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Definition of a Broadcasts and Accumulation

**Definition of Broadcast:**

Given are $G = (V, E)$ and $v \in V$.

- $v$ has information $I(v)$ and
- no node from $V \setminus \{v\}$ knows $I(v)$.
- Each node of $V \setminus \{v\}$ has to receive information $I(v)$.

**Definition of Accumulation:**

Given are $G = (V, E)$ and $v \in V$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\bigcup_{w \in V} I(w)$. 
Definition of a Gossip

**Definition of Accumulation:**

Given are $G = (V, E)$ and $v \in V$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Node $v$ should receive the information $\cup_{w \in V} I(w)$.

**Definition (Gossip):**

Given is $G = (V, E)$.

- Each node of $w \in V$ has information $I(w)$ and
- no node from $V \setminus \{w\}$ knows $I(w)$.
- Each node of $v \in V$ should receive the information $\cup_{w \in V} I(w)$. 
Types of Communication

- **Telegraph-Mode**: Communication is directed.
  - Is also called one-way communication.

- **Telephone-Mode**: Information is exchanged.
  - Is also called two-way communication.

- Communication only between neighbours.
- Communication is done in rounds.
- In each round the active edges are a matching.
- Each round uses one time-unit.
Types of Communication

- In the broadcast-problem the information of one node is transferred to all others.
- The accumulation-problem is a “inverse” broadcast.
- A gossip distributes the sum of all informations to all nodes.
- In each round the communication is done by a matching.
- The communication on an edge may be one-way or two-way, depending on the mode.
- The size of send date is ignored.
Definition

- By $\text{comm}(A)$ we denote the complexity (number of rounds) of a communication-algorithm.
- $r(G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the gossip-problem on } G\}$
- $r_2(G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the gossip-problem on } G\}$
- $b(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the broadcast-problem on } G \text{ and } v\}$
- $b_2(v, G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the broadcast-problem on } G \text{ and } v\}$
- $a(v, G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the accumulations-problem on } G \text{ and } v\}$
- $a_2(v, G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the accumulations-problem on } G \text{ and } v\}$
Definition

- $b(G) = \max\{b(v, G) \mid v \in V\}$
- $b_2(G) = \max\{b_2(v, G) \mid v \in V\}$
- $a(G) = \max\{a(v, G) \mid v \in V\}$
- $a_2(G) = \max\{a_2(v, G) \mid v \in V\}$
- $\min b(G) = \min\{b(v, G) \mid v \in V\}$
- $\min a(G) = \min\{a(v, G) \mid v \in V\}$
First Results

For each graph $G$ and $v \in V$ we have:

- $a_2(v, G) = b_2(v, G)$
- $a(v, G) = b(v, G)$
- $a(G) = b(G)$
- $\text{mina}(G) = \text{minb}(G)$
- $b(v, G) = b_2(v, G)$
- $b(G) = b_2(G)$

Note: reverse broadcast is accumulation.

There exists a graph $G$ with: $r(G) = 2 \cdot r_2(G)$.

Note: 2-clique or cycle of length four.

The following holds: $\text{minb}(G) \leq b(G) \leq r_2(G) \leq r(G) \leq 2 \cdot r_2(G)$.

The inequalities result from the definitions.

$\text{minb}(L(n)) = \lceil n/2 \rceil$

Optimal broadcast on a line start in the center of the line.

$b(L(n)) = n - 1$

A message from the left has to traverse all edges.
First Results II

**Lemma:**

For each graph $G$ with $|V| \geq 2$ we have:

- $b(G) \leq r(G) \leq 2 \cdot \min b(G)$
- $b(G) \leq r_2(G) \leq 2 \cdot \min b(G) - 1$

**Proof:** Consider the following steps.

- Let $v \in V$ with $b(v, G) = \min b(G) = \min a(G) = z$.
- Let $A = E_1, E_2, \cdots E_z$ be the corresponding one-way broadcast-algorithm.
- Let $B = F_1, F_2, \cdots F_z$ be the corresponding one-way accumulation-algorithm.
- Then is $F_1, F_2, \cdots F_z, E_1, E_2, \cdots E_z$ one-way gossip-algorithm.
- Note: in the two-way case holds: $F_z = E_1$.
- Note: For $L(2 \cdot n)$ we have equality.
Lemma:
For each even \( n \) with \( n \geq 8 \) exists a Graph \( G \) with \( n \) nodes and
\[
b(G) = r(G)
\]

Proof (for \( n = 8 \)):

Both broadcasts together are a gossip-algorithm.
First Results IV

- \( \text{rad}(G) \leq \text{minb}(G) \).
- \( \text{rad}(G) \leq \text{diam}(G) \leq b(G) \).
- Let \( G = (V, E) \) and \( H = (V, F) \) with \( F \subset E \). Then we have:
  - \( b(G) \leq b(H) \).
  - \( \text{minb}(G) \leq \text{minb}(H) \).
  - \( r(G) \leq r(H) \).
  - \( r_2(G) \leq r_2(H) \).
- \( \text{minb}(G) \leq (\deg(G) - 1) \cdot \text{rad}(G) + 1 \).
- \( b(G) \leq (\deg(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \deg(G) \cdot \text{rad}(G) \).
- \( r(G) \leq 2(\deg(G) - 1) \cdot \text{rad}(G) + 2 \)
- \( r_2(G) \leq 2(\deg(G) - 1) \cdot \text{rad}(G) + 1 \)
Lower Bound

Lemma

Let $G = (V, E)$ be a graph with $n$ nodes. Then we have:

- $b(G) \geq \min b(G) \geq \lceil \log n \rceil$

Proof:

- Let $A(t)$ be the number of informed nodes after $t$ rounds.
- $A(0) = 1$
- $A(t + 1) \leq 2 \cdot A(t)$
- $A(t) \leq 2^t$
- At the end $2^t \geq n$ must hold.
Optimal Broadcast-Tree

Each informed node has to send in each round the information to a non-informed node:

- A tree $T_i$ is a broadcast-tree, iff
  - the root of $T_i$ has $i$ successors $v_0, v_1, \ldots, v_{i-1}$ and
  - $v_j$ is the root of a $T_j$. 

Lemma

We have:

- \( \min b(K(n)) = b(K(n)) = \lceil \log n \rceil \) and
- \( \min b(HQ(m)) = b(HQ(m)) = m \).

Proof \((K(n))\):

\[
\text{for } t = 1 \text{ to } \lceil \log n \rceil \text{ do }
\text{for all } i \in \{0, 1, \cdots , 2^{t-1} − 1\} \text{ do in parallel}
\text{if } i + 2^{t-1} \leq n \text{ then}
\text{ } i \text{ sends to } i + 2^{t-1}
\]

Proof \((HQ(m))\):

\[
\text{for } i = 1 \text{ to } m \text{ do }
\text{for all } a_1, a_2, \cdots , a_{i-1} \in \{0, 1\} \text{ do in parallel}
\text{ } a_1a_2\cdots a_{i-1}00\cdots 0 \text{ sends to } a_1a_2\cdots a_{i-1}10\cdots 0
\]
Lemma

For all \( k, m \geq 2 \) we have: \( \min_b(T_k(m)) = k \cdot m \).

Idea of proof:

- \( b(\varepsilon, T_k(m)) = k \cdot m \).
- \( b(\varepsilon, T_k(m)) \leq b(\nu, T_k(m)) \).
- Note that \( \nu \) has to inform \( \varepsilon \)
- and \( \varepsilon \) has to inform the other successors.
Complexity

**Definition:**

The special Broadcast-Problem is:
- Given: \( G = (V, E), \ v \in V \) and \( k \in \mathbb{N} \).
- Question: Does \( b(v, G) \leq k \) hold?

**Definition:**

The Broadcast-Problem is:
- Given: \( G = (V, E) \) and \( k \in \mathbb{N} \).
- Question: Does \( b(G) \leq k \) hold?
Complexity

**Theorem:**

The special Broadcast-Problem on trees is in $\mathcal{P}$.

- The algorithm computes recursively the broadcast-time from a node (which we consider as root) in its subtree.
- For the leafs is this time 0.
- When all broadcast-times are computed for all successors of the root, we sort these times.
- After this we may compute the order of subtrees of the root in which we forward the information from the root.
- Example: 5 subtrees have broadcast-times 10, 10, 9, 9, 7. Then we inform these subtrees in the same order. The total broadcast-time from the root is $\max(10 + 1, 10 + 2, 9 + 3, 9 + 4, 7 + 5) = 13$.

**Theorem:**

The Broadcast-Problem on trees is in $\mathcal{P}$.
Complexity

Theorem:
The special Broadcast-Problem is in \( NP \).

Proof: simple exercise.

- IF a message from node \( v \) has to be send to node \( w \) and the remaining time is the same as the distance between \( v \) and \( w \), then we call this message critical.

- I.e. the messages has to be forwarded towards \( w \) without any delay.

- Is the shortest path between \( v \) and \( w \) unique, then we know precisely the way (times and places) the messages has to traverse towards \( w \).

- If there exists an other node \( w' \) with: \( \text{dist}(v, w) = \text{dist}(v, w') + 1 \) and the shortest path towards \( w' \) splits from the path from \( v \) to \( w \), then is the message also critical on this path.
Idea of the Proof

Broadcast from $a_0$ in 9 rounds:

Thus each node $a_i, b_i$ has to be informed in round $i$. 
Idea of the Proof (Part A)

Broadcast from $a_0$ in 9 rounds:

May be extended to any number of “paths”.
Idea for the Variables

Consider the following situation:

- There are unique shortest paths from $v$ to $w$, $w'$, $w''$, which share the same splitting node.
- Assume that $\text{dist}(v, w) - 2 = \text{dist}(v, w') = \text{dist}(v, w'')$ holds and that the message on the path from $v$ towards $w$ is critical.
- Then will be one of the other paths (i.e. from $v$ to $w'$) critical.
- The other path (i.e. from $v$ to $w''$) is not critical:
  - We may delay the message on that path one time or
  - we may inform an additional node in the last step. informieren.

- We have now the idea for the “variable”: one path from $v$ to $w'$ is critical or the other path from $v$ to $w''$ is critical.
Idea of the Proof (Part B)

Broadcast from $a_0$ in 9 rounds:

Thus we have a “Variable”.
3-SAT

Definition

A boolean formula $\mathcal{F}$ is in 3-CNF (EXACT-3-CNF):

$$\mathcal{F}(x_1, x_2, \ldots, x_r) = \bigwedge_{i=1}^{m} c_i$$

(clauses) $c_i = (l_i^1 \lor l_i^2 \lor l_i^3)$ $\forall 1 \leq i \leq m$

(literals) $l_i^j = \begin{cases} \neg x_k & \text{oder} \\ x_k & \text{für ein } k : 1 \leq k \leq r \end{cases}$ $\forall 1 \leq i \leq m$ $\forall 1 \leq j \leq 3$

An assignment is a function $W : \{x_1, x_2, \ldots, x_r\} \mapsto \{0, 1\}$. It is NP-complete to test, if there is an assignment which satisfies $F$. 
Thus we have many “variables”.

Idea of the Proof (Part C)
The last Step

- So far we are able to construct any number of variables.
- But the clauses are still missing.
- In 3-SAT a clause has to be satisfied by some variable.
- We may represent a clause by a node, which may only be informed the variables (paths), which are not critical (which represent the boolean value “true”). We have now the full idea for the reduction to 3-SAT.
Thus we have a “clause”.
Idea of the Proof

- Consider a boolean formula $\mathcal{F}$ from $3-SAT$:
- Generate for each of the $n$ variables from $\mathcal{F}$ a critical path (Part A).
- Generate for each of the above critical paths an alternative (Part B).
- Thus we have now all literals.
- Generate for each literal $x$ paths, if the literal occurs in $\mathcal{F}$ $x$ times (Part C).
- Generate for each clause a construction given by Part D.
Theorem:
The special broadcast-problem on graphs of degree 3 is in \( \mathcal{NP} \).

Proof: it is easy to build the above construction with nodes of degree \( \leq 3 \).

Theorem:
The special broadcast-problem on planar graphs of degree 3 is in \( \mathcal{NP} \).

Idea of proof: The planar 3-SAT is in \( \mathcal{NP} \). That is the dependency graph between clauses and variables is planar.

Definition:
Let \( \mathcal{F} \) be a boolean formula in KNF. Let \( V \) be the variables and \( C \) be the clauses. The dependency graph is:

\[
G_{\mathcal{F}} = (V, C, \{\{v, c\} \mid v \text{ is in } c\})
\]
Theorem:
The broadcast-problem on planar graphs of degree 3 is in \( \mathcal{NP} \).

Proof:
- Extend the above construction, such that there is a unique “hardest” node.
- Add to the above construction a very long path.
- Thus the broadcast from the start node of the long path is the hardest.
Complexity

Definition:
The gossip-problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{IN}$.
- Question: Does $r_2(G) \leq k$ hold?

Theorem:
The gossip-problem is in $\mathcal{NPC}$.

Proof: Extend the above construction, such that there is a unique “hardest” node.
Definition:
The one-way gossip-problem is:
- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r(G) \leq k$ hold?

Theorem:
The one-way gossip-problem is in $\mathcal{NP}$.

Proof: Extend the above construction, such that there is a unique “hardest” node.
And prevent the blocking of critical messages.
Lemma

We have:

- \( b(\text{CCC}(k)) \leq 5k + O(1) \)
- \( b(\text{BF}(k)) \leq 4.5k + O(1) \)
- \( b(\text{SE}(k)) \leq 4k + O(1) \)
- \( b(\text{DB}(k)) \leq 3k + O(1) \)

Proof: Use the following statements:

- \( b(G) \leq (\text{deg}(G) - 1) \cdot \text{diam}(G) + 1 \).
- \( b(G) \leq \text{deg}(G) \cdot \text{rad}(G) \).
Theorem:

We have: $\lceil \frac{5k}{2} \rceil - 2 \leq \min b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil \frac{5k}{2} \rceil - 1$.

- The following parts are proven:
  - $\min b(\text{CCC}(k)) \geq \lceil \frac{5k}{2} \rceil - 2$
  - Algorithm for $\lceil \frac{5k}{2} \rceil - 1$ will be presented.
CCC, Proof \( \text{minb}(\text{CCC}(k)) \geq \lceil 5 \cdot k/2 \rceil - 2 \)

- \( \text{diam}(\text{CCC}(k)) = \lceil 5/2 \cdot k \rceil - 2 \)
- The statement holds for even \( k \).
- Let \( k \) be odd.
- Let \((0,00 \cdots 0)\) be the origin of the message.
- The nodes \((\lfloor k/2 \rfloor, 11 \cdots 1)\) and \((\lfloor k/2 \rfloor + 1, 11 \cdots 1)\) are both in distance \((\lceil 5 \cdot k/2 \rceil - 2)\).
- Thus we need one round more than the diameter.
- The statement hold, because the CCC is node-symmetric.
Algorithm BROADCAST-CCC\(_k\)

(0, 00...0) sends to (0, 10...0);

for \(i = 0\) to \(k - 1\) do begin

    for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel
        \((i - 1, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \((i, a_0 \ldots a_{i-1}00 \ldots 0)\);

    for all \(a_0, \ldots, a_{i-1} \in \{0, 1\}\) do in parallel
        \((i, a_0 \ldots a_{i-1}00 \ldots 0)\) sends to \((i, a_0 \ldots a_{i-1}10 \ldots 0)\);

end;

for all \(\alpha \in \{0, 1\}^k\) do in parallel

    Broadcast on cycle \(C_\alpha(k)\) starting from \((k - 1, \alpha)\);
Theorem:

We have: \( \min_b(\text{CCC}(k)) = b(\text{CCC}(k)) \leq \lceil 5 \cdot k/2 \rceil - 2. \)

Idea of proof: Change the first phase and send in both directions.
Theorem:

We have: \( \min b(SE(k)) = b(SE(k)) = 2 \cdot k - 1 \)

Proof:

- The diameter provides the lower bound.
- Note \( SE(k) \) is not node-symmetric.
- We have to provide an algorithm for any node \( v \).
- Algorithm has to be without conflicts.
For each \( w = a_1 a_2 \ldots a_k \in \{0, 1\}^k \), let

- \( w_1 = a_1 \) and
- \( w(t) = a_t a_{t+1} \ldots a_k \) (for \( 1 \leq t \leq k \))
- \( w(k + 1) = \varepsilon \).

Let \( \alpha = a_1 a_2 \ldots a_k \) in \( SE_k \) be the origin.

\( \alpha = a_1 a_2 \ldots a_{k-1} a_k \) sends to \( a_1 a_2 \ldots a_{k-1} \bar{a}_k \) (exchange);

for \( t = 1 \) to \( k - 1 \) do

for all \( \beta \in \{0, 1\}^t \) do in parallel

begin

if \( \alpha(t) \notin \{\beta_1\}^+ \)

then \( \alpha(t) \beta \) sends to \( \alpha(t+1) \beta a_t \) (shuffle);

\( \alpha(t+1) \beta a_t \) sends to \( \alpha(t+1) \beta \bar{a}_t \) (exchange)

end;
α = a_1a_2 \ldots a_{k-1}a_k sends to a_1a_2 \ldots a_{k-1}\bar{a}_k \text{ (exchange)};

\begin{align*}
&\text{for } t = 1 \text{ to } k - 1 \text{ do} \\
&\quad \text{for all } \beta \in \{0, 1\}^t \text{ do in parallel begin} \\
&\quad\quad \text{if } \alpha(t) \not\in \{\beta_1\}^+ \\
&\quad\quad\quad \text{then } \alpha(t)\beta \text{ sends to } \alpha(t + 1)\beta a_t \text{ (shuffle)}; \\
&\quad\quad\alpha(t + 1)\beta a_t \text{ sends to } \alpha(t + 1)\beta\bar{a}_t \text{ (exchange) end;} \\
&\text{Show: There are no conflicts!}
\end{align*}

- There is no conflict for the exchange-edges, because the last bit give a unique sender and receiver.
- Assume there is a conflict by the shuffle-edges.
- We have \(\alpha(t)\beta = \alpha(t + 1)\gamma a_t\) for some \(\beta, \gamma \in \{0, 1\}^t\).
- Then we have:
  \[a_t\alpha(t + 1) = \alpha(t + 1)\gamma_1 \Rightarrow a_t = a_{t+1} = \cdots = a_k = \gamma_1 \Rightarrow \alpha(t) \in \{\gamma_1\}^+\.
- This is a contradiction: shuffle-edges for \(\alpha(t) \in \{\gamma_1\}^+\) are not used.
α = a₁a₂...aₖ₋₁aₖ sends to a₁a₂...aₖ₋₁āₖ (exchange);
for t = 1 to k − 1 do
    for all β ∈ {0, 1}^t do in parallel begin
        if α(t) /∈ {β₁}^+
            then α(t)β sends to α(t + 1)βaₜ (shuffle);
        α(t + 1)βaₜ sends to α(t + 1)βāₜ (exchange) end;
Show: All nodes are informed!

- Show by induction: After 2 ⋅ r + 1 rounds are all nodes
  α(r + 2)β, β ∈ {0, 1}^{r+1} informed.
- IS: r = 0 is obvious.
- All nodes α(r + 1) /∈ {β₁}^+, β ∈ {0, 1}^{r+1} will be informed, because all
  nodes α(r + 2)β have already received the information.
- If α(r + 1) ∈ {β₁}^+, β ∈ {0, 1}^{r+1} holds, then we have
  α(r + 2)βa_r+1 = α(r + 1)β₁βa_r+1.
- This node has been informed before.
Theorem:

We have: $\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$

- The diameter gives the lower bound.
- Algorithm will be provided in the following.
BF (Idea of proof)

- Distribute the information in two ways:
  - Prefer in the first strategy the cycle-edges.
  - Prefer in the second strategy the cross-edges.
- Split the butterfly into two isomorph parts.
- Choose for each part a different strategy.
- Distribute in the last phase on the cycles.

\[
\left\lfloor \frac{3m}{2} \right\rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m
\]
BF (Proof I)

Splitting of \( BF(m) \) in \( F_0 \) and \( F_1 \):
- \( F_0 \) has nodes: \( \{(l, \alpha 0) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\} \).
- \( F_1 \) has nodes: \( \{(l, \alpha 1) \mid 0 \leq l \leq m - 1, \alpha \in \{0, 1\}^{m-1}\} \).
- \( F_0 \) and \( F_1 \) are isomorph.
- \( \#_0(w) \) denotes the number of 0’en in \( w \).
- \( \#_1(w) \) denotes the number of 1’en in \( w \).
Consider $F_0$: from node $v_0 = (0, 00 \cdots 00)$ exists a unique path of length $m - 1$ to $w_0 = (m - 1, \alpha 0)$ for $\alpha \in \{0, 1\}^{m-1}$.

Consider $F_1$: from node $v_1 = (m - 1, 00 \cdots 01)$ exists a unique path of length $m - 1$ to $w_1 = (0, \alpha 1)$ for $\alpha \in \{0, 1\}^{m-1}$.

First step of the algorithm $v_0$ informs $v_1$.

Then we use in $F_0$ and $F_1$ two different strategies.
BF (Proof III)

\[ \lfloor \frac{3m}{2} \rfloor \leq \min_b(BF(m)) = b(BF(m)) \leq 2 \cdot m \]

- **Aim:** Inform in \( \lfloor \frac{3m}{2} \rfloor \) steps the nodes \( w_0 = (m - 1, \alpha_0) \) and \( w_1 = (0, \alpha_1) \) for \( \alpha \in \{0, 1\}^{m-1} \).

- If a node \( w_0 = (m - 1, \alpha_0) \) gets informed, then it informs in the next step \( w_1 = (0, \alpha_1) \) (if necessary).

- If a node \( w_1 = (0, \alpha_1) \) gets informed, then it informs in the next step \( w_0 = (m - 1, \alpha_0) \) (if necessary).
In $F_0$ a informed node $(l, a0)$ sends first to $(l + 1, a0)$ and then to $(l + 1, a(l)0)$. $[a(l) = a_1 \ldots \tilde{a}_l \ldots ]$

In $F_1$ a informed node $(l, a1)$ sends first to $(l + 1, a(l)1)$ and then to $(l + 1, a1)$.

The time to inform from $v_0 = (0, 00 \cdots 00)$ a node $w_0 = (m - 1, a0)$ is:

$$1 + \#_0(\alpha) + 2\#_1(\alpha) = m + \#_1(\alpha).$$

The time to inform from $v_1 = (m - 1, 00 \cdots 01)$ a node $w_1 = (0, a1)$ is:

$$1 + 2\#_0(\alpha) + \#_1(\alpha) = m + \#_0(\alpha).$$
Case 1: $m$ is odd:

Case 1.1: $\#_1(\alpha) < (m - 1)/2$:
Node $w_0$ will be informed from $v_0$ at time
$m + \#_1(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
After this $w_0$ sends to $w_1$.
$w_1$ is informed at time $\lfloor 3m/2 \rfloor$.

Case 1.2: $\#_0(\alpha) < (m - 1)/2$:
Node $w_1$ will be informed from $v_0$ at time
$m + \#_0(\alpha) < (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
$w_0$ will be informed from $w_1$ at time $\lfloor 3m/2 \rfloor$.

Case 1.3: $\#_0(\alpha) = \#_1(\alpha) = (m - 1)/2$:
$w_0$ is informed at time
$m + \#_1(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor$.
$w_1$ is informed at time $m + \#_0(\alpha) = (3m - 1)/2 = \lfloor 3m/2 \rfloor$. 

$\lfloor 3m/2 \rfloor \leq \min b(BF(m)) = b(BF(m)) \leq 2 \cdot m$
BF (Proof V)

Case 2: $m$ is even:

Case 2.1: $\#_1(\alpha) \leq (m - 2)/2$:
node $w_0$ will be informed from $v_0$ at time
$m + \#_1(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
Thus node $w_1$ will be informed at time $\lfloor 3m/2 \rfloor$.

Case 2.2: $\#_0(\alpha) \leq (m - 2)/2$:
node $w_1$ will be informed from $v_0$ at time
$m + \#_0(\alpha) \leq 3m/2 - 1 < \lfloor 3m/2 \rfloor$.
Thus node $w_0$ will be informed at time $\lfloor 3m/2 \rfloor$.

In the last phase we distribute the information on the cycles.

Running time is: $\lceil m/2 \rceil$ rounds.

Total running time: $\lfloor 3m/2 \rfloor + \lceil m/2 \rceil = 2m$
Theorem:

We have: \( d \leq \min b(DB(d)) = b(DB(d)) \leq \lfloor 3/2 \cdot (d + 1) \rfloor. \)

Proof:

- Idea \((y_1, y_2, \ldots, y_d)\) informs \((y_2, \ldots, y_d, y_1)\) and \((y_2, \ldots, y_d, \overline{y_1})\).
- The order is given by the parity.
- Let \(\alpha = \#_1(y_1, y_2, \ldots, y_d) \mod 2\).
- \((y_1, y_2, \ldots, y_d)\) informs first \((y_2, \ldots, y_d, \alpha)\) and then \((y_2, \ldots, y_d, \overline{\alpha})\).
- \((0011000)\) informs first \((0110000)\) and then \((0110001)\).
For $k \in \{0, 1\}$ consider the path $P_k$
from $(y_1, y_2, \ldots, y_d)$ to $(z_1, z_2, \ldots, z_{d-1}, z_d)$.

$$(y_1, \ldots, y_d), (y_2, \ldots, y_d, k), (y_3, \ldots, y_d, k, z_1), (y_4, \ldots, y_d, k, z_1, z_2), \ldots$$

$$\ldots, (y_d, k, z_1, \ldots, z_{d-2}), (k, z_1, \ldots, z_{d-1}), (z_1, \ldots, z_{d-1}, z_d)$$

Let $v_{0i} = (y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2})$ the i-th node on $P_0$.
Let $v_{1i} = (y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2})$ the i-th node on $P_1$.

We have different times (1 or 2) for sending:

- $(y_i, \ldots, y_d, 0, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 0, z_1, \ldots, z_{i-2}, z_{i-1})$
- $(y_i, \ldots, y_d, 1, z_1, \ldots, z_{i-2}) \rightarrow (y_{i+1}, \ldots, y_d, 1, z_1, \ldots, z_{i-2}, z_{i-1})$

Thus the sum of running times is on $P_0$ and $P_1$: $3(d + 1)$.
Thus the running time for the broadcast is: $\lfloor 3(d + 1)/2 \rfloor$. 
Degree of the Nodes

**Theorem:**

Let $n \geq 5$ and $G = (V, E)$ be a graph with $n$ nodes:

- If $\Delta(G) = 3$ holds, we have: $b(G) \geq \min b(G) \geq 1.4404 \log(n) - 3$.
- If $\Delta(G) = 4$ holds, we have: $b(G) \geq \min b(G) \geq 1.1374 \log(n) - 2$.

**Proof:**

- Let $A$ be a broadcast-algorithm.
- Let $\text{Broad}_i^A(v_0)$ be the set of nodes, which are informed from $v_0$ by $A$ in $i$ rounds.
- Let $\text{Rec}_i^A(v_0) = \text{Broad}_i^A(v_0) \setminus \text{Broad}_{i-1}^A(v_0)$.
- Let $\text{Rec}_0^A(v_0) = \{v_0\}$.
- We have: $|\text{Broad}_i^A(v_0)| = \sum_{s=0}^{i} |\text{Rec}_s^A(v_0)|$. 
Building the Idea

We consider here only the case $\Delta(G) = 3$. The case $\Delta(G) = 4$ is similar.

- The initial node may send at most three times.
- The initial node sends only in rounds 1, 2, 3.
- Any other nodes will be informed at time $t$ via an edge $e$.
- No further node may be informed via $e$.
- Thus any other node may send at most two times.
- If a node $v$ is informed in round $t$ by $w$, then did $w$ receive the information at round $t - 1$ or $t - 2$.
- Thus the number of newly informed nodes in round $t > 3$, is at most the number of nodes which got informed in rounds $t - 1$ and $t - 2$. 
Proof

- Let $A(i) = |Rec_i^A(v_0)|$.
- $A(0) = 1$
- $A(1) = 1$
- $A(2) = 2$
- $A(3) = 4$
- $A(i) = A(i - 1) + A(i - 2)$ für $i \geqslant 4$.
- Show by induction: $A(i) \leqslant 1.61804^i$ for $i \geqslant 0$. 

Proof

- $A(0) = 1 \leq 1 = 1.61804^0$
- $A(1) = 1 \leq 1.61804 = 1.61804^1$
- $A(2) = 2 \leq 2.61805 = 1.61804^2$
- $A(3) = 4 \leq 4.23612 = 1.61804^3$

Induction step ($i \geq 4$):
- We have: $A(j) \leq 1.61804^j$ for any $j \leq i - 1$.
- $A(i) = A(i - 1) + A(i - 2) \leq 1.61804^{i-1} + 1.61804^{i-2} \leq 1.61804^i$
- Note for this: $1.61804 + 1 \leq 1.61804^2$.

Thus we have: $n \leq |\text{Broadcast}^A_t(v_0)| = \sum_{i=0}^t |\text{Rec}^A_i(v_0)| \leq \sum_{i=0}^t A(i) \leq \sum_{i=0}^t 1.61804^i = \frac{1.61804^{t+1} - 1}{1.61804 - 1} \leq 3 \cdot 1.61804^t$

- $t \geq 1.4404 \cdot \log_2 n - 3$.

Proof of the second statement may be done in the same way.
More Results

Consequence:
\[ b(\text{DB}_k) \geq \min b(\text{DB}_k) \geq 1.1374 \cdot k - 2 \]

Theorem:
\[ b(\text{BF}_m) = \min b(\text{BF}_m) > 1.7396m \text{ for large enough } m. \]

Idea of Proof: Check the number of nodes in distance \( k \).

Theorem:
\[ b(\text{DB}_m) > 1.3042m \text{ for large enough } m. \]

Idea of Proof: Check the number of nodes in distance \( k \).
# Overview

| Graph    | $|V|$ | Diameter       | Lower Bound       | Upper Bound       |
|----------|------|----------------|-------------------|-------------------|
| $K_n$    | $n$  | 1              | $\lceil \log_2 n \rceil$ | $\lceil \log_2 n \rceil$ |
| $HQ_k$   | $2^k$ | $k$            | $k$               | $k$               |
| $CCC_k$  | $k \cdot 2^k$ | $\lceil 5k/2 \rceil - 2$ | $\lceil 5k/2 \rceil - 2$ | $\lceil 5k/2 \rceil - 2$ |
| $SE_k$   | $2^k$ | $2k - 1$       | $2k - 1$          | $2k - 1$          |
| $DB_k$   | $2^k$ | $k$            | $1.4404k$         | $\frac{3}{2}(k + 1)$ |
| $BF_k$   | $k \cdot 2^k$ | $\lceil 3k/2 \rceil$ | $1.7609k$         | $2k - \frac{1}{2} \log \log k + c$ |
J. Hromkovič, et al.:
Dissemination of Information in Communication Networks:
Broadcasting, Gossiping, Leader Election, and Fault-Tolerance.
Questions

- Give the idea for the NP-completeness proof for the broadcast problem?
- Give the idea for the broadcast on the following networks
  - CCC
  - BF
  - SE
  - DB
- What are the ideas for the lower bounds for the broadcast problem?
Legend

- : Not of relevance
- : implicitly used basics
- : idea of proof or algorithm
- : structure of proof or algorithm
- : Full knowledge