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Definition (Gossip):

Given is \( G = (V, E) \).

- Each node \( w \in V \) has some information \( I(w) \) and no node of \( V \setminus \{w\} \) knows \( I(w) \).
- Construct algorithm, where each node \( v \in V \) collects information \( \bigcup_{w \in V} I(w) \).

- By \( \text{comm}(A) \) we denote the complexity (number of rounds) of a communication-algorithm.
- \( r(G) = \min\{\text{comm}(A) \mid A \text{ is a one-way algorithm for the gossip-problem on } G\} \)
- \( r_2(G) = \min\{\text{comm}(A) \mid A \text{ is a two-way algorithm for the gossip-problem on } G\} \)
Motivation

- Broadcast is a part of gossip.
- Many broadcasts have to “cooperate”. This makes the problem interesting.
- More important for algorithms on networks.
- Example: Distribute lower bounds for “Branch and Bound”.
- For gossip we get a difference between telegraph- and telephone-mode.
- We start with gossiping in the telephone-mode.
Lemma:

Let $G = (V, E)$ a graph with $n$ nodes. Then we have:

$$r(G) \geq r_2(G) \geq \begin{cases} \lceil \log_2 n \rceil & n \text{ even}, \\ \lceil \log_2 n \rceil + 1 & n \text{ odd}. \end{cases}$$

Proof: Only the case, where $n$ is odd, has to be proven.

- **Show:** $r_2(G) \geq \lceil \log_2 n \rceil + 1$.

- Let $A$ be a communication-algorithm for the gossip-problem. $A$ has communication rounds (matchings) $E_1, E_2, \cdots, E_k$.

- Show by induction: After $i$ rounds has each node at most $2^i$ pieces of information.
  - $i = 0$: Each node has $2^0 = 1$ pieces of information.
  - $i - 1 \to i$: at most $2^{i-1} + 2^{i-1} = 2^i$ pieces of information may be collected by any node.

- In round $k$ is at least one node $v$ inactive.

- $v$ has after $k$ rounds at most $2^{k-1}$ pieces of information.
Lemma:

For any graph $G = (V, E)$ with $|V| = n$ we have:

- $r(G) \leq 2n - 2$, and
- $r_2(G) \leq 2n - 3$.

Proof: Follows from the following known statements:

- $\minb(G) \leq n - 1$ for any graph $G = (V, E)$ with $|V| = n$.
- $r(G) \leq 2 \cdot \minb(G)$
- $r_2(G) \leq 2 \cdot \minb(G) - 1$
Simple Algorithm (Continuation)

Lemma:

We have:
- $r(T_k(1)) = 2k$
- $r_2(T_k(1)) = 2k - 1$

Proof:

- Show: $r(T_k(1)) \geq 2k$.
- $r(T_k(1))$ has one root and $k$ leaves.
- The maximal matching is 1.
- In each round is only one leaf active.
- Each leaf has to send at least once.
- Each leaf has to receive at least once.
- Thus in total $2k$ rounds necessary.
- $r_2(T_k(1)) \geq 2k - 1$, is a simple exercise.
Gossip on Lines

**Theorem:**

We have:

- \( r_2(L(n)) = n - 1 \) for any even number \( n \geq 2 \),
- \( r_2(L(n)) = n \) for any odd number \( n \geq 3 \),
- \( r(L(n)) = n \) for any even number \( n \geq 2 \) and
- \( r(L(n)) = n + 1 \) for any odd number \( n \geq 3 \).

**Proof:**

- Show: \( r_2(L(n)) \geq n - 1 \).
- Note: \( r_2(L(n)) \geq b(L(n)) \geq diam(L(n)) = n - 1 \)
Gossip on Lines (Proof 1)

- Show: $r_2(L(n)) \leq n - 1$ for $n$ even.

- Consider algorithm $A$, given by the following matchings:
  
  1. $\{\{0, 1\}, \{n - 1, n - 2\}\}$
  2. $\{\{1, 2\}, \{n - 2, n - 3\}\}$
  3. $\{\{2, 3\}, \{n - 3, n - 4\}\}$
  4. ...
  5. $\{\{n/2 - 1, n/2\}\}$
  6. ...
  7. $\{\{2, 3\}, \{n - 3, n - 4\}\}$
  8. $\{\{1, 2\}, \{n - 2, n - 3\}\}$
  9. $\{\{0, 1\}, \{n - 1, n - 2\}\}$

\[
\begin{align*}
  r_2(L(n)) &= n - 1 & (n \equiv 0 \text{ (mod 2)}) \\
  r_2(L(n)) &= n & (n \equiv 1 \text{ (mod 2)}) \\
  r(L(n)) &= n & (n \equiv 0 \text{ (mod 2)}) \\
  r(L(n)) &= n + 1 & (n \equiv 1 \text{ (mod 2)})
\end{align*}
\]
Gossip on Lines (Proof II)

Show: \( r_2(L(n)) \leq n \) for \( n \) odd.

Consider algorithm \( A \), given by the following matchings:

1. \( \{0, 1\} \),
2. \( \{1, 2\}, \{n - 1, n - 2\} \),
3. \( \{2, 3\}, \{n - 2, n - 3\} \),
4. \( \ldots \)
5. \( \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\} \}
6. \( \ldots \)
7. \( \{2, 3\}, \{n - 2, n - 3\} \),
8. \( \{1, 2\}, \{n - 1, n - 2\} \),
9. \( \{0, 1\} \)
Gossip on Lines (Proof II)

Show: $r_2(L(n)) \geq n$ for $n$ odd.

Consider the flow of messages from the left to the right node.
These could not be forwarded without delay.
Because we would get a time-conflict in the center.
Thus at least one message has to be delayed.
This provides the lower bound.
Gossip on Lines (Proof III)

- Show: \( r(L(n)) \leq n \) for \( n \) even.

- Consider algorithm \( A \), given by the following matchings:

\[
\begin{align*}
1 & \{(0, 1), (n - 1, n - 2)\}, \\
2 & \{(1, 2), (n - 2, n - 3)\}, \\
3 & \{(2, 3), (n - 3, n - 4)\}, \\
4 & \ldots \\
5 & \{(n/2 - 1, n/2)\} \\
6 & \{(n/2, n/2 - 1)\} \\
7 & \ldots \\
8 & \{(3, 2), (n - 4, n - 3)\}, \\
9 & \{(2, 1), (n - 3, n - 2)\}, \\
10 & \{(1, 0), (n - 2, n - 1)\} \\
\end{align*}
\]
Gossip on Lines (Proof IV)

- Show: \( r(L(n)) \geq n \) for \( n \) even.
- The proof is similar to the above one:
- Consider the flow of messages from the left to the right node.
- These could not be forwarded without delay.
- Because we would get a time-conflict in the center.
- Thus at least one messages has to be delayed.
- This provides the lower bound.

\[
\begin{align*}
  r_2(L(n)) &= n - 1 \quad (n \equiv 0 \pmod{2}) \\
  r_2(L(n)) &= n \quad (n \equiv 1 \pmod{2}) \\
  r(L(n)) &= n \quad (n \equiv 0 \pmod{2}) \\
  r(L(n)) &= n + 1 \quad (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on Lines (Proof V)

- Show: $r(L(n)) \leq n + 1$ for $n$ odd.

- Consider algorithm $A$, given by the following matchings:

1. $\{(0,1)\}$,
2. $\{(1,2), (n-1, n-2)\}$,
3. $\{(2,3), (n-2, n-3)\}$,
4. $\ldots$
5. $\{([n/2], \lceil n/2 \rceil)\}$
6. $\{([n/2], \lceil n/2 \rceil)\}$
7. $\ldots$
8. $\{(3,2), (n-3, n-2)\}$,
9. $\{(2,1), (n-2, n-1)\}$,
10. $\{(1,0)\}$
Gossip on Lines (Proof VI)

- Show: \( r(L(n)) \geq n + 1 \) for \( n \) odd.
- The proof is similar to the above one:
- Consider the flow of messages from the left to the right node.
- These could not be forwarded without delay.
- Because we would get a time-conflict in the center.
- Thus at least one messages (w.l.o.g. the right) has to be delayed.
- Now the right message has to move, because otherwise we would have already a delay of two.
- But now we still do get a further delay.
- Thus we have proven the lower bound.

\[
\begin{align*}
r_2(L(n)) &= n - 1 & (n \equiv 0 \pmod{2}) \\
r_2(L(n)) &= n & (n \equiv 1 \pmod{2}) \\
r(L(n)) &= n & (n \equiv 0 \pmod{2}) \\
r(L(n)) &= n + 1 & (n \equiv 1 \pmod{2})
\end{align*}
\]
Gossip on arbitrary Trees

Lemma:
For any tree $T$ we have:
- $r(T) = 2 \cdot \min b(T)$
- $r_2(T) = 2 \cdot \min b(T) - 1$

Idea of the proof:
- We have already for any graph $G$: $r(G) \leq 2 \cdot \min b(G)$.
- We have to show: $r(G) \geq 2 \cdot \min b(G)$.
- Let $W = \bigcup_{v \in V} I(v)$ be the total information.
- Let $A$ be any communication algorithm on $T$.
- Let $t$ be the point in time, when some node knows $W$.
- Let $v$ one node, which after $t$ steps know $W$.
- Show: at time $t$ only node $v$ knows $W$. 
Gossip on arbitrary Trees (Proof I)

- Let $u \neq v$ be an other node which knows $W$ after $t$ steps.
- Let $(u, y_1, y_2, \cdots, y_k, v)$ be the unique path connecting $u$ and $v$.
- If $v$ sends to $y_k$ at time $t$, then $v$ did know $W$ at time $t-1$.
- So we have to consider the case: $y_k$ sends to $v$ at time $t$:
  - In this case $y_k$ sends $v$ some missing information.
  - $y_k$ knows at time $t-1$ the full information, which has to be send from $y_k$ to $v$.
  - The information, which has to be send from $v$ to $y_k$, is already send.
  - Then the node $y_k$ know $W$ at time $t-1$.

- Contradiction, the node $u$ does not exist.

- Thus we have: $t \geq \min b(T) = b(v, T)$.
Gossip on arbitrary Trees (Proof II)

- Consider the situation at node $v$ after round $t$.
- Let w.l.o.g. $v$ be the root of $T$.
- Let $v_1, v_2, \cdots, v_k$ be the successors of $v$.
- Let $T_1, T_2, \cdots, T_k$ be the subtrees with roots $v_1, v_2, \cdots, v_k$.
- In each subtree $T_i$ is some information $w_i$ missing.
- Only the node $v$ knows $\bigcup_{j=1}^{k} w_j$.
- Thus there are $b(v, T)$ steps to be done.
- We finally have $r(T) \geq \min b(T) + b(v, T) \geq 2 \cdot \min b(T)$
Consider the two-way mode: by a similar way we may prove:

At time $t$ only two neighbours nodes $u$ and $v$ know the total information. We get in the similar way the second statement.
Implication

Lemma:
For all $m \geq 1$ and $k \geq 2$ we have:

- $r(T_k(m)) = 2 \min b(T_k(m)) = 2 \cdot k \cdot m$.
- $r_2(T_k(m)) = 2 \min b(T_k(m)) - 1 = 2 \cdot k \cdot m - 1$. 
**Lemma:**

Let $G = (V, E)$ be a graph with bridge $e \in E$, which is separated by $e$ in components $G_1$ and $G_2$, then we have

$r(G) \geq \min b(G) + 1 + \min \{\min b(G_1), \min b(G_2)\}$

**Proof:** Let $W = \bigcup_{v \in V} I(v)$ be the total information. Let $t \geq \min b(G)$ the time, when a node $w$ knows $W$.

- If $w \in G_1$ hold, then do no node from $G_2$ know $W$.
- Then there are still $1 + \min b(G_2)$ steps to do.
- If $w \in G_2$ hold, then do no node from $G_1$ know $W$.
- Then there are still $1 + \min b(G_1)$ steps to do.
- Thus we have: $r(G) \geq \min b(G) + 1 + \min \{\min b(G_1), \min b(G_2)\}$. 

\[ \Sigma = 0 \]
Lemma:

Let $G = (V, E)$ be a graph with bridge $e \in E$, which is separated by $e$ in components $G_1$ and $G_2$, then we have:

$$r_2(G) \geq \min b(G) + \min \{\min b(G_1), \min b(G_2)\}$$

Proof: Let $t \geq \min b(G)$ be the time, when node $w$ knows $W$ the first time. As before we may prove:

- Let $i \in \{1, 2\}$. If $w \in G_i$ and $v_{3-i}$ does not know $W$, then no node from $G_{3-i}$ knows $W$. There are still $1 + \min b(G_{3-i})$ steps to do.

- If $v_1$ and $v_2$ know $W$ at time $t$, then no other node knows $W$. There are still $\min \{\min b(G_1), \min b(G_2)\}$ Steps to do.

- Thus we have: $r_2(G) \geq \min b(G) + \min \{\min b(G_1), \min b(G_2)\}$. 

\[ \begin{array}{c}
G_1 & v_1 & v_2 & G_2 \\
\end{array} \]
Gossip on Cycles

**Theorem:**

We have:

- \( r_2(C(k)) = \frac{k}{2} \) for even \( k \).
- \( r_2(C(k)) = \lceil \frac{k}{2} \rceil + 1 \) for odd \( k \).

Idea of the proof (\( k \) even): [\( k \) odd: an easy exercise]

- Let \( k \) be even.
- \( r_2(C(k)) \geq \frac{k}{2} \) results by the diameter.
- \( r_2(C(k)) \leq \frac{k}{2} \) is true by the following algorithm:
  1. \{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i + 1\}, \ldots, \{n - 2, n - 1\}
  2. \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\}
  3. \{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2i, 2i + 1\}, \ldots, \{n - 2, n - 1\}
  4. \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2i - 1, 2i\}, \ldots, \{n - 1, 0\}
  5. \ldots
- Note: After \( i \) rounds knows each node \( 2 \cdot i \) Informationen.
1-Way Gossip on Cycles (Idea)

- Messages should traverse in both directions.
- Activate each $f(n)$-th node on the cycle.
- This will result in an additional $\Theta(f(n))$ steps.
- During the distribution we get $\Theta\left(\frac{n}{2 \cdot f(n)}\right)$ delays.
- Thus we will choose $f(n) = \Theta(\sqrt{n})$.
- By this idea we may get a lower and upper bound.
Gossip on Cycles (Idea)
Gossip on Cycles (Idea of the algorithm)

- Split the cycle in $\Theta(\sqrt{n})$ blocks $B_i$.
- Within block $B_i$ ($i \in \{1, 2, 3, \cdots , k\}$ with $k \in \Theta(\sqrt{n})$) do the following:
  - Phase 1:
    - The nodes $v_i$ and $u_i$ start a “wave” to the left [right].
    - The messages of $v_i$ and $u_i$ are delayed $\Theta(\sqrt{n})$ times by the other messages.
    - After $n/2 + \Theta(\sqrt{n})$ round know nodes $z_i$ the total information.
  - Phase 2:
    - Each node $z_i$ distribute the total information to $\Theta(\sqrt{n})$ nodes.
- Note: If $n$ is even, we have always a delay of one and the synchronization is easy.
Gossip on Cycles (Idea)

Theorem:

We have:

- \( r(C(n)) \leq \frac{n}{2} + \sqrt{2n} - 1 \) for even \( n \).
- \( r(C(n)) \leq \lceil \frac{n}{2} \rceil + \lceil 2 \cdot \sqrt{\lceil \frac{n}{2} \rceil} \rceil - 1 \) for odd \( n \).
- \( r(C(n)) \geq \frac{n}{2} + \sqrt{2n} - 1 \) for even \( n \).
- \( r(C(n)) \geq \lceil \frac{n}{2} \rceil + \lceil \sqrt{2n} - 1/2 \rceil - 1 \) for odd \( n \).

Proof: See literature.
Gossip on the Hypercube

Theorem:

For all \( m \in \mathbb{N} \) we have: \( r_2(HQ(m)) = m \)

Proof:

- The lower bound is the diameter.
- Upper bound by the following algorithm:

  for \( i = 1 \) to \( m \) do
    for all \( a_1, a_2, \ldots, a_{m-1} \in \{0,1\} \) do in parallel
      \[
      a_1a_2\cdots a_{i-1}0a_ia_{i+1}\cdots a_{m-1} \text{ sends to}
      a_1a_2\cdots a_{i-1}1a_ia_{i+1}\cdots a_{m-1}
      \]

Corollary:

For all \( m \in \mathbb{N} \) we have: \( r_2(K(2^m)) = m \)
Consider one-way mode:

- Start with the first phase of the gossip-algorithm for cycles on all cycles.
- Then each $\Theta(\sqrt{n})$-th node on each cycle knows the total information of its cycles.
- In $\Theta(\sqrt{n})$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each $\Theta(\sqrt{n})$-th node of each cycle the total information.
- The final part is the second phase of the gossip-algorithm of cycles on all cycles.
- All nodes know now the total information.
Consider two-way mode:

- Start with the gossip algorithm for cycles on all cycles.
- Each node of the cycle knows now the total information of its cycle.
- In $\Theta(n/2)$ waves distribute this information down and between the cycles.
- After $\Theta(n)$ steps knows each node the total information.
Theorem:

Let $k \geq 3$, then we have:

- $r(CCC(k)) \leq r(C(k)) + 3k - 1 \leq \left\lceil \frac{7k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 2$.

- $r(BF(k)) \leq r(C(k)) + 2k \leq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil 2\sqrt{\left\lceil \frac{k}{2} \right\rceil} \right\rceil - 1$.

- $r_2(CCC(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.

- $r_2(CCC(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$.

- $r_2(BF(k)) \leq \frac{k}{2} + 2k = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for even $k$.

- $r_2(BF(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 2k + 2 = 5 \cdot \left\lceil \frac{k}{2} \right\rceil$ for odd $k$. 
Complexity

Definition:
The two-way gossip-problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r_2(G) \leq k$ hold.

Definition:
The one-way gossip-problem is:

- Given: $G = (V, E)$ and $k \in \mathbb{N}$.
- Question: Does $r(G) \leq k$ hold.
Complexity

Theorem:
The two-way and one-way gossip-problem on trees is in $\mathcal{P}$

Proof: simple exercise.

Theorem:
The two-way and one-way gossip-problem is in $\mathcal{NPC}$

Proof: Same way as the for the broadcast-problem.
Gossip on Graphs with $2 \cdot m$ Nodes (0. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes (1. Idea)

**Implication:**

- For all $m \in \mathbb{N}$ we have: $r_2(K(2^m)) = m$
- For all $m \in \mathbb{N}$ we have: $r_2(K(m)) \leq \lceil \log m \rceil + 1$
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)

- Too many nodes where inactive for too long time.
- These nodes could not double their information.
- Idea: Try to double the information of any node.
- Detailed idea: In each step each node has an “interval” of information.
- To make the doubling easy split the nodes into two groups.
- Both groups should be the same size.
- In the first step pairs of node from each group share their information.
Gossip on Graphs with $2 \cdot m$ Nodes (2. Idea)
Gossip on Graphs with $2 \cdot m$ Nodes

Theorem:
For all $m \in \mathbb{N}$ we have: $r_2(K(2m)) = \lceil \log 2m \rceil$

Proof: Split the nodes in groups $Q[i]$ and $R[i]$ ($0 \leq i \leq m - 1$).

- algorithm:
  
  for all $i \in \{0, \cdots, m-1\}$ do in parallel
  
  Exchange the information between $Q[i]$ and $R[i]$

  for $t = 1$ to $\lceil \log_2 m \rceil$ do
    
    for all $i \in \{0, \ldots, m-1\}$ do in parallel
    
    Exchange the information between $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$

- Invariant:
  
  Let $\alpha[i]$ be the information of $Q[i]$ and $R[i]$ after their initial exchange.

  After round $t$ know nodes $Q[i]$ and $R[(i + 2^{t-1}) \mod m]$: $\bigcup_{0 \leq j \leq 2^t - 1} \alpha[(i + j) \mod m]$

  The invariant is easy to be shown.
Gossip on Graphs with $2 \cdot m + 1$ Nodes (a try)

- We need an extra round.
- A nice proof with this idea will become complicated.
- We will try to put some structure into the proof.
How could this be an idea?

We only have the edges of the first step.

Idea: We could now choose a small even number of Nodes, which together have the total information.

These nodes may perform the above gossip algorithm.

In the last step we repeat the first round.
Gossip on Graphs with $2 \cdot m + 1$ Nodes

- Let $n = 2 \cdot m + 1$.
- Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be all nodes.
- For all $i \in \{0, 1, \cdots, m-1\}$ the node $v_{m+2+i}$ sends to $v_i$.
- The node $\{v_0, v_1, v_2, \cdots, v_m\}$ have now the total information.
- If $m + 1$ is even, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_m\}$.
- If $m + 1$ is odd, perform a gossip on the nodes $\{v_0, v_1, v_2, \cdots, v_{m+1}\}$.
- For all $i \in \{0, 1, \cdots, m-1\}$ the nodes $v_i$ send to $v_{m+2+i}$.
- Correctness follows direct by the construction.

Running time for $m + 1$ even:

$$r_2(K(m + 1)) + 2 = \lceil \log_2(m + 1) \rceil + 2 = \lceil \log_2(n + 1) \rceil + 1 = \lceil \log_2 n \rceil + 1$$

Running time for $m + 1$ odd:

$$r_2(K(m + 2)) + 2 = \lceil \log_2(m + 2) \rceil + 2 = \lceil \log_2(n + 3) \rceil + 1 = \lceil \log_2 n \rceil + 1$$
**1st Idea (Let the Knowledge grow)**

We need more rounds.

A nice proof with this idea will become complicated.

We will try to put some structure into the proof.
2nd Idea (Let the Knowledge grow in a structured way)

- We need an additional two rounds.
- $v_x$ and $w_y$ alternate as sender and receiver.
- The information grows in blocks (intervals) in the nodes.
- With this idea we may do the proof.
- Only the first two rounds are special.
2nd Idea (Let the Knowledge grow in a structured way)

- After the first two rounds some node-pairs share their information.
- Consider this situation as the start:
  - All $v_x$ and $w_x$ have one information pair.
  - $v_i$ sends to $w_j$ and the $w_x$ have 2 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 3 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 5 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 8 information pairs.
  - $v_i$ sends to $w_j$ and the $w_x$ have 13 information pairs.
  - $w_i$ sends to $v_j$ and the $v_x$ have 21 information pairs.
- Thus the grow-rate and the algorithm is clearly visible.
Let $n = 2m$.

Gossip-Algorithm:

$\begin{align*}
 & t := 0; \\
 & \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel } R[i] \text{ sends to } Q[i]; \\
 & \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel } Q[i] \text{ sends to } R[i]; \\
 & \text{while } fib(2t+1) < m \text{ do begin} \\
 & \hspace{1em} t := t + 1; \\
 & \hspace{1em} \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel} \\
 & \hspace{2em} R[(i + fib(2t-1)) \mod m] \text{ sends to } Q[i]; \\
 & \hspace{2em} \text{if } fib(2t) < m \text{ then} \\
 & \hspace{3em} \text{for all } i \in \{0, \ldots, m-1\} \text{ do in parallel} \\
 & \hspace{4em} Q[(i + fib(2t)) \mod m] \text{ sends to } R[i] \\
 & \text{end;} \\
\end{align*}$

\begin{align*}
\text{fib}(0) &= \text{fib}(1) = 1 \\
\text{fib}(i) &= \text{fib}(i-1) + \text{fib}(i-2)
\end{align*}
One-Way-Gossip

Theorem:
Let \( n = 2m \) and \( k = \min\{x \mid \text{fib}(x) \geq m\} \). Then we have \( r(K(n)) \leq k + 1 \).

Proof:
- The algorithm stops, if \( \text{fib}(2t + 1) \geq m \) or \( \text{fib}(2t) \geq m \) holds.
- The number of rounds within the loop is \( 2t \) or \( 2(t - 1) + 1 \).
- The total number of rounds is \( (k - 1) + 2 \).
- Correctness may be proven by the following invariant:
- Let \( a[i] \) be the information, which share \( R[i] \) and \( Q[i] \) after two rounds.
- After \( t \) loops we have:
  - \( Q[i] \) knows \( \bigcup_{0 \leq j \leq \text{fib}(2t+1)-1} \alpha[(i + j) \mod m] \)
  - \( R[i] \) knows \( \bigcup_{0 \leq j \leq \text{fib}(2t+2)-1} \alpha[(i + j) \mod m] \)
- The correctness is a direct result of this.
One-Way-Gossip

**Theorem:**

Let \( n = 2m - 1 \) and \( k = \min\{x \mid \text{fib}(x) \geq m\} \). Then we have \( r(K(n)) \leq k + 2 \).

**Proof:** Using the same idea as for the two-way mode.

**Theorem:**

Let \( n \) even. Then we have: \( r(K(n)) \geq 2 + \lceil \log_{\frac{1}{2}(1+\sqrt{5})} \frac{n}{2} \rceil \).

**Proof:** See literature (Idea is given the following).

<table>
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<th>6</th>
<th>8</th>
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<th>12</th>
<th>14</th>
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<tbody>
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<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
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</tr>
<tr>
<td>Lower Bound</td>
<td>2</td>
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</tr>
</tbody>
</table>
Idea for the lower Bound

- **Situation:**
  - Algorithm with “fibonacci growth”.
  - No idea to enlarge this growth.

- **Construction of a lower bound:**
  - Start with an arbitrary algorithm.
  - Use only the restriction of the algorithm.
  - Abstract.

  - We will now try to do the abstraction.

- **Try the get the core-problem.**

- **The core-problem ist:**
  - “Fibonacci growth” could not be improved.
1. Abstraction

Definition:

The Network Counting Problem:

- Given a directed graph $G = (V, E)$.
- Each node stores a number.
- Initial just the number 1 is stored.
- The receiver add the number from the sender to his number after one communication.
- The objective is: all nodes should store a number larger than $|V|$.
- With $nc(G)$ we denote the minimal rounds to achieve this objective.

Lemma:

For any graph $G$ we have: $r(G) \geq nc(G)$. 
2. Abstraction

- Let \( G = (\{v_1, v_2, v_3, \ldots, v_n\}, E) \) be a directed Graph.
- Each node \( v_i \) stores after \( t \) rounds the number \( z^t_i \).
- One situation of the network counting problem could be described by a vector:
  - Initial: \((1, 1, 1, \ldots, 1)^T\).
  - After \( t \) rounds: \((z^t_1, z^t_2, z^t_3, z^t_n)^T\).
- One round of an algorithm for the network counting problem is given by a matrix \( B \):
  - \( A \) is a \( n \times n \) matrix.
  - \( a_{ij} = 1 \) node \( j \) sends to node \( i \).
  - \( A \) contains on the diagonal only ones.
  - \( A \) has in each row at most two ones.
  - \( A \) has in each column at most two ones.
  - If \( a_{ij} = a_{kl} = 1 \) \((i \neq j \neq k \neq l)\), then we have \( l \neq i \neq k \) and \( l \neq j \neq k \).
  - Thus we get: \( A \cdot (z^t_1, z^t_2, z^t_3, z^t_n)^T = (z^{t+1}_1, z^{t+1}_2, z^{t+1}_3, z^{t+1}_n)^T \).
2. Abstraction (Continuation)

- We consider now matrices of the above form.
- These are matrices $A$, for which there is a transformation $T$ with:

$$TAT^{-1} = \begin{pmatrix} B & 0 \\ B & 0 \\ \vdots & \ddots & B \\ 0 & \cdots & 1 \\ 0 & \cdots & 1 \end{pmatrix}.$$  

and $B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

- We will estimate the growth, which these matrices provide for the network counting problem.
Recollection (Norm, 3. Abstraction)

- Let $\|\cdot\|$ be the vector norm over $\mathbb{R}^n$. Then we have:
  - $\|x\| = 0 \iff x = 0^n$,
  - $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$,
  - $\|x + y\| \leq \|x\| + \|y\|$
  - this holds for all $\alpha \in \mathbb{R}, x, y \in \mathbb{R}^n$

- The matrix norm for a vector norm $\|\cdot\|$ is defined by $\|A\| = \sup_{x \neq 0} \frac{|Ax|}{\|x\|}$. Then we have:
  - $\|A\| = 0 \iff A = 0$
  - $\|A + B\| \leq \|A\| + \|B\|$
  - $\|\alpha A\| = \alpha \cdot \|A\|$
  - $\|A \cdot B\| \leq \|A\| \cdot \|B\|$
  - $\|A \cdot x\| \leq \|A\| \cdot \|x\|$
  - this holds for all $A, B \in \mathbb{R}^{n^2}, x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \alpha \geq 0$.

- Here we use: $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ for $\in \mathbb{R}^n$.

- Known: $\|A\| = \text{Spectral Norm}(A) = \sqrt{|\lambda_{\text{max}}(A^T \cdot A)|}$ with: $\lambda_{\text{max}}$ is the largest Eigenvalue.
2. Abstraction (Continuation)

- We compute the spectral norm:
  \[ \|A\| = \|TAT^{-1}\| = \|B\| . \]

- \[ B^T \cdot B = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \begin{pmatrix} 11 \\ 01 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix} . \]

- \[ \Rightarrow (2 - \lambda)(1 - \lambda) - 1 = 0 \]

- \[ \Rightarrow \lambda^2 - 3\lambda + 1 = 0 \]

- \[ \Rightarrow \lambda_{\text{max}}(B^TB) = \frac{3}{2} + \sqrt{\frac{5}{4}} \]

- \[ \|A\| = \sqrt{\lambda_{\text{max}}(A^TA)} = \frac{1}{2}(1 + \sqrt{5}) \]
Theorem:

A algorithm, solving the network counting problem needs $2 + \lceil \log_{1/2}(1 + \sqrt{5}) \frac{n}{2} \rceil$ rounds.

Proof:

- Let $A_j, 1 \leq j \leq r$ be matrices, which solve the problem in $r$ rounds.
- $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = A_{r-2} \cdot \cdots \cdot A_2 \cdot A_1 \cdot (1, 1, \cdots, 1)$.
- $||\alpha|| \leq \left( \prod_{i=1}^{r-2} ||A_i|| \right) \cdot ||(1, \ldots, 1)|| \leq (\frac{1}{2}(1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}$
- Let $\inf(i, t)$ be the number, which have the nodes $v_i$ after $t$ rounds.
- After round $t$ we have: $\inf(i, t) \geq n$ for all $i \in \{1, 2, \cdots, n\}$.
- After round $t - 1$ we have: $\inf(i, t - 1) \geq n$ for at least $n/2$ nodes.
- There could be some $i$ with: $\inf(i, t - 2) \geq n$.
- But if $\alpha_i < n$ and $\inf(i, t - 1) \geq n$, then there exists $j$ with: $\alpha_i + \alpha_j \geq n$. 
Let

- $c_1$ be the number of cases with: $\alpha_i \geq n$,
- $c_2$ be the number of cases with: $\alpha_i < n$ and $\alpha_j \geq n$,
- $c_3$ be the number of cases with: $\alpha_i < n$, $\alpha_j < n$ and $\alpha_i + \alpha_j \geq n$.

Then we have: $c_1 \geq c_2$ and $c_1 + c_2 + c_3 \geq n/2$.

Thus we also get: $2c_1 + c_3 \geq \frac{n}{2}$

$||\alpha|| = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \sqrt{c_1 n^2 + c_3 \cdot 2 \cdot \frac{n^2}{4}} \geq n \cdot \sqrt{\frac{1}{2} (2c_1 + c_3)} \geq \frac{n}{2} \sqrt{n}$.

We already have:

$||\alpha|| \leq (\prod_{i=1}^{r-2} ||A_i||) \cdot ||(1, \ldots, 1)|| \leq (\frac{1}{2} (1 + \sqrt{5}))^{r-2} \cdot \sqrt{n}$.

And we get:

$\frac{n}{2} \cdot \sqrt{n} \leq ||\alpha|| \leq \Phi^{r-2} \cdot \sqrt{n}$,

From which we conclude:

$r \geq 2 + \lceil \log_{\frac{1}{2} (1 + \sqrt{5})} \frac{n}{2} \rceil$
Quality of these Bounds

Lemma:

Let \( n = 2m \) and let:

- \( t_1 := 1 + k \), with \( k \) is the smallest number with \( m \leq F(k) \) and
- \( t_2 := 2 + \lceil \log_{\frac{1}{2}}(1 + \sqrt{5}) \cdot m \rceil \).

Then we have \( t_1 = t_2 \) for infinite many \( m \) and \( t_1 \leq t_2 + 1 \) for all \( m \).

Proof:

- Let \( \Phi = \frac{1}{2} (1 + \sqrt{5}) \).
- Then we have: \( \Phi^2 = \Phi + 1 \).
- Furthermore we have \( \Phi^{i-2} \leq F(i) \leq \Phi^{i-1} \) for all \( i \geq 2 \).
- Consider \( n \in \mathbb{N} \) with: \( n = 2 \cdot F(k) \) for some \( k \).
  - Then we have: \( t_1 = k + 1 \) and \( t_2 = 2 + \lceil \log_{\Phi} F(k) \rceil = 2 + k - 1 = k + 1 \).
  - From which we get: \( t_1 = t_2 \) for these \( n \).
Quality of these Bounds (Part 2)

Lemma:

Let $n = 2m$ and let:

- $t_1 := 1 + k$, with $k$ is the smallest number with $m \leq F(k)$ and
- $t_2 := 2 + \lceil \log_{\frac{1}{2}}(1 + \sqrt{5}) m \rceil$.

Then we have $t_1 = t_2$ for infinite many $m$ and $t_1 \leq t_2 + 1$ for all $m$.

Proof:

- Setze $\Phi = \frac{1}{2}(1 + \sqrt{5})$.
- Then we have $\Phi^{i-2} \leq F(i) \leq \Phi^{i-1}$ for all $i \geq 2$.
- Let $n = 2 \cdot m$ arbitrary.
  - Let $i$ be defined by: $\Phi^{i-1} < m \leq \Phi^i$, then we have: $t_2 = 2 + i$.
  - Let $k$ be the smallest number with $F(k) \geq m$.
  - Note: $\Phi^{k-2} \leq F(k) \leq \Phi^{k-1}$.
  - Then we have: $i = k - 1$ oder $i = k - 2$.
  - From which we conclude: $t_1 = k + 1 \leq i + 3$. 
### Summary (Telefon-Mode)

| Graph   | |V| | diam | Lower Bound | Upper Bound |
|---------|----------------|-------|--------|-------------|-------------|
| $K_n$   | $n$            | 1     | $\lceil \log_2 n \rceil + odd(n)$ | $\lceil \log_2 n \rceil + odd(n)$ |
| $H_k$   | $2^k$          | $k$   | $n - even(n)$ | $n - even(n)$ |
| $P_n$   | $n$            | $n - 1$ | $\lceil n/2 \rceil + odd(n)$ | $\lceil n/2 \rceil + odd(n)$ |
| $C_n$   | $n$            | $\lfloor n/2 \rfloor$ | $\lceil 5k/2 \rceil + 1$, $k$ odd | $\lceil 5k/2 \rceil - 2$, $k$ even |
| $CCC_k$ | $k \cdot 2^k$ | $\lfloor 5k/2 \rfloor - 2$ | $2k - 1$ | $2k + 5$ |
| $SE_k$  | $2^k$          | $2k - 1$ | $1.9770k$ | $2.25 \cdot k + o(k)$ |
| $BF_k$  | $k \cdot 2^k$ | $\lfloor 3k/2 \rfloor$ | $1.5965k$ | $2k + 5$ |
| $DB_k$  | $2^k$          | $k$   | $1.9770k$ | $2k + 5$ |
### Summary (Telegraph-Mode)

| Graph    | $|V|$          | diam $	ext{diam}$ | Lower Bound  | Upper Bound  |
|----------|---------------|--------------------|--------------|--------------|
| $K_n$    | $n$           | 1                  | $1.44 \log_2 n$ | $1.44 \log_2 n$ |
| $H_k$    | $2^k$         | $k$                | $1.44k$      | $1.88k$      |
| $P_n$    | $n$           | $n - 1$            | $n + \text{odd}(n)$ | $n + \text{odd}(n)$ |
| $C_n$    | $n$ even      | ⌊$\frac{n}{2}$⌋   | $\frac{n}{2} + \lceil \sqrt{2n} \rceil - 1$ | $\frac{n}{2} + \lceil \sqrt{2n} \rceil - 1$ |
|          | $n$ odd       | ⌊$\frac{n}{2}$⌋   | $\lceil \frac{n}{2} \rceil + \lceil \sqrt{2n} - \frac{1}{2} \rceil - 1$ | $\lceil \frac{n}{2} \rceil + \lceil \sqrt{\frac{n}{2}} \rceil - 1$ |
| $CCC_k$  | $k \cdot 2^k$| ⌊$\frac{5k}{2}$⌋  | ⌊$\frac{5k}{2}$⌋  | ⌊$\frac{7k}{2}$⌋  |
|          |               | $- 2$              | $- 2$        | $- 2$        |
| $SE_k$   | $2^k$         | $2k - 1$           | $2k - 1$    | $3k + 3$    |
| $BF_k$   | $k \cdot 2^k$| ⌊$\frac{3k}{2}$⌋  | $1.9770k$   | $\lceil \frac{5k}{2} \rceil + \lceil \sqrt{\frac{k}{2}} \rceil - 1$ |
|          | $2^k$         | $k$                | $1.5965k$   | $3k + 3$    |
| $DB_k$   | $2^k$         | $k$                | $1.5965k$   | $3k + 3$    |
J. Hromkovič, et al.:
Dissemination of Information in Communication Networks:
Broadcasting, Gossiping, Leader Election, and Fault-Tolerance.
Legend

- : Not of relevance
- : implicitly used basics
- : idea of proof or algorithm
- : structure of proof or algorithm
- : Full knowledge