Evolutionary Game Theory

Informatik 1
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Note: Only some aspects of Section 2 were addressed in the lecture. Section 3 was not treated at all and is included only as a reference.
2. Modeling Agents’ Behavior

1 Introduction

So far we have studied various aspects of equilibria. However, these results are based on standard assumptions of game theory that can hardly be justified when it comes to games involving something as large and inscrutable as the Internet. We assume full rationality, complete and accurate knowledge about network topology, latency functions, and the behavior of the other agents, and expect the agents to make quite some computational effort to find equilibria.

Here, we want to get rid of most of these assumptions by modeling the agents’ behavior by a process requiring very little knowledge and computational effort, if any. Furthermore, this allows for the analysis of dynamical properties of repeatedly played games.

While the main part of this manuscript will be applicable to games in general, the last section will focus on applications to selfish routing. For a more comprehensive introduction to evolutionary game theory and population dynamics see for example [5, 11, 13].

2 Modeling Agents’ Behavior

We start by describing two simple discrete stochastic processes which will – in the so-called fluid limit – lead to a system of differential equations which we will examine throughout the rest of this chapter and which will also give rise to a new concept of stability.

2.1 Discrete Processes

The aspect in which classical and evolutionary game theory differ most is the setting in which games are played. Whereas in classical game theory games are usually played only once, evolutionary game theory assumes that there is an infinite number of players forming a population. These players are repeatedly and randomly paired to play the game under study. Thus, players can learn to perform well in the game by successively gathering information about their payoff depending on the strategy chosen.

Consider a symmetric two-player game in normal form with an $m \times m$ payoff matrix $A$. By $[m] = \{1, \ldots, m\}$ we denote the set of strategies. We want to describe the dynamics of such a population of agents playing pure strategies. Denote the fraction or share of agents playing the pure strategy $i \in [m]$ by $x_i$. Clearly, playing against an agent drawn uniformly at random from population $\vec{x} = (x_i)_{i \in [m]}$ is equivalent to playing against an agent playing the mixed strategy $\vec{x}$ and yields expected payoff $(A\vec{x})_j$ for every pure strategy $j \in [m]$. The expected payoff of the population playing against itself is $\vec{x}^T A \vec{x}$. We denote the set of possible populations by the simplex $\Delta := \{\vec{x} \mid \sum_{i \in [m]} x_i = 1, \forall i \in [m] : x_i \geq 0\}$. 

2.1 Discrete Processes

We want to assume as little as possible about the knowledge and computational capabilities of the agents. In particular, the agents do not know the payoff matrix $A$. What would be a reasonable behaviour of agents to perform well in this setting?

Assume that the agents play the game repeatedly against random opponents. What one would reasonably do is to keep on playing the own strategy as long as one wins, and switch to a new strategy when the own strategy turns out to perform poorly. Consider the following decision rule. An agent plays the pure strategy $i$ against a random opponent which in turn plays strategy $j$. If $i$ earns more than $j$, nothing happens. However, if $i$ observes, that $j$ earns more than herself, our agent imitates her opponent with a probability proportional to the payoff difference. Given this discrete process and an initial population $\mathbf{x}$, we can analyze the expected population shares $\mathbf{x}'$ after one step. The expected growth of population share $x_i$ in one step is $E[x'_i - x_i] = E[x'_i] - x_i$:

$$
E[x'_i] - x_i = -x_i \sum_{j \in [m]: (A \mathbf{x})_j > (A \mathbf{x})_i} x_j \cdot \lambda(\mathbf{x}) \cdot ((A \mathbf{x})_j - (A \mathbf{x})_i) + \sum_{j \in [m]: (A \mathbf{x})_j < (A \mathbf{x})_i} x_j x_i \cdot \lambda(\mathbf{x}) \cdot ((A \mathbf{x})_i - (A \mathbf{x})_j)
$$

for some function $\lambda : \Delta \rightarrow \mathbb{R}$. The first sum is the expected number of agents lost to other strategies with higher payoff. The probability that our agent samples an agent playing strategy $j$ is given by $x_j$. Finally, $(A \mathbf{x})_j - (A \mathbf{x})_i$ is the payoff difference, and multiplying with some factor $\lambda(\mathbf{x})$ gives the probability to imitate the opponent. Multiplying the overall probability with the number of agents currently playing strategy $i$ yields the expected number of agents switching from strategy $i$ to some other, better strategy. The function $\lambda$ accounts for normalizing the probabilities such that they are bounded from above by 1 as well as for any other scalar factors like the rate at which players are paired. Conversely, the second sum is the number of agents that formerly played a different strategy and now change to strategy $i$. It is constructed in a similar way. We see that we can combine the two sums, yielding

$$
E[x'_i] - x_i = \sum_{j \in [m]} x_i x_j \cdot \lambda(\mathbf{x}) \cdot ((A \mathbf{x})_i - (A \mathbf{x})_j)
$$

$$
= \lambda(\mathbf{x}) \cdot x_i \cdot \left( (A \mathbf{x})_i \sum_{j \in [m]} x_j - \sum_{j \in [m]} x_j \cdot (A \mathbf{x})_j \right)
$$

$$
= \lambda(\mathbf{x}) \cdot x_i \cdot \left( (A \mathbf{x})_i - \mathbf{x}^T \mathbf{A} \mathbf{x} \right),
$$

since $\sum_{j \in [m]} x_j = 1$. 
2. Modeling Agents’ Behavior

One may argue that this process still requires the agents to have too much information, e.g. their opponents’ payoff. The following process does not even use this information.

Again, an agent plays against a random opponent. However, before doing this, she chooses an aspiration level at random in the range \( [\underline{a}, \overline{a}] \), where \( \underline{a} = \min_{i,j} \{ a_{ij} \} \) and \( \overline{a} = \max_{i,j} \{ a_{ij} \} \). If the agent’s payoff is above its aspiration level, she keeps on playing the old strategy. If she falls short of this level, she imitates her opponent, no matter what their payoff is (after all, the opponent’s payoff is not even observed). Again, we can calculate the expected population shares after one step. Let \( L \) be the random variable describing the aspiration level.

\[
E[x'_i] - x_i = -x_i \cdot \Pr((A\vec{x})_i < L) + \sum_{j \in [m]} x_i \cdot x_j \cdot \Pr((A\vec{x})_j < L)
\]

\[
= -x_i \cdot \frac{(\overline{a} - a) - ((A\vec{x})_i - a)}{\overline{a} - a} + \sum_{j \in [m]} x_i x_j \cdot \frac{(\overline{a} - a) - ((A\vec{x})_j - a)}{\overline{a} - a}
\]

\[
= \frac{1}{\overline{a} - a} \left( -x_i \overline{\pi} + x_i (A\overline{x})_i + x_i \overline{\pi} \sum_{j \in [m]} x_j - x_i \sum_{j \in [m]} x_j (A\overline{x})_j \right)
\]

\[
= \frac{1}{\overline{a} - a} \cdot x_i \cdot ((A\vec{x})_i - \vec{x}' A\vec{x})
\]

Again, the first term gives the number of agents leaving strategy \( i \) due to falling short of the random aspiration level \( L \) and the second term gives the number of agents switching to strategy \( i \) from other strategies. Of course, other proportionality factors are also possible, again introducing an arbitrary function \( \lambda(\vec{x}) \) into this equation.

Surprisingly, up to a scalar factor, both processes lead to the same expected population shares after one step. There are even more processes yielding the same result.

2.2 Fluid Limit – The Replicator Dynamics

Based on the expectation values derived in the preceding section, we now go to the fluid limit, i.e., we describe the population vector as a function of time \( \vec{x} : t \mapsto \Delta \) and, letting the number of agents go to infinity, we replace the random variables by their expectation values. Thus, we obtain the following system of differential equations which is the well-known replicator dynamics.

**Definition 1 (Replicator dynamics).** For any function \( \lambda(\vec{x}) \), the replicator dynamics is given by the system of differential equations

\[
\dot{x}_i(t) = \lambda(\vec{x}(t)) \cdot x_i(t) \cdot ((A\vec{x}(t))_i - \vec{x}(t) \cdot A\vec{x}(t)).
\]

for all \( i \in [m] \).
2.2 Fluid Limit – The Replicator Dynamics

Here, $\dot{x}_i$ is the derivative with respect to time. In the following we omit the argument $t$. The function $\lambda(\vec{x})$ is a free parameter in the equation. If we are only interested in the solution orbits of this equation, i.e., the set $\{\vec{y} | \exists t \geq 0 : \vec{y} = \vec{x}(t)\}$ then the choice of $\lambda$ is irrelevant. It merely determines the speed at which the orbit is traversed. In evolutionary game theory, the function $\lambda(\vec{x})$ is usually set to the constant function $\lambda(\vec{x}) = 1$.

We should check that solutions to equation (1) actually exist and produce legal population dynamics. Denote the right hand side of the equation by $F_i(\vec{x}) = \lambda(\vec{x}) \cdot x_i \cdot ((\vec{\mathbf{A}} \vec{x}))_i - \vec{x}^T \vec{\mathbf{A}} \vec{x})$. The partial derivatives $\partial F_i / \partial x_j$ exist for all $i, j \in [m]$ and are bounded for the entire domain $\Delta$. Therefore, by the Picard–Lindelöf Theorem [2], a solution to (1) exists and is unique. Summing up the $\dot{x}_i$, we see that equation (1) satisfies $\sum_{i \in [m]} \dot{x}_i = 0$ and therefore, solutions of the differential equations stay inside $\Delta$, i.e., $\vec{x}(t) \in \Delta$ for all $t$.

If we assume that agents are homogeneous, i.e., behave in the same way, then it is useful to write the replicator dynamics as

$$\dot{x}_i = x_i \cdot g_i(\vec{x}),$$

where $g_i(\vec{x}) = \lambda(\vec{x}) \cdot ((\vec{\mathbf{A}} \vec{x}))_i - \vec{x}^T \vec{\mathbf{A}} \vec{x})$ is the growth rate of strategy $i$ per agent. In the following we collect some basic properties of the replicator dynamics.

1. *Monotonicity*. The growth rates $g_i$ satisfy a monotonicity condition in the sense that higher payoff causes faster growth and vice versa, i.e.,

$$((\vec{\mathbf{A}} \vec{x}))_i > ((\vec{\mathbf{A}} \vec{x}))_j \iff g_i(\vec{x}) > g_j(\vec{x})$$

for all $i \in [m]$.

2. *Aggregate monotonicity*. Furthermore, (sub-)populations with an overall higher payoff have an overall faster growth rate.

$$\vec{y}^T \vec{\mathbf{A}} \vec{x} > \vec{z}^T \vec{\mathbf{A}} \vec{x} \iff \vec{y} \cdot g(\vec{x}) > \vec{z} \cdot g(\vec{x}).$$

It has been shown that every aggregate monotonic dynamics can be expressed in the form of equation (1) for some choice of $\lambda(\vec{x})$ [9].

3. *Invariance of interior and boundary*. The interior and the boundary of the simplex are invariant, i.e., if $x_i(t) = 0$ for some $t$, then $x_i(t) = 0$ for all $t$, and if $x_i(t) > 0$ for some $t$, then $x_i(t) > 0$ for all $t$. This means that no new strategies are discovered by the replicator dynamics and strategies present in the initial population never get completely extinct. One could consider this lack of innovation the most important drawback of the replicator dynamics.

**Example 1 (Orbits for the rock-scissors-paper game).** The rock-scissors-paper game is a two-player game often played by children. Each player simultaneously shows one out of the three symbols “rock”, “scissors”, and “stone”. Each symbol wins two points against exactly one other symbol: scissors cut paper, stone breaks scissors and paper wraps around the stone. Two identical symbols are a draw (one point each). The loser gets no points. The payoff matrix looks as follows:

$$
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1 \\
\end{bmatrix}
$$

Clearly, any biased mixed strategy, or population, cannot be a Nash equilibrium, since playing the respective winning symbol as a pure strategy would yield a higher payoff. The only Nash equilibrium is therefore the vector $\bar{x} = (1/3, 1/3, 1/3)$. Then, playing an arbitrary strategy against $\bar{x}$ yields expected payoff 1.

How do solutions of the replicator dynamics with respect to this game look like? One could expect that the solution orbit sooner or later converges towards $\bar{x}$. However, that is not the case. Figure 1(a) shows the solution orbit. Topologically, this orbit is a circle. The Nash equilibrium is not approached. However, a slight modification to the game can change the situation dramatically. If we change the winner’s payoff to $2 + \epsilon$, we get the modified rock-scissors-paper game with payoff matrix

$$
\begin{bmatrix}
1 & 2 + \epsilon & 0 \\
0 & 1 & 2 + \epsilon \\
2 + \epsilon & 0 & 1 \\
\end{bmatrix}
$$

As depicted in Figure 1(b), if $\epsilon > 0$, the orbit converges towards $\bar{x}$ as we might have already expected for the standard game. However, when we choose $\epsilon < 0$, the orbit even diverges towards the boundary of the simplex, as depicted in Figure 1(c). Topologically, these two orbits are simply lines.

We see that the concept of Nash equilibria is no longer appropriate in this new dynamical scenario.

### 2.3 New Concepts for Stability

As the previous example has shown, the concept of Nash equilibria has to be refined in order to be meaningful in the framework of evolutionary game theory. We must define new criteria for stability.
2.3 New Concepts for Stability

Figure 1: Solutions for three versions of the rock-scissors-paper game. The corners of the simplex are the unit vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ corresponding to the three pure strategies.
When talking about differential equations, a natural concept of stability is the concept of critical points, also called fixed points or rest points. A critical point of the replicator dynamics is a vector $\mathbf{x}^* \in \Delta$ for which every right hand side of the system of differential equations vanishes, i.e., $\dot{x}_i = 0$ for all $i \in [m]$. A critical point $\mathbf{x}^*$ may be stable in the sense that for every sphere $S_1$ around $\mathbf{x}^*$ with radius $\epsilon > 0$ there exists a (possibly smaller) sphere $S_2$ with radius $\delta > 0$ such that for all starting points inside $S_2$ solutions do not leave $S_1$. It may also be asymptotically stable in that furthermore there exists a sphere $S_3$ with radius $\delta_0$, $0 < \delta_0 < \delta$, such that all solutions with starting points inside $S_3$ approach $\mathbf{x}^*$ as $t$ goes to infinity. A critical point that is not stable is unstable.

The above concepts are known from the theory of differential equations and are not directly related to evolutionary game theory. In evolutionary game theory, the following definition of stability was established. Later we will see how these two concepts are related.

**Definition 2 (Evolutionary stable).** A population vector $\mathbf{x}$ is said to be evolutionary stable if for every strategy $\mathbf{y} \neq \mathbf{x}$ there exists some constant $\epsilon_0 \in (0, 1)$ such that for every $\epsilon \in (0, \epsilon_0)$ it holds that $\mathbf{x}^T \mathbf{Ax} > \mathbf{y}^T \mathbf{Ax}$ where $\mathbf{x}' = \epsilon \cdot \mathbf{y} + (1 - \epsilon) \cdot \mathbf{x}$.

Intuitively, this requires the overall payoff of an invasive population $\mathbf{y}$ against the population $\mathbf{x}'$ containing both $\mathbf{x}$ and $\mathbf{y}$ proportionately to be less than the overall payoff of the incumbent population $\mathbf{x}$ against $\mathbf{y}$. A useful characterization of evolutionary stable strategies is the following [13].

**Proposition 1.** A vector $\mathbf{x} \in \Delta$ is evolutionary stable iff (1) it is a Nash equilibrium and (2) for all best replies $\mathbf{y}$ to $\mathbf{x}$, $\mathbf{y} \neq \mathbf{x}$ it holds that $\mathbf{x}^T \mathbf{Ay} < \mathbf{y}^T \mathbf{Ay}$.

The second condition says the following. Assume that we have an incumbent population $\mathbf{x}$ which is a Nash equilibrium and an invasive population $\mathbf{y}$. When population $\mathbf{y}$ replaces the old population $\mathbf{x}$, afterwards it has to play against itself. If the overall payoff $\mathbf{y}^T \mathbf{Ay}$ of $\mathbf{y}$ playing against itself is less than the overall payoff of $\mathbf{x}$ playing against $\mathbf{y}$, then $\mathbf{y}$ will not be able to replace $\mathbf{x}$.

**Example 2 (Evolutionary stability of the rock-scissors-paper game).** We have already seen that solutions of the replicator dynamics for the rock-scissors-paper game do not converge towards the unique Nash equilibrium. Assume that the incumbent strategy $\mathbf{x}$ is the Nash equilibrium $(1/3, 1/3, 1/3)$. Clearly, whatever the invasive population $\mathbf{y}$ plays, the overall expected payoff that $\mathbf{x}$ earns against $\mathbf{y}$ is exactly 1. However, the expected payoff that $\mathbf{y}$ earns against itself is also 1. Therefore, in the standard rock-scissors-paper game the Nash equilibrium is not evolutionary stable.
Let us look at the modified versions with payoff $2+\epsilon$ for the winner. The payoff of $\tilde{x}$ against any invasive population $\tilde{y}$ is $1+\epsilon/3$, independently of $\tilde{y}$. However, the payoff of $\tilde{y}$ against itself is $y_1^T \tilde{A} \tilde{y} = 1 + (y_1 + y_2 - y_1^2 - y_2^2 - y_1 y_2) \epsilon$. (Note that $y_3$ is determined by $y_1$ and $y_2$.) Since $y_1 + y_2 - y_1^2 - y_2^2 - y_1 y_2 \leq 1/3$ with equality only for $y_1 = y_2 = y_3 = 1$, the payoff $y_1^T \tilde{A} \tilde{y}$ is less than $\tilde{x}^T \tilde{A} \tilde{x} = 1 + \epsilon/3$ if $\epsilon > 0$ and greater than $1 + \epsilon/3$ if $\epsilon < 0$. Therefore, in the modified game with $\epsilon > 0$ the Nash equilibrium is evolutionary stable whereas in the modified game with $\epsilon < 0$ it is not. This corresponds to the solution orbits we already observed.

2.4 Convergence of Orbits – Lyapunov’s Method

One way to prove convergence towards a critical point is Lyapunov’s second method. Since this method allows for such a proof without explicitly knowing the actual solution of the differential equation, it is also known as Lyapunov’s direct method. We give a brief sketch of the geometric proof of its correctness here. For a detailed proof see, e.g., [2]. Consider a system of differential equations $\frac{\text{d}x}{\text{d}t} = f(x)$. For this system we try to exhibit a function $V(x)$ with the following properties:

1. $V$ has continuous first partial derivatives,
2. $V$ is positive definite, i.e., $V(\bar{0}) = 0$ and $V(x) > 0$ for $x \neq \bar{0}$,
3. $\dot{V}$ is continuous, and
4. $\dot{V}$ is negative definite, i.e., $\dot{V}(\bar{0}) = 0$ and $\dot{V}(x) < 0$ for $x \neq \bar{0}$.

We call this function a Lyapunov function. The existence of such a function $V$, defined on a neighbourhood of the origin, implies that solutions $\frac{\text{d}x}{\text{d}t} = f(x)$ converge towards $\bar{0}$ when starting close enough to $\bar{x}$, i.e., $\bar{x}$ is asymptotically stable. Of course we can also show convergence towards points other than $\bar{0}$ by translating coordinates such that the critical point coincides with the origin.

Now, how can we show the correctness of this method? Recall that asymptotic stability requires the existence of a sphere $S_3$ with radius $\delta_0$ such that starting inside $S_3$ the solution orbit approaches the origin. We choose the radius $\delta_0$ in the following manner. First we look at the sets $V_\epsilon := \{x : V(x) = \epsilon\}$. Once the solution orbit is inside $V_\epsilon$, it cannot escape from it, since $V \leq 0$ and by continuity of the orbits. For every sphere $S_1$ we can choose the radius $\delta$ of $S_2$ such that $S_1$ is entirely contained within $\sup_{x \in S_1} \{V(x)\}$. Then, solutions cannot escape $S_2$. This way, we can already establish stability.

To prove asymptotic stability, consider a solution $\tilde{x}(t)$ of the differential equation with boundary condition $\tilde{x}(0) = \tilde{x}_0$ with $\tilde{x}_0 \in S_3$. We show that
2. Modeling Agents’ Behavior

Figure 2: The Lyapunov method: Every solution enters the set $C$ and remains inside.

the distance of the solution $\vec{\xi}(t)$ to the origin, denoted by $|\vec{\xi}(t)|$, falls below every value $d$ and remains there for all future time. Denote the sphere with radius $d$ around the origin by $D := \{ \vec{x} \in S_3 : |\vec{x}| \leq d \}$. Denote the infimum of all values of $V$ outside this sphere by $I := \inf_{\vec{x} \in S_3 \setminus D} V(\vec{x})$. Finally, denote the set of all vectors for which the value of $V$ is bounded by $I$ by $C := \{ \vec{x} \in S_3 | V(\vec{x}) \leq I \}$. The situation is depicted in Figure 2. Clearly, $C$ is entirely contained within $D$, and the borders of $D$ and $C$ may partially coincide. Furthermore, $\sup_{\vec{x} \notin C} \{ V(\vec{x}) \} \leq -\epsilon < 0$ for some $\epsilon > 0$ by negative definiteness and continuity of $V$ and since $C$ contains a neighborhood of $\vec{0}$. This implies that, as long as we are outside $C$, the value of $V$ decreases by at least $\epsilon$ per time unit, and therefore at some point of time $V(\vec{\xi}(t)) \leq I$ which means that $\vec{\xi}(t) \in C$ and $\vec{\xi}$ will never leave $C$ again. Especially, $\vec{\xi}(t) \in D$ and thus $|\vec{\xi}(t)| \leq d$, which is our claim.

By using more properties of the Lyapunov function, we will see, that it is also possible to show global convergence to the origin applying this method.

**Example 3 (Convergence of the rock-scissors-paper game).** We apply Lyapunov’s method to the rock-scissors-paper game. Let $\vec{\xi}(t)$ be a solution to the replicator dynamics starting in the interior of the simplex. Denote the Nash equilibrium by $\vec{x}^* = (1/3, 1/3, 1/3)$. We use the conditional entropy as a Lyapunov function $V$:

$$H(\vec{\xi}) = \sum_{i=1}^{3} x_i^* \cdot \ln \frac{x_i^*}{\xi_i}.$$ 

The entropy is 0 if $\vec{\xi} = \vec{x}^*$ and positive otherwise. Applying the chain rule
we get
\[ \dot{H}(\xi) = -\frac{1}{3} \sum_{i=1}^{3} \frac{\dot{\xi}_i}{\xi_i} \]

Now we substitute the replicator dynamics into this equation and get
\[ \dot{H}(\xi) = -\frac{1}{3} \sum_{i=1}^{3} (\xi_i^2 \cdot A^\xi A^\xi) \]

and setting \( \xi_3 = 1 - \xi_1 - \xi_2 \) we obtain
\[ \dot{H}(\xi) = -(1/3 + \xi_1 \xi_2 + \xi_1^2 + \xi_2^2 - \xi_1 - \xi_2) \epsilon. \]

If \( \xi_1 = \xi_2 = \xi_3 = 1/3 \), i.e., \( \xi = \bar{x}^* \), then this term is 0. Otherwise it is non-zero and has the opposite sign of \( \epsilon \).

Consider the case \( \epsilon > 0 \). Recall that \( \bar{x}^* \) is evolutionary stable in this case. After translating the system such that \( \bar{x}^* \) coincides with the origin, \( H \) is positive definite and \( \dot{H} \) is negative definite. Therefore, Lyapunov’s method applies and we know that all solutions \( \xi \) must approach \( \bar{x}^* \).

The following theorem summarizes the relationship between equilibria on the one hand and critical points of the replicator dynamics on the other hand. It is sometimes referred to as the folk theorem of evolutionary game theory (see, e.g., [5, 12]) and can be proved using similar methods.

**Theorem 1.** The following statements hold.

1. Nash equilibria are critical points.
2. Strict Nash equilibria are asymptotically stable.
3. Critical points that are limits of interior orbits are Nash equilibria.
4. Stable critical points are Nash equilibria.

### 2.5 Other Dynamics

Several other types of dynamics are discussed in evolutionary game theory, all of which have different advantages and drawbacks. For more details on the dynamics considered below, see [6].

**Imitation dynamics** The replicator dynamics which was discussed in the preceding section is the most popular variant of a wider class of dynamics known as imitation dynamics. Recall our process mentioned at the beginning where agents are assigned a random partner, play the game and imitate the opponent’s strategy with a probability proportional to the payoff difference. A more general approach would be to assume that agents imitate their
2. Modeling Agents’ Behavior

opponent with a probability which is a function $f_{ij} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ depending on the payoffs $(A\vec{x})_i$ and $(A\vec{x})_j$. Then we get the dynamics

$$\dot{x}_i = x_i \sum_{j \in [m]} x_j \cdot (f_{ij}((A\vec{x})_i, (A\vec{x})_j) - f_{ji}((A\vec{x})_j, (A\vec{x})_i))$$

A reasonable selfish behavior would be to imitate the opponent iff its payoff is greater than the own payoff, i.e., $f(u, v) = 1$ if $u < v$ and $f(u, v) = 0$ otherwise. Note that this causes a discontinuity in the right hand side of the differential equation. We call this dynamics the better response dynamics.

One could also require that the function $f(u, v)$ is invariant under addition of terms to $u$ and $v$ yielding a function $f(u - v)$ only depending on the payoff difference. Then, choosing $f(w) = \max(0, \lambda \cdot w)$ yields the replicator dynamics.

**Best response dynamics** All of these imitation dynamics require agents simply to observe the payoff difference. We could give the agents a slightly stronger computational power enabling them to base their decisions on competent play instead of imitation only. The ability to compute the best reply, which is often cheap, gives rise to the best response dynamics. Here, at every step of time, agents choose a best reply to the current population from $\beta(\vec{x})$. Since best replies are not necessarily unique, this does not yield a differential equation, but rather a differential inclusion:

$$\dot{\vec{x}} \in \beta(\vec{x}) - \vec{x}.$$ 

Generally, solutions to this differential inclusion are not unique. Assume that there exists a uniquely determined best reply $\vec{\beta}$. Then, as long as $\vec{\beta}$ stays a best reply, $\dot{\vec{x}} = \vec{\beta} - \vec{x}$ and we get a linear solution orbit

$$\vec{x}(t) = (1 - e^{-t})\vec{\beta} + e^{-t}\vec{x}$$

which approaches $\vec{\beta}$ exponentially fast. Since there always exists a best reply which also is a best reply to itself, there is a piecewise linear solution orbit by iterating the above construction. In contrast to all dynamics of the class of imitation dynamics, the best response dynamics is innovative in that it discovers unused strategies.

**Logit dynamics** A drawback of the best response dynamics is that it has no unique solution. One way out of this is a smoothing of the best-response dynamics which leads, among other possibilities, to the logit dynamics

$$\dot{x}_i = \frac{e^{(A\vec{x})_i/\epsilon}}{\sum_{j \in [m]} e^{(A\vec{x})_j/\epsilon}} - x_i.$$
If the pure strategy $k$ is a unique best reply to $\bar{x}$ then the $k$-th term dominates the sum in the denominator and the fraction approaches 1 for $i = k$ and 0 for $i \neq k$ as $\epsilon > 0$ approaches 0. Therefore, the logit dynamics is an approximation of the best response dynamics if $\epsilon$ is small.

If payoffs are observed under the influence of noise, this can be modeled by adding a random perturbation vector $\epsilon$ to the payoff vector. Then, the probability that strategy $i$ outperforms strategy $j$ is $\Pr((\mathbf{A}\bar{x})_i + \epsilon_i > (\mathbf{A}\bar{x})_j + \epsilon_j)$. If the $\epsilon_i$ are i.i.d. with the extreme value distribution $F(x) = \exp(-\exp(-x))$ then the best response dynamics under the influence of noise reduces to the logit dynamics. For details, see [4].

Brown-von Neumann-Nash dynamics Another dynamics which chooses better replies, is innovative, and has a continuous right-hand side is the Brown-von Neumann-Nash dynamics (BNN), which is a so-called excess payoff dynamics. Denote the positive excess payoff of strategy $i$ by

$$k_i(\bar{x}) = \max(0, (\mathbf{A}\bar{x})_i - x^T \mathbf{A}\bar{x}).$$

Then the BNN dynamics is given by

$$\dot{x}_i = k_i(\bar{x}) - x_i \sum_{j \in [m]} k_j(\bar{x}).$$

In contrast to all of the above dynamics, where players consider revising their strategies at constant rates, the class of excess payoff dynamics describes processes where players revise their strategies at a rate depending on the excess payoffs. Only when all excess payoffs are 0, i.e., at an equilibrium, this rate vanishes. Similarly, the choice functions defining a probability distribution on the strategies also depend on the excess payoffs. For choice functions where the probability to choose strategy $i$ is proportional to the excess payoff $k_i(\bar{x})$ of this strategy and the revision rate is proportional to the overall excess payoff $\sum_{j \in [m]} k_j(\bar{x})$, we get precisely the BNN dynamics [10].

Mutator dynamics The replicator dynamics can also be made innovative by adding a random mutation term. Assume that agents behave as described for the replicator dynamics but with probability $\mu$ they decide to choose a random strategy. Then we obtain the dynamics

$$\dot{x}_i = x_i \cdot ((\mathbf{A}\bar{x})_i - x^T \mathbf{A}\bar{x}) + \mu (1/n - x_i).$$

When the mutating agent does not pick a new strategy uniformly, as it is often the case in biological applications, the probability for mutating from strategy $i$ to strategy $j$ may also be given explicitly by the matrix $(q_{ij})_{i,j \in [m]}$ where typically, $q_{ii} \approx 1$ and $q_{ij} \approx 0$ if $i \neq j$. We obtain

$$\dot{x}_i = \sum_{j \in [m]} x_j q_{ji} (\mathbf{A}\bar{x})_j - x_i \cdot x^T \mathbf{A}\bar{x}.$$
3. Evolutionary Selfish Flow

Here, \( x_j q_j \) is the fraction of agents mutating from strategy \( j \) to strategy \( i \). In this dynamics, in general, critical points are not, Nash equilibria and Nash equilibria are no critical points.

We have seen that there exist a variety of dynamics all of which have plausible motivations, advantages, and drawbacks. We remark that there seems to be no general consensus as to how dynamics and evolutionary stability should be defined in the case of asymmetric games. For a more comprehensive introduction we refer to [6].

3 Evolutionary Selfish Flow

In this section we present applications of evolutionary game theory to the model of selfish routing initially introduced by Roughgarden and Tardos [8]. This was first done in [3]. So far we looked at dynamics for symmetric two-player games. Now, what would be a two-player game in this context? There is no obvious answer to this question and our proposal is to forget about games and to stick to the replicator dynamics independently of an underlying game. Still, the motivation of the replicator dynamics is valid.

A reasonable behavior for an individual agent would be the following. In each step she observes her own latency and compares it to the average latency or to the latency of a randomly chosen agent which is equivalent in the fluid limit. This information is clearly available at the source node. If her own latency is at most the average latency, she sticks to the old strategy. If it is worse, she samples a random agent and adopts its strategy with probability proportional to the difference between own and sampled latency. This yields precisely the replicator dynamics

\[
\dot{f}_P = \lambda(f) \cdot f_P \cdot (\bar{\ell}(f) - \ell_P(f)),
\]

where \( \bar{\ell}(f) \) is the average latency and \( \lambda(f) \) is some positive real function defined on \( \Delta \). Recall that \( f = (f_P)_{P \in \mathcal{P}} \) is the vector of flow paths which corresponds to the population vector \( \bar{x} = (x_i)_{i \in [m]} \). This works for the case of a single commodity. Can we generalize this to the multi-commodity case? In evolutionary game theory there is no general consensus as to how one should model such multi-population games. However, the most generally accepted model is a model in which players of the same subpopulation do not play against each other. While this may be useful in biological applications, it is not in our context. However, there is a very obvious generalization of dynamics (2) by taking the average over the paths belonging to the own commodity only. Denote the commodity of path \( P \in \mathcal{P} \) by \( i(P) \). In the case that this commodity is not uniquely defined, we can simply add another copy of the path to the network for each commodity that uses it. We get

\[
\dot{f}_P = \lambda_{i(P)}(f) \cdot f_P \cdot (\bar{\ell}_{i(P)}(f) - \ell_P(f))
\]
3.1 Basics

where we have different functions \( \lambda_i(f) \) for each commodity \( i \in I \). In Section 3.2 we will analyze this dynamics with respect to stability and convergence. We will see that the choice of the \( \lambda_i \) has no influence on the convergence behavior. In Section 3.3, where we will give bounds on the time of convergence, we will specify a reasonable choice of the \( \lambda_i \).

3.1 Basics

Since we are dealing with differential equations here, we have to take some precautions before we can proceed any further. First we must be sure that solutions to (3) do actually exist. As noted earlier in the introduction of the replicator dynamics, existence and uniqueness of solutions are guaranteed by the Picard-Lindelöf Theorem if the right hand sides of the differential equations are Lipschitz continuous. Therefore, we also have to assume that the latency functions are Lipschitz continuous, i.e., for every \( x \in [0,1] \) there exist an \( \epsilon > 0 \) and an \( \alpha > 0 \) such that for every \( x' \in [x-\epsilon,x+\epsilon] \cap [0,1] \) it holds that \( |\ell_e(x) - \ell_e(x')| \leq \alpha|x-x'| \).

In the multi-commodity case it does not suffice to show that the solution orbit does not leave the simplex. We must also show, that the overall flow sums up to the demand or rate \( r_i \) for each commodity \( i \) individually. We show that at any time the total growth rate of \( \sum_{P \in \mathcal{P}_i} f_P \) is zero for all \( i \in I \).

\[
\sum_{P \in \mathcal{P}_i} \dot{f}_P = \lambda_i(f) \cdot \sum_{P \in \mathcal{P}_i} f_P \cdot (\ell_P(f) - \bar{\ell}_i(f)) \\
= \lambda_i(f) \cdot \left( r_i \sum_{P \in \mathcal{P}_i} \frac{f_P}{r_i} \cdot \ell_P(f) - \bar{\ell}_i(f) \sum_{P \in \mathcal{P}_i} f_P \right) \\
= \lambda_i(f) \cdot (r_i \cdot \bar{\ell}_i - \bar{\ell}_i \cdot r_i) \\
= 0
\]

This implies, that \( \sum_{P \in \mathcal{P}_i} f_P = r_i \) for all times.

Showing global convergence to a Wardrop equilibrium is only possible if the Wardrop equilibrium is unique. In order to ensure this, we also assume that latency functions are strictly increasing. As we already know, Wardrop equilibria have a formulation as a convex program with variables \( f_e, e \in E \) and \( f_P, P \in \mathcal{P} \) where the objective function only depends on the \( f_e \)'s. When latency functions are strictly increasing, then the objective function is strictly convex in the \( f_e \) variables and therefore, it has a unique optimum with respect to the \( f_e \) variables. The \( f_P \) variables, however, are not necessarily uniquely defined at a Wardrop equilibrium and the set \( \{ (f_P)_{P \in \mathcal{P}} | f \text{ is a Wardrop equilibrium} \} \) is a convex set in \( \mathbb{R}^{|P|} \). Similarly, if the latency functions are only non-decreasing (but not strictly increasing), then also the \( f_e \) are not uniquely defined, but the set \( \{ (f_e)_{e \in E} | f \text{ is a Wardrop equilibrium} \} \) is a convex set in \( \mathbb{R}^{|E|} \).
Keeping this in mind, we assume that latency functions are strictly increasing in the following. It is then guaranteed, that Wardrop equilibria are essentially unique in the above sense.

**Definition 3.** We say that two flow vectors \( f \) and \( \tilde{f} \) are essentially equal, written \( f \equiv \tilde{f} \), if they are equal with respect to edge flows, i.e., \( f_e = \tilde{f}_e \), for all \( e \in E \).

### 3.2 Stability

We begin with the single-commodity scenario. We want to show that Wardrop equilibria are evolutionary stable. Our characterization of evolutionary stability translates naturally into the scenario of selfish routing.

**Definition 4 (Evolutionary stable).** A flow vector \( f \in \Delta \) is called evolutionary stable iff (1) it is a Wardrop equilibrium and (2) for all best replies \( \tilde{f} \) to \( f \), \( \tilde{f} \neq f \) it holds that \( \tilde{f} \cdot \tilde{l}(\tilde{f}) > f \cdot \tilde{l}(\tilde{f}) \). It is essentially evolutionary stable if (2) holds for all best replies \( \tilde{f} \) to \( f \) with \( \tilde{f} \neq f \).

**Theorem 2.** Wardrop equilibria are essentially evolutionary stable.

**Proof.** Consider a Wardrop equilibrium \( f \). We show that condition (2) of Definition 4 even holds for an arbitrary invasive population \( \tilde{f} \neq f \), not only for best replies. At a Wardrop equilibrium, all latencies of used strategies are equal and the latencies of unused strategies may be greater or equal to, but not less than the latency of used strategies. Assume that the latencies are fixed to the values induced by \( f \). Then, the average latency of \( \tilde{f} \) with respect to these fixed latencies cannot be less than the average latency of \( f \) with respect to these latencies. More precisely, \( \tilde{f} \cdot \tilde{l}(f) \geq f \cdot \tilde{l}(f) \). Therefore we have

\[
\tilde{f} \cdot \tilde{l}(f) \geq f \cdot \tilde{l}(f) + \tilde{f} \cdot \tilde{l}(\tilde{f}) - \tilde{f} \cdot \tilde{l}(f) = f \cdot \tilde{l}(f) + \sum_{P \in P} \tilde{f}_P \cdot (\ell_P(\tilde{f}) - \ell_P(f)) = f \cdot \tilde{l}(f) + \sum_{e \in E} \tilde{f}_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e))
\]

Now we want to find a lower bound on the terms in the sum. For every edge \( e \) there are three cases.

1. \( \tilde{f}_e > f_e \). Then \( \ell_e(\tilde{f}_e) > \ell_e(f_e) \) by strict monotonicity of the latency functions and therefore \( \tilde{f}_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) > f_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) \).
2. \( \tilde{f}_e < f_e \). Now, \( \ell_e(\tilde{f}_e) < \ell_e(f_e) \), but again \( \tilde{f}_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) > f_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) \).
3. \( \tilde{f}_e = f_e \). This implies that \( \tilde{f}_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) = f_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) \).
3.2 Stability

Since \( f \neq \tilde{f} \), we are in one of the first two cases at least once. Altogether we have

\[
\sum_{e \in E} \tilde{f}_P \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) > \sum_{e \in E} f_e \cdot (\ell_e(\tilde{f}_e) - \ell_e(f_e)) = f \cdot \bar{\ell}(\tilde{f}) - f \cdot \bar{\ell}(f)
\]

and therefore \( \tilde{f} \cdot \bar{\ell}(\tilde{f}) > f \cdot \bar{\ell}(f) \).

The well-known entropy function which was used in the analysis of the rock-scissors-paper game can also be used to show convergence in our model if the number of commodities is limited to one.

**Lemma 1.** Let \( f \in \Delta \) be a Wardrop equilibrium. The conditional entropy

\[
H_f(\xi) = \sum_{P \in \mathcal{P}} f_P \ln \frac{f_P}{\xi_P}
\]

is a Lyapunov function for the replicator dynamics (2) on the interior of \( \Delta \) if \( \lambda(f) \) is positive definite.

**Proof.** Since the interior of the simplex \( \Delta \) is invariant, \( \xi_P(t) > 0 \) for all \( P \in \mathcal{P} \) and for all \( t \). Therefore, the entropy is well-defined. Again, we translate the coordinate system such that \( f \) coincides with the origin. Then, \( H_f \) is positive definite. We must show that \( \dot{H}_f \) is negative definite.

Using the chain rule, the time-derivative of \( H_f \) is

\[
\dot{H}_f(\xi) = - \sum_{P \in \mathcal{P}} f_P \xi_P \frac{1}{\xi_P}
\]

Substituting the replicator dynamics for \( \dot{\xi}_P \) and canceling out \( \xi_P \) we get

\[
\dot{H}_f(\xi) = \lambda(\xi) \sum_{P \in \mathcal{P}} f_P \cdot (\ell_P(\xi) - \bar{\ell}(\xi))
\]

\[
= \lambda(\xi) \left( \sum_{P \in \mathcal{P}} f_P \cdot \ell_P(\xi) - \sum_{P \in \mathcal{P}} \xi_P \cdot \ell_P(\xi) \right)
\]

\[
= \lambda(\xi) \left( \sum_{P \in \mathcal{P}} (f_P - \xi_P) \cdot \ell_P(\xi) \right)
\]

\[
= \lambda(\xi) \cdot (f - \xi) \cdot \bar{\ell}(\xi).
\]

Since \( f \) is a Wardrop equilibrium, the proof of Theorem 2 asserts that \( (f - \xi) \cdot \bar{\ell}(\xi) < 0 \) if \( \xi \neq f \). Hence, \( \dot{H}_f \) is negative definite. 

\[
\square
\]

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Corollary 1. For single-commodity networks, solutions of the replicator dynamics (2) with initial condition \( f(0) \in \text{int} \, \Delta \) converge towards a Wardrop equilibrium if \( \lambda(f) \) is positive definite, i.e. for a given Wardrop equilibrium \((f_e)_{e \in E}\) and a solution \( \tilde{f}(t) \) it holds that \( \lim_{t \to \infty} \| (f_e)_{e \in E} - (\tilde{f}_e)_{e \in E} \| = 0 \).

We remark, that with minor changes this proof would also work for the multi-commodity case, as long as the functions \( \lambda_i(f) \) are identical for all commodities \( i \in \mathcal{I} \). If the number of commodities is greater than one, however, we have to take into account, that, in general, the \( \lambda_i \) may be different for every commodity \( i \in \mathcal{I} \).

We consider a Lyapunov function defined on the space of edge flows \( \mathbb{R}^{|E|} \). A well known function is the continuous generalization of the potential function introduced by Rosenthal [7] for congestion games

\[
\Phi(f) := \sum_{e \in E} \int_0^{f_e} \ell_e(u) \, du + c.
\]

It is known that Wardrop equilibria minimize \( \Phi \) [1]. We choose the constant \( c \) such that \( \inf_{f \in \Delta} \{ \Phi(f) \} = 0 \).

Lemma 2. The function \( \Phi(f) \) is a Lyapunov function for the replicator dynamics (3) in \( \mathbb{R}^{|E|} \), if the \( \lambda_i(f) \) are positive definite for all commodities \( i \in \mathcal{I} \).

Proof. After translating the coordinate space such that the Wardrop equilibrium (which is unique with respect to edge flows) coincides with the origin, \( \Phi \) is positive definite, since it has a unique minimum.

Let \( L_e \) be an antiderivative of \( \ell_e \). The time-derivative of \( \Phi \) is

\[
\dot{\Phi} = \sum_{e \in E} \dot{L}_e(f_e) = \sum_{e \in E} \dot{f}_e \cdot \ell_e(f_e) = \sum_{e \in E} \sum_{P \ni e} \dot{f}_P \cdot \ell_e(f_e).
\]
3.3 Time of Convergence

Now we substitute the replicator dynamics (3) into this equation and obtain

\[\dot{\Phi} = \sum_{e \in \mathcal{E}} \sum_{P \ni e} (\lambda_i(P)(f) \cdot f_P \cdot (\bar{\ell}_i(P)(f) - \ell_P(f))) \cdot \ell_e(f_e)\]

\[= \sum_{i \in \mathcal{I}} \lambda_i(f) \cdot \sum_{P \in \mathcal{P}_i} \sum_{e \in P} f_P \cdot (\bar{\ell}_i(f) - \ell_P(f)) \cdot \ell_e(f_e)\]

\[= \sum_{i \in \mathcal{I}} \lambda_i(f) \cdot \sum_{P \in \mathcal{P}_i} f_P \cdot (\bar{\ell}_i(f) - \ell_P(f)) \cdot \ell_P(f)\]

\[= \sum_{i \in \mathcal{I}} \lambda_i(f) \left( r_i \cdot (\bar{\ell}_i(f))^2 - \sum_{P \in \mathcal{P}_i} f_P \cdot \ell_P(f)^2 \right)\]

\[= \sum_{i \in \mathcal{I}} \lambda_i(f) \cdot r_i \cdot \left( \bar{\ell}(f)^2 - \sum_{P \in \mathcal{P}} \frac{f_P}{r_i} \ell_P(f)^2 \right) .\]

By Jensen’s inequality this difference is bounded from above by 0. Furthermore it can only be zero if all latencies of used paths are equal or if \(f_P = 0\) for some \(P \in \mathcal{P}\). Therefore, \(\Phi\) fulfills all properties of a Lyapunov function.

Applying Lyapunov’s method only asserts asymptotic stability. However, since the Lyapunov function \(\Phi\) is defined on the entire simplex \(\Delta\) and the vector field of the differential equation always points strictly into the interior of every contour set of \(\Phi\), the Wardrop equilibrium attracts the entire interior of the simplex.

**Corollary 2.** For multi-commodity networks, solutions of the replicator dynamics (3) with initial condition \(f(0) \in \text{int } \Delta\) converge towards a Wardrop equilibrium in terms of edge flows if the \(\lambda_i(f)\) are positive definite for all commodities \(i \in \mathcal{I}\).

3.3 Time of Convergence

From the computer scientist’s perspective, one is usually interested in the running time of algorithms. One could also consider the process of evolutionary selfish routing as a kind of parallel randomized algorithm. Then a natural question would be: What is the time until an equilibrium is reached or approximated.

Before we can analyze this, recall, that we still have to specify values for the functions \(\lambda_i(f)\). When talking about convergence, this does not matter much. As long as the number of commodities is only 1, the orbit is invariant under a change of \(\lambda\), and \(\lambda\) merely determines the speed at which the orbit
3. Evolutionary Selfish Flow

is traversed. Since we do not want the speed to depend on the scale by
which we measure latency, we must normalize the term $\ell(f) - \ell_p$ in some
way. Therefore we propose to choose $\lambda_i(f) = 1/\ell_i(f)$.

Obviously the replicator dynamics will never actually reach a Wardrop
equilibrium, unless it starts at one. Therefore we have to define approximate
equilibria. In evolutionary game theory, approximate equilibria are some-
times defined as a population vector where no strategy’s payoff exceeds any
other used strategy’s payoff by more than an additive term $\epsilon$. In computer
science approximations up to additive terms are rarely considered. Since
the quality of an approximation should be independent of scale, one usually
defines relative approximation rates. We propose a definition where agents
are allowed to derive by a factor of $(1 + \epsilon)$ from the average latency. Denote
the set of expensive paths by

$$\mathcal{P}_e = \{ P \in \mathcal{P} : \ell_P(f) > (1 + \epsilon) \cdot \ell_i(f) \}$$

We could also require all used paths to be non-expensive. However, by
choosing a starting vector with $f_P$ arbitrarily close to 0 for some path $P$
which is used in the Wardrop equilibrium we can make the time to reach an
approximate equilibrium in this sense arbitrarily large. Therefore we allow
an $\epsilon$-fraction of agents to travel on expensive paths. Denote the number of
agents using paths from $P_e$ by $f_e$.

**Definition 5.** A population vector $f$ is an $\epsilon$-approximate equilibrium iff
$f_e \leq \epsilon$.

Note that once an $\epsilon$-approximate equilibrium is reached, it can be left
again, when sleeping minorities on small links start to grow.

We are interested in the time it takes to reach an $\epsilon$-approximate equi-
librium. We will give bounds on the time to reach an $\epsilon$-equilibrium in
terms of maximal and minimal latency. Let $\ell_{\text{max}} := \max_{e \in E} \ell_e(1)$ and
$\ell^* = \min_{i \in I} \ell_i(f)$ where $f$ is a Wardrop equilibrium. Furthermore let
$r^* := \min_{i \in \mathcal{I}} r_i$ be the demand of the smallest commodity. The following
bounds are proved in [3].

**Theorem 3.** Given a network $G = (V, E)$, latency functions $\ell_e$, $e \in E$, an
arbitrary flow vector $f_0 \in \Delta$, and a number $\epsilon > 0$, let $\xi(t)$ be a solution to
the replicator dynamics (3) with boundary condition $f(0) = f_0$. Then upper
bounds for the first $t$ for which $\xi(t)$ is an $\epsilon$-approximate equilibrium are given
by the following table.

<table>
<thead>
<tr>
<th>Topology of $G$</th>
<th>Commodities</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>Single</td>
<td>$O(\epsilon^{-3} \cdot \ln \frac{\ell_{\text{max}}}{\ell^*})$</td>
</tr>
<tr>
<td>Parallel Links</td>
<td>Multi</td>
<td>$O(\epsilon^{-2} \cdot \ln \frac{\ell_{\text{max}}}{\ell^*})$</td>
</tr>
<tr>
<td>General</td>
<td>Multi</td>
<td>$O(\epsilon^{-2} \cdot r^* - 1 \cdot \frac{\ell_{\text{max}}}{\ell^*})$</td>
</tr>
</tbody>
</table>
Note that in the context of algorithms, where the network parameters are the input, one would consider the bounds for general single-commodity networks and for multi-commodity parallel links networks to be polynomial, since they are polynomial in the length of the encoding of the input parameters $\ell_{\text{max}}$ and $\ell^*$. On the contrary, the upper bound for general multi-commodity links would be considered to be pseudo-polynomial.

There is also a worst case example showing a tight lower bound for a single-commodity parallel links network.

**Theorem 4.** For any $\alpha := \frac{\ell_{\text{max}}}{\ell^*}$ there exists a single-commodity network of parallel links $G$ such that the first $t$ for which the solution to the replicator dynamics (3) reaches an $\epsilon$-approximate equilibrium is bounded from below by $\Omega(\epsilon^{-2} \cdot \ln \alpha)$.

The bound is tight for this type of network. It is an open question if the linear term of $\ell_{\text{max}}$ in the bound for general multi-commodity networks is really necessary.

### 4 Summary

We have seen that evolutionary game theory can overcome the assumptions that classical game theory imposes on the agents – rationality, complete and accurate knowledge about the game and the other players. This is done by modeling the agents’ behavior by reasonable processes, leading – in the fluid limit – to differential equations, the most famous of which is the replicator dynamics. In the context of differential equations, certain kinds of critical points take the role of equilibria. We have seen that Nash equilibria do not necessarily correspond to asymptotically stable critical points. However, in the case of selfish routing games we have seen that the replicator dynamics always converges towards Wardrop equilibria. In the case of single commodity networks and multi-commodity parallel links, approximate equilibria are also approached rapidly.

### References


REFERENCES


