Algorithmic Graph Theory (SS2016)
Chapter 5
Perfect Graphs

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Reminder 1

- Colouring is hard!
- Colouring is NP-complete.
- Colouring is not approximable.
- There are no good bounds known.
- Question: is there a graph class with good bounds?
Definition

Let $G = (V, E)$ be a graph.

$$
\alpha(G) = \max \{ |V'| : V' \subset V \land \forall a, b \in V' : (a, b) \notin E \} \\
\omega(G) = \max \{ |V'| : V' \subset V \land \forall a, b \in V' : (a, b) \in E \} \\
\chi(G) = \min \{ k : \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \notin E \} \\
\bar{\chi}(G) = \min \{ k : \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \in E \} 
$$

Further notations:
$$
\omega(G) = \bar{\alpha}(G), \\
\alpha(G) = \bar{\omega}(G) = \beta_0(G), \\
\kappa(G) = \bar{\chi}(G)
$$
Statements I

Theorem

Let $G = (V, E)$ be a graph. Then we have:

$$\alpha(G) = \overline{\alpha(G)} \text{ and } \chi(G) = \overline{\chi(G)}$$

Proof:

$$\alpha(G) = \max\{ |V'| \mid V' \subseteq V \land \forall a, b \in V' : (a, b) \not\in E \}$$

$$\omega(G) = \max\{ |V'| \mid V' \subseteq V \land \forall a, b \in V' : (a, b) \in E \}$$

$$\chi(G) = \min\{ k \mid \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^k V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \not\in E \}$$

$$\overline{\chi}(G) = \min\{ k \mid \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^k V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \in E \}$$
Introduction
Theorems
Chordal Graphs
Clique-Separators

Statements II

Theorem

Let $G = (V, E)$ be a graph with $n = |V|$. Then we have:

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1.$$ 

Proof:

$$\alpha(G) = \max \{ |V'| ; \ V' \subset V \land \forall a, b \in V' : (a, b) \notin E \}$$

$$\chi(G) = \min \{ k ; \ \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \notin E \}$$
Statements III

Theorem

Let $G = (V, E)$ be a graph with $n = |V|$. Then we have:

$$2\sqrt{n} \leq \chi(G) + \overline{\chi}(G) \leq n + 1$$

$$n \leq \chi(G) \cdot \overline{\chi}(G) \leq \left(\frac{n+1}{2}\right)^2.$$

Idea of proof:

$$\chi(G) = \min\{ k \mid \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i: 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \notin E \}$$

$$\overline{\chi}(G) = \min\{ k \mid \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^{k} V_i = V \land \forall i: 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \in E \}$$

Consider the two Coverings as a grid.
Statements III

\[2 \sqrt{n} \leq \chi(G) + \chi(G) \leq n + 1\]
\[n \leq \chi(G) \cdot \chi(G) \leq \left(\frac{n+1}{2}\right)^2.\]
Definition

A graph $G = (V, E)$ is called:

1. **χ**-perfect, iff for all node-induced subgraphs $H$ of $G$ holds: $\chi(H) = \omega(H)$.
2. **α**-perfect, iff for all node-induced subgraphs $H$ of $G$ holds: $\kappa(H) = \alpha(H)$.
3. perfect, if it is **χ**-perfect [and **α**-perfect].

\[
\begin{align*}
\alpha(G) &= \max \{ |V'| ; V' \subseteq V \land \forall a, b \in V' : (a, b) \notin E \} \\
\omega(G) &= \max \{ |V'| ; V' \subseteq V \land \forall a, b \in V' : (a, b) \in E \} \\
\chi(G) &= \min \{ k ; \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^k V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \notin E \} \\
\overline{\chi}(G) &= \min \{ k ; \exists V_1, V_2, \ldots, V_k : \bigcup_{i=1}^k V_i = V \land \forall i : 1 \leq i \leq k : \forall a, b \in V_i : (a, b) \in E \}
\end{align*}
\]
Definitions

Definition

A graph $G = (V, E)$ is called:

1. $\chi$-perfect, iff for all node-induced subgraphs $H$ of $G$ holds: $\chi(H) = \omega(H)$.
2. $\alpha$-perfect, iff for all node-induced subgraphs $H$ of $G$ holds: $\kappa(H) = \alpha(H)$.
3. perfect, if it is $\chi$-perfect [and $\alpha$-perfect].

Definition

A property $\mathcal{E}$ of a graph $G = (V, E)$ is called hereditary, iff the property holds for each node-induced subgraph of $G$. 

\[ \omega(G) = \overline{\omega}(G), \quad \alpha(G) = \overline{\alpha}(G) = \beta_0(G), \quad \kappa(G) = \overline{\chi}(G) \]
Examples ($\chi$-perfect)

- Planar graphs: no
- Intervall-graphs: yes
- Arc-graphs: no
- Permutation-graphs: yes
- Outerplanar graphs: no
- Maximal outerplanar graphs: yes
- Maximal planar graphs: no (following slide)
- Bipartite graphs: yes
- K-Trees: yes
- Complement of a bipartite graph: yes (following slides)
- Cycles of odd length $\geq 5$: no
- Linegraphs of bipartite graphs: yes (following slides)
Example Planar

\[ \omega(G) = \overline{\alpha}(G), \alpha(G) = \overline{\omega}(G) = \beta_0(G), \kappa(G) = \overline{\chi}(G) \]
Complement of a bipartite Graph

Lemma

The complement of a bipartite graph is \(\chi\)-perfect.

Proof:

- Note, that the class is hereditary.
- Show \(\chi(\overline{G}) = \omega(\overline{G})\).
- So we have to prove: \(\kappa(G) = \alpha(G)\).
- By the theorem of König we get:
  - Take a maximum matching \(M\) with \(|M| = a\).
  - Assume that \(b\) nodes are not covered by \(M\).
  - Then we have: \(\alpha(G) = a + b\) and
  - \(\kappa(G) = a + b\).
Lemma

Linegraphs of bipartite graphs are $\chi$-perfect.

Proof:

- Note, that the class is hereditary.
- Let $G$ bipartite graph and $H = L(G)$.
- Then we have by the construction of the linegraph:
  - $\omega(H) = \Delta(G)$ and
  - $\chi(H) = \chi'(G)$.
- Furthermore is already known: $\chi'(G) = \Delta(G)$.
- Thus we have: $\omega(H) = \Delta(G) = \chi'(G) = \chi(H)$. 

\[ \omega(G) = \overline{\omega}(G), \ \alpha(G) = \overline{\omega}(G) = \beta_0(G), \ \kappa(G) = \overline{\chi}(G) \]
Definition

A relation $\leq$ is called partial ordering, iff:

- Reflexive: $x \leq x$
- Transitive: $x \leq y \land y \leq z \implies x \leq z$
- Antisymmetric: $x \leq y \land y \leq x \implies x = y$

- Two elements are called comparable, if $x \leq y$ oder $y \leq x$.
- A set of pairwise comparable elements is called a chain.
- A set of pairwise not comparable elements is called an anti-chain.
- $y$ covers $x$ ($x \preceq y$), if $x \leq y$ and $x \leq a \leq y \implies a \in \{x, y\}$.
- This is called a PO-set
- The PO-set is denoted by $P_{\preceq}$.

\[ \omega(G) = \overline{\omega}(G), \alpha(G) = \overline{\alpha}(G) = \beta_0(G), \kappa(G) = \overline{\chi}(G) \]
Definition

A graph $G = (V, E)$ is called **comparability graph**, if there is a partial ordering $\leq$ on $V$, with:

$\{x, y\} \in E$ iff. $x$ and $y$ are comparable.

- Example: bipartite graphs.
- Comparability graphs are transitive orientable.
- Example: transitive orientation of a bipartite graph.
Lemma

Let $P\leq$ be a PO-set. The maximal length of a chain is equal to the minimal number of anti-chains in which $P\leq$ may be partitioned.

$\leq$ : Clear!

$\geq$ :

- $x$ minimal: $\forall a \in P_{\leq} : a \leq x \implies a = x$
- From this we may define a height function $h(x)$.
- Let $x = z_1 \leq z_1 \leq \ldots \leq z_{h_y} = y$ be the longest chain of length $h(y)$.
- The elements of the same height form an anti-chain.
- We have defined a partition of $h(y)$ anti-chains.

$$\omega(G) = \overline{\omega}(G), \alpha(G) = \overline{\omega}(G) = \beta_0(G), \kappa(G) = \overline{\chi}(G)$$
**Theorem**

*Comparability graphs are* $\chi$-*perfect.*

*Proof:* clear!

*Note:* $\chi(G) \leq \omega(G)$ holds.

**Lemma**

*Let* $P \leq$ *be a PO-set. The maximal length of a anti-chain is equal to the minimal number of chains in which* $P \leq$ *may be partitioned.*

**Definition**

*A topological ordering of* $G = (V, A)$ *is an ordering of the nodes* $\rho : V \mapsto \{1, 2, \ldots, n\}$ *with:*

$(u, v) \in A \implies \rho(u) < \rho(v).$

**Lemma**

*The colouring problem may be solved in linear time on comparability graphs by using a topological ordering.*
Statements

Theorem

*Interval graphs are χ-perfect.*

Theorem

*The complement of an interval graph is a comparability graph.*

Theorem

*For a graph G are the following statements equivalent:
  * G is an interval graph.
  * G contains no induced $C_4$ and $\overline{G}$ is a comparability graph.
  * The maximal cliques of G may be ordered such that, the cliques which have a common node, follow in the ordering each other.*
First Observations

**Theorem**

The disjoint union of \( \chi \)-perfect graphs is a \( \chi \)-perfect graph.

**Theorem**

The identification of two \( \chi \)-perfect graphs at a clique gives a \( \chi \)-perfect graph.

**Theorem**

A graph \( G \) is \( \chi \)-perfect, iff in all induced subgraphs exists an independent set, which hits all maximum-cliques: \( \forall H \subset G : \exists I : \omega(H - I) \leq \omega(H) - 1 \) and \( I \) is an independent set.
Theorem

A graph $G$ is $\chi$-perfect, iff in all induced subgraphs exists an independent set, which hits all maximum-cliques: $\forall H \subseteq G : \exists I : \omega(H - I) \leq \omega(H) - 1$.

Proof:

$\implies$:
- Because $\chi(G) = \omega(G)$ holds,
- will each colour-class hit all maximum-cliques.

$\impliedby$:
- We may show by induction over $|V(H)|$:

$$\chi(H) \leq \chi(H - I) + 1 \overset{\text{V.}}{\iff} \omega(H - I) + 1 \leq \omega(H).$$
Strong perfect Graphs

Definition

A graph $G = (V, E)$ is called strong perfect, iff for each node-induced subgraph exists an independent set, which hits all maximal cliques.

Theorem

A strong perfect graph is also perfect.

Theorem

The problems for $\chi(G), \alpha(G), \omega(G), \kappa(G)$ are on $\chi$-perfect graphs solvable in polynomial time.

Note: Proof uses the Ellipsoid Method.
Theorem

The following statements are equivalent for graphs $G = (V, E)$:

1. $G$ is $\chi$-perfect.
2. $G$ is $\alpha$-perfect
3. For all node-induced subgraphs $H = (V', E')$ of $G$ holds: $\alpha(H) \cdot \omega(H) \geq |V'|$.

Theorem

Perfect Graphs are closed under complement.
Statements II

**Lemma**

*If a node* $x$ *of a* $\chi$-*perfect graph* $G$ *is substituted by a* $\chi$-*perfect graph* $H$, *then we get a* $\chi$-*perfect graph* $G_H$.

**Proof:**

- Construct an independent set $I$, which hits all maximum cliques.
- Colour $G$ with $\chi(G)$ colours.
- Let $I_x$ be the set of nodes with the same colour as $x$.
- Let $I_H$ be an independent set in $H$, which hits all maximum-cliques in $H$.
- Let: $I = I_x \setminus \{x\} \cup I_H$
- Let $C$ be a maximum-clique in $G_H$.
  - If $C \cap V(H) = \emptyset$ holds, then is $C$ in $G - x$ and
  - because $\omega(G) \geq \chi(G)$ holds, we get $C \cap I_x \neq \emptyset$.
  - If $C \cap V(H) \neq \emptyset$, than contains $C$ a maximum-clique of $H$
  - and therefore hits $I_H$ also $C$. 
Lemma

If a node $x$ of a $\alpha$-perfect graph $G$ is substituted by an independent set $S$, then we get a $\alpha$-perfect graph $G_S$.

- It is sufficient to add just one node $y$ as a copy of $x$.
- We consider two cases:
  - $x$ is in an independent set $S$ of size $\alpha(G)$.
  - $x$ is not in an independent set $S$ of size $\alpha(G)$. 
Let $\mathcal{K}$ be a clique cover of $G$ with $|\mathcal{K}| = \kappa(G) = \alpha(G)$.

$x$ is in an independent set $S$ of size $\alpha(G)$.

- Thus $S \cup \{y\}$ is an independent set and $\alpha(G_{\{y\}}) = \alpha(G) + 1$ holds.
- Because $\mathcal{K} \cup \{y\}$ is a clique cover of $G_{\{y\}}$, we get:
  - $\kappa(G_{\{y\}}) \leq \kappa(G) + 1 = \alpha(G) + 1 = \alpha(G_{\{y\}}) \leq \kappa(G_{\{y\}})$. 

$\omega(G) = \omega(G) = \omega(G) = \omega(G) = \omega(G) = \omega(G) = \omega(G)$
Let $\mathcal{K}$ be a clique cover of $G$ with $|\mathcal{K}| = \kappa(G) = \alpha(G)$.

$x$ is not in an independent set $S$ of size $\alpha(G)$.

- Thus we have $\alpha(G_{\{y\}}) = \alpha(G)$.
- Because of $\kappa(G) = \alpha(G)$ each clique from $\mathcal{K}$ hits each maximum independent set.
- Therefore hits $K_x$ (the clique, which contains $x$) each maximum independent set precisely once.
- And $D = K_x \setminus \{x\}$ hits each maximum independent set precisely once.
- Thus we get: $\alpha(G[V \setminus D]) = \alpha(G) - 1$.
- By induction we get:
  \[ \kappa(G[V \setminus D]) = \alpha(G[V \setminus D]) = \alpha(G) - 1 = \alpha(G_{\{y\}}) - 1. \]
- Thus there is a clique cover of $G[V \setminus D]$ of size $\alpha(G_{\{y\}}) - 1$.
- Finally we get $\kappa(G_{\{y\}}) = \alpha(G_{\{y\}})$ (Covering: $D \cup \{y\}$).
Theorem (Lovász)

The complement of a perfect graph is perfect.

Proof (we will show that $\alpha$-perfect induces $\chi$-perfect):

- Let $G$ be a $\alpha$-perfect graph.
- We will use induction over $n = |V(G)|$.
- The statement holds clearly for $n \leq 3$. Let $n \geq 4$.
- For all induces real subgraphs of $G$ holds the statement.
- Thus we have to show $\chi(G) \leq \omega(G)$.
- If $G$ has an independent set $S$, which hists all maximum cliques,
  then $\omega(G \setminus S) = \omega(G) - 1$ holds.
- Thus we get: $\chi(G) \leq \chi(G \setminus S) + 1 = \omega(G \setminus S) + 1 \leq \omega(G)$.
- Therefore we assume in the following, that $G$ has not an independent set $S$, which hists all maximum cliques.
Proof

- $G$ has not an independent set $S$, which hists all maximum cliques.
- For each independent set $S$ holds: $G \setminus S$ contains a clique $C_S$, with $C_S \cap S = \emptyset$ and $|C_S| = \omega(G)$.
- Let $S$ be the set of independent sets in $G$.
- For $v_i \in V(G)$ let $h_i = |\{S \in S \mid v_i \in C_S\}|$.
- We replace each node $v_i \in V(G)$ by an independent set of size $h_i$.
- This new graph $H$ is also $\alpha$-perfect.
- Furthermore we get:

\[
|V(H)| = \sum_{v_i \in V(G)} h_i = \sum_{v_i \in V(G)} \sum_{S \in S} |v_i \cap C_S| = \sum_{S \in S} \sum_{v_i \in V(G)} |v_i \cap C_S| = \sum_{S \in S} |C_S| = \omega(G) \cdot |S|
\]
Proof

\begin{itemize}
\item By Construction of \( H \) we have \( \omega(H) \leq \omega(G) \).
\item Then it holds (note in the following: \( |T \cap C_S| \leq 1 \) and \( |S \cap C_S| = 0 \)):

\[
\alpha(H) = \max_{T \in S} \sum_{x_i \in T} h_i = \max_{T \in S} \sum_{S \in S} |T \cap C_S| \leq |S| - 1
\]

\item Furthermore we get:

\[
\kappa(H) \geq \frac{|V(H)|}{\omega(H)} = \frac{|V(H)|}{\omega(G)} = |S|.
\]

\item Thus we get the following contradiction:

\[
\kappa(H) \geq |S| > |S| - 1 \geq \alpha(H).
\]
\end{itemize}
Definition

A graph $G = (V, E)$ is called minimal imperfect, iff it is not perfect and each node induced real subgraph is perfect.

Strong Perfect Graph Theorem

A minimal imperfect graph is either an odd cycle of length $\geq 5$ or its complement.

Theorem

The Recognition of perfect graphs is in $\mathcal{P}$.
Definition
A graph $G = (V, E)$ is called minimal imperfect, iff it is not perfect and each node induced real subgraph is perfect.

Strong Perfect Graph Theorem
A minimal imperfect graph is either an odd cycle of length $\geq 5$ or its complement.

Theorem
*The Recognition of perfect graphs is in $\mathcal{P}$.***
Definition

A graph $G$ is called chordal, iff it induces no $C_k$ for $k \geq 4$.

Note: I.e. $G$ does not contain a $C_k$ as induced subgraph.
Note: are sometimes also called triangulated.
Examples:

- Intervall-graphs
- Maximal outer-planar graphs
- K-trees
Theorem

A graph $G$ is chordal, iff each inclusion minimal separator is a clique.

Proof (\(\implies\)):
- Let $S$ be a inclusion minimal separator is a clique.
- $S$ separates $H_1$ and $H_2$.
- All nodes from $S$ have neighbours in $H_1$ and $H_2$.
- Let $u, v$ be from $S$.
- There is shortest path $P_i$ from $u$ to $v$ in $H_i$.
- Thus three is a cycle given by $P_1$ and $P_2$.
- There is an edges $\{u, v\}$. 
Theorem

A graph $G$ is chordal, iff each inclusion minimal separator is a clique.

Proof ($\Longleftarrow$):

- Let $C$ be a cycle of length $\geq 4$.
- Let $u, v$ non-neighboured nodes in $C$.
- If $\{u, v\} \in E$, the statement holds.
- On the other side:
  - Let $S$ be a minimal separator for $u$ and $v$.
  - This separator is a clique.
  - This contains two other nodes from $C$.
  - These other nodes are connected.
Simplicial Nodes

Definition

A node is called simplicial, iff all its neighbours induce a complete subgraph.

Theorem

*Each Clique has a simplicial node and each chordal graph, who is not a clique, has two simplicial nodes, which are not connected.*

- Proof by induction. (Statement holds for $|V| \leq 3$.)
- Let $u, v$ be two non-neighboured nodes.
- Identify a minimal separator $S$ for $u, v$.
- $G - S$ splits into components $H_i$, with $i \geq 2$.
- $S$ is a clique.
- $H_i \cup S$ contains a simplicial node.
- This node is also simplicial node in $G$. 

![Diagram of simplicial nodes](image.png)
**Theorem**

*Chordal graphs and their complements are perfect.*

- **Proof (just using chordal graphs):**
  - By induction.
  - Let $G$ be no clique.
  - Then contains $G$ a separating clique $C$.
  - $G - C$ splits into components $H_i$, with $i \geq 2$.
  - $H_i \cup C$ are perfect.
  - Thus $G$ is perfect.

- **Proof (using the complement of chordal graphs):**
  - Identify clique in $G$, which hists all independent sets.
  - Choose simplicial node $s$, i.e. $C = \{s\} \cup \Gamma(s)$. 
Definition

Let $G = (V, E)$ be a graph with $|V| = n$. A total ordering $\rho : V \mapsto \{1, \ldots, n\}$ is called perfect node-elimination scheme, iff each node $v$ is a simplicial node in $G[\{u \in V \mid \rho(u) \geq \rho(v)\}]$. 
Chordal Graphs and PES

**Theorem**

*A graph is chordal, iff it has a PES.*

**Show:** \(\iff\).

- Let \(C\) be a cycle in \(G\).
- Let \(u\) be the first node in \(C\) under the ordering \(\rho\).
- Thus the neighbours of \(u\) are connected.
- Thus \(G\) is chordal.

**Show:** \(\implies\).

- Choose simplicial node \(v\) and let \(\rho(v) = 1\).
- Compute recursively more nodes of \(G - v\).
Theorem

Chordal graphs could be recognized in polynomial time.

Proof: determine a PES (on the next slides).

Theorem

Chordal graphs could be recognized in time $O(n^2 \cdot m)$.

Theorem

Chordal graphs could be recognized in time $O(n + m)$. 
Overview and Simple Algorithm

- Compute an ordering for $G$.
- Compute this ordering simply by using the node degrees.
- Show that this ordering is always a PES, if $G$ is chordal.
- We will get the following algorithm:
  - Compute ordering using the node degrees.
  - Test if this ordering is a PES.

Simple Algorithm:

- Compute the PES in a reverse fashion.
- Start with an arbitrary node $v_n$.
- Choose $v_{i-1}$ such that $v_{i-1}$ is connected to as many as possible nodes from $v_i, v_{i+1}, \ldots, v_n$.
- Show $v_1, v_2, \ldots, v_n$ is a PES.
A total ordering $\rho$ on $V$ is a PES, iff for all pairs of nodes $v_i, v_j$, which are connected by a path, for which for all inner nodes $u$ $\rho(u) < \min(\rho(v_i), \rho(v_j))$ holds, then follows that these nodes $v_i, v_j$ are connected by an edge.

Proof $\Rightarrow$ by contradiction.

Let $v_i, v_j$ be as above with $\{v_i, v_j\} \notin E$.

Let $P$ the shortest path from $v_i$ to $v_j$ and let $u$ be the leftmost node from $P$ in $\rho$.

The neighbours of $u$ on $P$ are connected by an edge.

Contradiction to the minimality of the path $P$.

Proof $\Leftarrow$ is simple.
Recognition

Theorem

*The simple algorithm computes for chordal graphs a PES.*

Claim

- Assume $\rho(u) < \rho(v) < \rho(w)$ holds, with
- $\{u, w\} \in E$ and $\{v, w\} \notin E$.
- Then there is a node $z$ with:
  - $\rho(v) < \rho(z)$, $\{u, z\} \notin E$ and $\{v, z\} \in E$.

Proof:

- Holds due to the chosen ordering.
- $v$ has at least as many neighbours as $u$. 
Recognition (Show, $\rho$ defines a PES)

- Assume that this does not hold:
- There are $v, w$ with $\{v, w\} \notin E$ and
- for all inner nodes $u$ on the path $P$ of $v, w$ holds:
  - $\rho(u) < \min(\rho(v), \rho(w))$.
- Choose $\rho(w)$ maximal and after that $\rho(v)$ maximal.
- Choose shortest path $P$ from $w$ to $v$.
- This path contains inner node $u$.

- There exists $z$ with: $\rho(v) < \rho(z)$, $\{u, z\} \notin E$ and $\{v, z\} \in E$.
- Therefore is $w$ with $z$ connected by a path.
- Because of the choosing of $v$ and $w$ holds $\{z, w\} \in E$.
- There is a cycle traversing $P$, $\{v, z\}$ and $\{z, w\}$.
- Choose the shortest path between $u$ and $v$.
- Thus we have a non chordal cycle containing $\geq 4$ nodes.

![Graph Diagram](attachment:image.png)
Recognition (Running Time)

- The test of the clique property may be more consuming.
- Test of the clique property may be done just by using data from the leftmost node of the clique.
- Therefore the edges are considered only once.
- Thus the recognition could be done in linear time.
Test PES Property

- The algorithm:
  - Start with an arbitrary node \( v_n \).
  - Choose \( v_{i-1} \) such that is connected with as many as possible nodes \( v_i, v_{i+1}, \ldots, v_n \).
  - Show \( v_1, v_2, \ldots, v_n \) is a PES.

- What is necessary to compute the ordering:
  - \( N_i = \{ v_j \in \Gamma(v_i) \mid j > i \} \)
  - \( R_i = |\{ v_j \in \Gamma(v_i) \mid j > i \}| \)

- What is necessary to do the following test:
  - Test \( N_i = \{ v_j \in \Gamma(v_i) \mid j > i \} \) induces a clique.
Compute $R_i$

- Let $B_0 = V$, $D = \emptyset$ and $l = n$.
- Let for $1 \leq i \leq n - 1$ be: $B_i = \emptyset$.
- Let for all $v \in V$ be: $R(v) = 0$.
- While $B_i \neq \emptyset$ for an $i$ do for the minimal $i$:
  1. Choose $x \in B_i$.
  2. Let $v_l = x$ and $D = D \cup \{x\}$.
  3. Let $\rho(x) = l$.
  4. Let $l = l - 1$.
  5. Let $B_i = B_i \setminus \{x\}$.
  6. For all $v \in \Gamma(x) \setminus D$ do:
     - Let $B_{R(v)} = B_{R(v)} \setminus \{v\}$.
     - Let $R(v) = R(v) + 1$.
     - Let $B_{R(v)} = B_{R(v)} \cup \{v\}$.
- Task was to compute: $R_i = |\{v_j \in \Gamma(v_i) \mid j > i\}|$.
- If a node $x = v_i$ as chosen, then $R(x)$ is not changed any more.
- Then: $R_i = R(x) = |\{v_j \in \Gamma(v_i) \mid j > i\}|$ holds.
Test $N_i$:

- Getting the idea:
- Check the nodes from left to right.
- For some node $v_i$ do not at once the test of $N_i$ to be a clique.
- Instead delay the test on for each neighbour $v_j$ of $v_i$.
- But prepare, the set of neighbours which $v_j$ should have.
- Store this in tables $T[v_j]$. 

Test $N_i = \{ v_j \in \Gamma(v_i) \mid j > i \}$ induces a clique.
Test $N_i$

- For all $v_j \in V$ do $T[v_j] = \emptyset$.
- For all $i$ from 1 to $n$ do:
  1. Consider the node $v_i$.
  2. Let $N = \{v_j \in \Gamma(v_i) \mid j > i\}$.
  3. If $T[v_i] \not\subset N$ holds, the stop with message “no PES”.
  4. If $N \neq \emptyset$ the do:
     - Let $v_l$ be the first (left) node of $N$.
     - Let $T[v_l] = T[v_l] \cup (N \setminus \{v_l\})$.

- Output: the ordering is a PES.
Teste $N_i$

- For all $v_j \in V$ do $T[v_j] = \emptyset$.
- For all $v_j \in V$ do $S[v_j] = 0$.
- For all $i$ from 1 to $n$ do:
  1. Consider the node $v_i$.
  2. Let $N = \{v_j \in \Gamma(v_i) \mid j > i\}$.
  3. For all $v \in N$ do $S[v] = 1$.
  4. For all $u \in T[v_i]$ do
     - If $S[u] = 0$ holds, then stop with message "No PES".
  5. For all $v_j \in V$ do $S[v_j] = 0$.
  6. If $N \neq \emptyset$ the do:
     - Let $v_l$ be the first (left) node of $N$.
     - Let $T[v_l] = T[v_l] \cup (N \setminus \{v_l\})$.

- Output: the ordering is a PES.
Algorithms for Graph Problems

- The standard graph problems could be solved in polynomial time.
- Idea: Greedy algorithm using the PES ordering.
- Note: Chordal Graphs have at most $|V|$ maximum cliques.
- Thus only the simplicial nodes have to be considered for the clique problem.
- For the colouring problem use greedy on the reverse PES ordering.
- Similar ideas work for the other problems.
Lemma

Let $T = \{ T_i \mid 1 \leq i \leq n \}$ be a family of subtrees of some base tree and each pair of trees from $T$ intersect each other.

- Then they have a common node.
- I.e. $\cap_{1 \leq i \leq n} T_i \neq \emptyset$

- The union of all subtrees $T_i$ induces a subtree $T'$.

- A leave of $T'$ which is not in all $T_i$ could be deleted without changing the intersections of the $T_i$.

- By repeating we find a node which is common to all $T_i$. 
Theorem

Let $G = (\{v_1, v_2, \ldots, v_n\}, E)$ be a Graph. The following statements are equivalent:

1. $G$ is chordal.

2. $G$ is the intersection graph of a family of subtrees.

3. There is a tree $B$ on the set of maximal cliques of $G$ such that for a pair of cliques $C', C''$ holds:
   - The clique $C' \cap C''$ is part of each maximal clique, which
   - is on the path from $C'$ to $C''$ in $B$. 
Proof 1

Show: $G$ is chordal $\implies G$ is intersection graph of a family of subtrees.

- Proof by Induction.
- $n = 1$ clear.
- Induction step: $n - 1 \to n$
  - Nodes $v_1, v_2, \ldots, v_n$ and $s = v_n$ a simplicial node.
  - Let $(B_{n-1}, \{T_1, T_2, \ldots, T_{n-1}\})$ intersection graph representation for $v_1, v_2, \ldots, v_{n-1}$
  - $\Gamma(s) \setminus \{s\}$ is a clique.
  - There is a common node $a$ in $\cap_{v \in \Gamma(s)} V(T_v)$.
  - Add to $B_{n-1}$ a new leaf $b$ for $a$.
  - And generate a new subtree, which consists of $b$.
  - And enlarge each subtree from $\Gamma(s)$ with $b$. 

![Diagram](attachment:image.png)
Proof II

Show: $G$ is intersection graph of a family of subtrees $\implies G$ is chordal.

- Let $C = (v_0, v_1, \ldots, v_{k-1})$ cycle of length $k \geq 4$.
- Let $T_0, T_1, \ldots, T_{k-1}$ be the corresponding trees.
- These subtrees will form a cycle in the base tree.

The other part of the proof follow in a similar way.
Simple Statements

**Lemma**

Let $G$ be a chordal graph. A node $v$ of $G$ is simplicial, iff it is contained in only one maximal clique.

**Lemma**

Let $G$ be a chordal graph and $C$ a clique in $G$. Then exitst a PES, which enumerates the nodes from $C$ last.
Theorem

*Any chordal graph with $n$ nodes has a $(\omega(G), 1/2)$-separator, which is a clique.*

- Note: A separator of size $\omega(G)$ must not be a Clique.
- Note: A clique-separator must not be minimal separating.
Proof

- Algorithm to compute a chordal separator:
  - \( C := \emptyset \)
  - As long a component \( A \) in \( G[V \setminus C] \) exists with \( |A| > n/2 \) do:
    - \( C := \{ c \in C \mid \Gamma(c) \cap A \neq \emptyset \} \)
    - Choose \( a \in A \) with: \( C \subset \Gamma(a) \)
    - \( C := C \cup \{ a \} \)

- There is at most one component \( A \) with: \( |A| > n/2 \).
- At each round, one node will be removed from that component.
- There are at most \( \lceil n/2 \rceil \) iterations.
- Show \( \exists a : C \subset \Gamma(a) \).
- Note:
  - At the start \( a \) is freely chosen.
  - \( C \) is always minimal separating for \( A \) and \( V \setminus (C \cup A) \).
  - All nodes from \( C \) have neighbours in \( A \).
Proof

- $C := \emptyset$
- As long a component $A$ in $G[V \setminus C]$ exists with $|A| > n/2$ do:
  - $C := \{c \in C \mid \Gamma(c) \cap A \neq \emptyset\}$
  - Choose $a \in A$ with: $C \subseteq \Gamma(a)$
  - $C := C \cup \{a\}$

- Show $\exists a : C \subseteq \Gamma(a)$.
- Let $\rho = (a_1, a_2, \ldots, a_{|A|}, c_1, c_2, \ldots, c_{|C|})$ be a PES for $G[A \cup C]$.
- Consider now $a = a_{|A|}$:
  - Each node from $C$ is connected by a path with $a$.
  - Thus each node from $C$ is directly connected with $a$.
  - Furthermore $\{a\} \cup C$ is a clique.
  - The computation could be done in time $O(n \cdot m)$.
  - Using an other algorithm a linear running-time is possible.
Introduction

Definition (Clique-Separator)

Clique $C$ in $G = (V, E)$ is called Clique-Separator, iff $G[V \setminus C]$ is disconnected.

Definition (Clique-Separator-Tree)

A clique-separator-tree $T$ is defined recursively:

- If $G = (V, E)$ contains no clique-separator:
  - $T$ consists only of the node $w$.
  - To $w$ is the set $V$ associated.

- If $G = (V, E)$ has a clique-separator $C$:
  - Let $A_1, A_2, \ldots, A_l$ be the components of $G[V \setminus C]$.
  - $T$ consists of the root $w$ and subtrees $T_1, T_2, \ldots, T_l$.
  - To a tree $T_i$ is the graph $G[A_i \cup C]$ associated.
  - To $w$ is the set $C$ associated.

- The leaves of the clique-separator-tree are called atoms.
Basics, Motivation

- A clique-separator-tree has at most $\binom{n}{2} - m$ atoms (Exercise).
- Each chordal graph has a clique-separator-tree, where all atoms are cliques.
- If the atoms are “simple”, then many problems become easy solvable.
- We will now introduce the MES, which is similar to PES.
Reminder

**Definition**

A node is called simplicial, iff all its neighbours are connected by an edge.

**Theorem**

*Each Clique has a simplicial node and each chordal graph, who is not a clique, has two simplicial nodes, which are not connected.*

**Definition**

Let $G = (V, E)$ be a graph with $|V| = n$. A total ordering $\rho : V \mapsto \{1, \ldots, n\}$ is called perfect node-elimination scheme, iff each node $v$ is a simplicial node in $G[\{u \in V \mid \rho(u) \geq \rho(v)\}]$.

**Theorem**

*A graph is chordal, iff it has a PES.*
**Definition (Fill-in)**

Let $G = (V, E)$ be a graph with $|V| = n$ and $\rho : V \mapsto \{1, \ldots, n\}$ an ordering of the nodes. The fill-in for $\rho$ is:

$$F_\rho := \left\{ \{v, w\} : v \neq w \land \{v, w\} \not\in E \land \text{there is a path } v = x_1x_2 \ldots x_l = w \text{ with: } \rho(x_i) < \min(\rho(v), \rho(w)) \forall i = 2, 3, \ldots, l - 1 \right\}$$

- Notation: $G_\rho = (V, E \cup F_\rho)$
- Any ordering $\rho$ is a PES for $G_\rho$.
- The fill-in for $\rho$ in $G_\rho$ is the empty set.
- Thus $G_\rho$ is chordal.
- $\Gamma_{\rho,F}(v) := \{w \mid \{v, w\} \in E \cup F \land \rho(w) > \rho(v)\}$
- $m_F(v)$ the node $u$ with: $\rho(u) = \min\{\rho(w) \mid w \in \Gamma_{\rho,F}(v)\}$. 
Lemma

Let $G = (V, E)$ be graph and $\rho$ a ordering.
Then is the fill-in $F_\rho$ the smallest set $F$, such that for all $v \in V$ holds:

$$\Gamma_{\rho,F}(v) \subseteq \Gamma_{\rho,F}(m_F(v)) \cup m_F(v)$$

Proof:

- Show that for $F = F_\rho$ the above equation holds.
  - Let $v$ be a node.
  - Let $w \in \Gamma_{\rho,F_\rho}(v)$ and $w \neq m_F(v) = x$.
  - Then is $m_F(v), v, w$ a path in $G_\rho$ with $\rho(v) < \min(\rho(m_F(v)), \rho(w))$.
  - Thus $\{w, m_F(v)\} \in E \cup F_\rho$ holds.
  - And $w \in \Gamma_{\rho,F_\rho}(m_F(v))$ holds.
Proof (Let $F$ be as defined, show that $F_\rho \subseteq F$ holds)

- Show by induction over $i$:
  \[ \forall \{v, w\} \in F_\rho \text{ with } \rho(v) \leq i : \{v, w\} \in F \]
- Assume the above holds for $i \leq i_0$.
- Let $\{v, w\} \in F_\rho$ with $\rho(v) = i_0 + 1 \leq \rho(w)$.
- Thus there is a path $v = x_1 x_2 \ldots x_k = w$ in $G_\rho = (V, E \cup F_\rho)$ with:
  - $k \geq 3$ and $\rho(x_j) < \min(\rho(v), \rho(w))$ for $j = 2, 3, \ldots k - 1$.
- Let $k$ be minimal.
- If $k > 3$ holds, let $l \geq 2$ be with $\rho(x_l) \geq \rho(x_j)$ for $j = 2, 3, \ldots k - 1$.
- Then is $v = x_1, x_2, \ldots, x_l$ a path in $G_\rho$ with $\rho(x_j) < \min(\rho(v), \rho(w))$ for $j = 2, 3, \ldots l - 1$.
- Thus $\{v, x_l\} \in F_\rho$ holds.
- This is a contradiction to the minimality of the path.

![Graph Diagram](image)
Proof (Let $F$ be a set satisfying the above equation, show that $F_\rho \subseteq F$ holds)

- Let $k = 3$ and $u = x_2$ with: $v, w \in \Gamma_{\rho,F_\rho}(u)$.
- Choose $u$ such that $\rho(u)$ is maximal.
- By induction and $\rho(u) < \rho(v)$ does $v, w \in \Gamma_{\rho,F}(u)$ hold.
- If $v \neq m_F(u)$ then we would get $v, w \in \Gamma_{\rho,F}(m_F(u))$.
- But this is a contradiction to the maximality of $\rho(u)$.
- Thus we have $v = m_F(u)$.
- But then is $w \in \Gamma_{\rho,F}(m_F(u))$.
- And also $\{v, w\} = \{m_F(u), w\} \in F$.
- Thus we get by induction: $F_\rho \subseteq F$. 

![Diagram](image-url)
Lemma

For a graph $G$ and a ordering $\rho$ is the fill-in computable in time $O(n + m + |F_\rho|)$.

Algorithm $Fill\_In(G, \rho)$

- For all $v \in V$ do:
  - $A(v) := \Gamma_{\rho, \emptyset}(v) = \{w \in \Gamma(V) \mid \rho(w) > \rho(v)\}$
- For $i := 1$ bis $n - 1$ do:
  - $v := \rho^{-1}(i)$
  - $m(v) := \rho^{-1}(\min\{\rho(u) \mid u \in A(v)\})$
  - $A(m(v)) := A(m(v)) \cup \{w \in A(v) \mid w \neq m(v)\}$
- $F_\rho = \emptyset$
- For all $v \in V$ and $w \in A(v) \setminus \Gamma(v)$ do:
  - $F_\rho = F_\rho \cup \{v, w\}$
Definition

An ordering \( \rho \) for \( G = (V, E) \) is called minimal elimination schema (MES), iff the Fill-in \( F_\rho \) is minimal, i.e. \( \nexists \rho' : F_{\rho'} \subset F_\rho \).

- Aim: clique-separator for \( G \) should also be clique-separator for \( G_\rho \), if \( \rho \) is a MES.
- Note: to find the smallest MES is in NPC.
- But here we only need a MES.
- This is possible in polynomial time:
  - Lexicographical breath-first-search
  - Comparing sets by their lexicographical order:
  - Thus \( \{2, 5\} < \{2, 4, 5\} \)
  - And \( \emptyset < \{2\} \)
Algorithm

- For all $v \in V$ do:
  - $pr(v) := \emptyset$
  - $\rho(v) := 0$

- For $i := n$ down to 1 do:
  - Choose node $v$ with $pr(v)$ maximal and $\rho(v) = 0$
  - $\rho(v) := i$
  - For all $w$ with $\rho(w) = 0$ do
    - If there is a path $v = v_1, v_2, \ldots, v_k = w$ with:
      - $\rho(v_i) = 0$ and $pr(v_j) < pr(v_w)$
      - for $j = 2, 3, \ldots, k - 1$, do:
        - $pr(w) := pr(w) \cup \{i\}$

Proof of correctness is complicated.

Running-time $O(n(m + n))$
Theorem

Let ρ be a MES for $G = (V, E)$. Then a clique-separator for $G$ is also a clique-separator for $G_ρ$.

- Let $V_1, \ldots, V_k$ be the node sets of the components from $G[V \setminus C]$.
- Delete from $F_ρ$ all edges, which connect two components.
- Call this new edge set $F, F \subset F_ρ$.
- Show: $G' = (V, E \cup F)$ is chordal.
  - Let $K$ be a cycle in $G'$ of length $\geq 4$.
  - If $K \subset G[V_i \cup C]$, then has $K$ a chord in $F_ρ$, because $G_ρ$ is chordal.
  - This chord is in $E \cup F$.
  - If $K$ goes through different $V_i$, then has $K$ two nodes in $C$, which are not connected in $C$.
  - Thus $K$ has a chord in $G'$. 
### Statements

**Theorem**

Let $\rho$ be a MES for $G = (V, E)$. Then a clique-separator for $G$ is also a clique-separator for $G_\rho$.

- Let $V_1, \ldots, V_k$ be the node sets of the components from $G[V \setminus C]$.
- Delete from $F_\rho$ all edges, which connects two components.
- Call this new edge set $F$, $F \subseteq F_\rho$.
- Shown on the last slide: $G' = (V, E \cup F)$ is chordal
- Thus $G'$ is chordal and has PES $\rho'$ with $F_{\rho'} = F$.
- $\rho$ is a MES, thus: $F_{\rho'} = F_\rho = F$.
- This ends the proof.
Clique-Separator-Tree Algorithm

\[ \rho := \text{LexBFS}(G) \]
\[ F_{\rho} := \text{Fill}_n(G, \rho) \]

For all \( v \in V \) do:
- \( C(v) := \emptyset \)
- For all \( w \in V \) do:
  - If \( \rho(w) > \rho(v) \) and \( \{v, w\} \in E \cup F_{\rho} \) holds, then do:
    - \( C(v) := C(v) \cup \{w\} \)

\( k := 1 \)

For all \( i := 1 \) bis \( n - 1 \) do:
- \( v := \rho^{-1}(i) \)
- Let \( A \) be a component in \( G[V \setminus C(v)] \) which contains \( v \).
- Let \( B = V \setminus (A \cup C(v)) \)
- If \( B \neq \emptyset \) and \( C(v) \) is a clique:
  - \( \text{Atoms}(k) := A \)
  - \( k := k + 1 \)
  - \( G := G[B \cup C(v)] \)
- \( \text{Atoms}(k) := V(G) \)
Correctness

Theorem

*If* $G$ *has a clique-separator. Then is this separator $C(v)$ for some node* $v$.

- Let $\rho$ a MES as computed by the above slides.
- Let $C$ be a inclusion minimal clique-separator.
- Let $A, B$ be two components from $G[V \setminus C]$.
- Thus each node from $C$ has a neighbour in $A$ and $B$.
- Let $x, y$ be nodes with the largest $\rho$ values in $A$ and $B$.
- Show now: there is no node $z \in C$ with: $\rho(z) \leq \min \{\rho(x), \rho(y)\}$.
  - By contradiction
  - on the next slide.
Correctness (intermediate step)

Assume: There is a node $z \in C$ with: $\rho(z) \leq \min\{\rho(x), \rho(y)\}$.

- Let $x = x_1, x_2, \ldots, x_{j-1}, x_j = z$ be the shortest path in $G_\rho$ with $x_1 x_2 \ldots x_{j-1} \in A$.
- If there is an $i$ with $i \leq j - 1$ and $\rho(x_i) \leq \rho(x_{j-1})$,
  then choose such $i$ maximal.
- Thus we have $i \geq 2$ (Note: $\rho(z) \leq \min\{\rho(x), \rho(y)\}$)
- And $\{x_{i-1}, x_{i+1}\} \in F_\rho$ holds, because of $\rho(x_i) \leq \min\{\rho(x_{i-1}), \rho(x_{i+1})\}$ and the definition of Fill-In
- This is a contradiction to the minimality of the path.
Correctness (intermediate step)

Assume: There is a node \( z \in C \) with: \( \rho(z) \leq \min \{ \rho(x), \rho(y) \} \).

- Thus there is a path \( x = x_1x_2 \ldots x_{j-1}x_j = z \) in \( G_\rho \) with \( \rho(x_i) > \rho(x_{i+1}) \) for \( i = 1, 2, \ldots, j - 1 \).
- Thus there a path \( y = y_1y_2 \ldots y_{l-1}y_l = z \) in \( G_\rho \) with \( \rho(y_i) > \rho(y_{i+1}) \) for \( i = 1, 2, \ldots, l - 1 \).
- Thus \( \{x, y\} \in F_\rho \) holds, which is a contradiction.
Correctness (Continuation)

- W.l.o.g. let now be $\rho(x) < \rho(y)$.
- Then holds: $\max\{\rho(v) \mid v \in A\} = \rho(x) < \rho(z)$ for all $z \in C$.
- Show now $C(x) = C$
- I.e. show: $\forall z \in C : \{x, z\} \in E \cup F_{\rho}$.
- Let $x = x_1x_2 \ldots x_{j-1}x_j = z$ be the shortest path in $G_{\rho}$ with $x_1, x_2, \ldots, x_{j-1} \in A$.
- If $j \geq 3$ holds, then we have $\rho(x_i) > \rho(x_{i+1})$ for $i = 1, 2, \ldots, j-1$.
- This is contradiction to $\rho(z) > \rho(x)$.
- Thus $j = 2$ and $\{x, z\} \in E \cup F_{\rho}$.
Theorems

The above algorithm has running-time $O(n(n + m))$ for computing the clique-separator-tree.

By using the clique-separator-tree are the following problems are reduced to the atoms:

- Clique-Problem
- Independent-Set Problem
- Colouring-Problem
Clique-Separable

Definition

A graph $G = (V, E)$ is of type $T_1$, iff:

- $V$ could be partitioned in $V_1, V_2$.
- $G[V_1]$ is a bipartite graph.
- $G[V_2]$ is a clique.
- Between $V_1$ and $V_2$ exist all possible edges.

Definition

A graph $G = (V, E)$ is of type $T_2$, iff it is complete $k$-partite.
Clique-Separable

**Definition**

A graph $G = (V, E)$ is clique-separable, iff all Atoms are of Type $T_1$ or $T_2$.

**Theorem**

*Clique-separable graphs could be recognized in time $O(n^4)$. The Clique-Problem, Independent-Set Problem and Colouring-Problem are solvable in polynomial time on clique-separable graphs.*
Questions

- What is a perfect graph?
- Which graph classes are perfect?
- How hard is the recognition of perfect graphs?
- How hard is the colouring on perfect graphs?
- What is a minimal imperfect graph?
- Which graphs are minimal imperfect?
- What is a chordal graph?
- What is known about chordal graph?
- Why are chordal graphs not perfect?
Questions

- How hard is the recognition of chordal graphs?
- What is a PES?
- Which problems are easy on chordal graphs?
- Give an alternative representation for chordal graphs?
- What are comparability graphs?
- What is known about comparability graphs and interval graphs?
- What is the idea of the proof to show that perfect graphs are closes under complement?
Legend

n : Not of relevance

z : implicitly used basics

i : idea of proof or algorithm

s : structure of proof or algorithm

w : Full knowledge