l.o.g.: $\text{int}(C_e) > 2/3 \cdot n$.
- Let $e = \{x, y\}$ and $z$ be the missing node of the window attached at $e$ and in the inside of $C_e$.
  - If $e' = \{x, z\}$ on the cycle $C_e$, continue with considering $C_{e''}$.
  - If $e'' = \{y, z\}$ on the cycle $C_e$, continue with considering $C_{e'}$.
  - Otherwise let w.l.o.G. $\text{int}(C_{e'}) \leq \text{int}(C_{e''})$ and consider now $C_{e''}$.

- In the last step $\text{int}(C_e) \leq 2/3 \cdot n$ und $\text{int}(C_e) \geq 1/3 \cdot n$ holds.
- It follows that $\text{int}(C_e) \leq 2/3 \cdot n$ und $\text{ext}(C_e) \leq 2/3 \cdot n$ holds.
Proof (continued)

- Last step in detail:
  - The inside of $e = \{x, y\}$ is too large:
    
    $$
    2/3 \cdot n < \text{int}(C_{\{x,y\}}) = \text{int}(C_{\{x,z\}}) + \text{int}(C_{\{y,z\}}) + |C_{\{x,z\}} \cap C_{\{y,z\}}| - 1
    $$

  - The inside of $e'' = \{x, z\}$ is the larger part of ($\text{int}(C_{e'}) \leq \text{int}(C_{e''})$):
    
    $$
    2/3 \cdot n < \text{int}(C_{\{x,z\}}) + \text{int}(C_{\{y,z\}}) + |C_{\{x,z\}} \cap C_{\{y,z\}}| - 1
    \leq 2 \cdot \text{int}(C_{\{x,z\}}) + |C_{\{x,z\}}|
    $$

- This way we get:
  
  $$
  \text{ext}(C_{\{x,z\}}) = n - |C_{\{x,z\}}| - \text{int}(C_{\{x,z\}})
  \leq n - 1/3 \cdot n
  = 2/3 \cdot n
  $$
Example

Start BFS from some node $r$. If the radius is smaller than $\sqrt{2n}$ we apply the lemma.
Example

Consider the case that the radius is large than $\sqrt{2n}$. Each intermediate level disconnects the graph. We could only hope for a small separator.
Example

None of the levels is a separator.

Check set of levels of distance $s = \lceil \sqrt{n/2} \rceil$.

One set is smaller than $\lfloor n/s \rfloor$. 
Example

If this set is no separator, consider the largest component.
And apply the lemma.
Planer Separator (Teil 1)

Theorem (Lipton, Tarjan 1979)

Any planar Graph with \( n \) nodes has a \((2 \cdot \sqrt{2n}, 2/3)\)-separator.

Proof:

- Choose node \( w \) as the root.
- Determine \( S_i \) (\( 1 \leq i \leq l \)) the set of nodes at distance \( i \) from \( w \).
- If \( 2 \cdot l + 1 \leq 2\sqrt{2n} \) holds, the proof follows from the above Lemma.
- Otherwise let \( s = \lceil \sqrt{n/2} \rceil \).
- Define \( L_j = \bigcup_{i \equiv j \mod s} S_i \) for \( 0 \leq j < s \).
- For a \( k \) hold: \(|L_k| \leq \lceil n/s \rceil\).
- Consider \( H = G[V \setminus L_k] \).
- Assume now, that one component of \( H \) has more than \( 2/3 \cdot n \) nodes.
Proof (continued)

- $H$ contains at most $s - 1$ continuous levels $S_i$.
- Let $S_l, S_{l+1}, \ldots, S_{l+s-2}$ be those levels.
- Show that $H$ could be embedded as a planar graph $H'$ with radius $s - 1$.
  - If $l = 0$ holds, is $w$ part of $H$ and we have an embedding.
  - Otherwise $l > 0$ holds and we connect all nodes from $S_l$ with a node $w'$.
- We have by the above lemma for $H'$ a $(2 \cdot s - 1, 2/3)$-separator $C'$.
- The separator for $G$ is $C = C' \cup L_k$.
- $|C| \leq \lceil n/s \rceil + 2 \cdot s - 1$.
- Note: $s = \lceil \sqrt{n/2} \rceil$.
- Thus we have: $|C| \leq \sqrt{2n} + \sqrt{2n}$. 
Theorem 2

Theorem (Lipton, Tarjan 1979)

A $(2 \cdot \sqrt{2n}, 2/3)$-separator for a planar graph may be computed in time $O(n)$.

- Computing the levels: breath-first-search.
- Counting the nodes: run through a tree.
- Planar embedding: depth-first-search.
- Triangulation: local search.
- Construction of the tree of windows: depth-first-search.
- Counting the nodes: run through the tree of windows
- Used also: dynamic programming.
Resultats

Theorem

Any graph with genus $g$ and $n$ nodes has a $(6 \cdot \sqrt{gn} + 2 \cdot \sqrt{2n} + 1, 2/3)$-separator, which could be computed in time $O(n + g)$.

Theorem

Any graph without $H$-minor and $n$ nodes has a $(|H|^{3/2} \sqrt{n}, 2/3)$-separator.
The independent set problem on planar Graphs of degree three is NP-complete.

Proof:

- Construct component for the crossing of two edges.
- This component will increase the size of the independent set by six.
- We could replace a polynomial number of crossing with this component.
- Replace a node of degree $\geq 4$ by a special binary tree.
- The leaves will take the role of the original node.
- There could be two cases:
  - All leaves are in the independent set and the total number within the tree is $x$.
  - No leaf is in the independent set and the total number within the tree is $x - 1$. 
Crossings (1)
Crossings (2a)
Crossings (2b)
Crossings (3b)
Crossings (4a)
Crossings (4b)
Gradkomponente

Diagramm mit Knoten und Kanten.
The independent set problem on planar graphs is solvable in time $2^{O(\sqrt{n})}$.

**Algorithm:**

- Compute a $C$.
- For each independent set $I$ on $C$:
  - Remove all nodes $\Gamma(I)$ from the components of $G[V \setminus C]$.
  - Solve the independent set problem recursively on each component.

**Running time:** $t(n) \leq O(n) + 2^{\sqrt{8n}} + O(n) \cdot t(2/3 \cdot n)$.

Let $k_1, k_2, n_0$ be the constants of the $O$ terms,

- i.e. $(k_1 + k_2) \cdot n \leq 2^{\sqrt{8n}}$ for all $n > n_0$.
- Let $L \geq t(n)$ for all $n \leq n_0$.

Show by induction: $t(n) \leq L \cdot 2^{\frac{2\sqrt{8n}}{1-\frac{2}{3}}}$.
Proof (Continuation)

- Show by induction: \( t(n) \leq L \cdot 2^{\frac{2\sqrt{8n}}{1-\sqrt{2/3}}} \).
- Holds for \( n \leq n_0 \).
- Let \( n > n_0 \):
  - Reminder: \((k_1 + k_2) \cdot n \leq 2^{\sqrt{8n}}\).
  - \[
  t(n) \leq k_1 \cdot n + 2^{\sqrt{8n}} + k_2 \cdot n \cdot t\left(\frac{2}{3} \cdot n\right)
  \]
  - \[
  \leq 2^{2 \cdot \sqrt{8n}} \cdot t\left(\frac{2}{3} \cdot n\right)
  \]
  - \[
  \leq 2^{2 \cdot \sqrt{8n}} \cdot L \cdot 2^{\frac{2\sqrt{2/3 \cdot 8n}}{1-\sqrt{2/3}}}
  \]
  - \[
  = L \cdot 2^{\frac{2\sqrt{8n}}{1-\sqrt{2/3}}}
  \]